Malaya Journal of Matematik, Vol. 6, No. 3, 521-529, 2018

https://doi.org/10.26637/MJM0603/0010

Energy decay of solutions for viscoelastic wave equations with a dynamic boundary and delay term

Mahdi Fatima Zohra[1], Ferhat Mohamed [2] and Hakem Ali[3]*

Abstract

In this paper, we establish a general decay result by using Nakao's technique for a system of multi-dimensional viscoelastic wave equations with dynamic boundary conditions related to the Kelvin Voigt damping and delay term acting on the boundary.

Keywords

Global existence, energy decay, blow up of solutions, nonlinear damping.

AMS Subject Classification

35L60, 35K55, 26A33, 35B44, 35B33.

[1,2,3] *Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, ALGERIA.*

*Corresponding author: [3] hakemali@yahoo.com; [1]mahdifatimazohra@yahoo.fr; [2]ferhat22@hotmail.fr

Article History: Received **12** January **2018**; Accepted **18** May **2018**

Contents

1. Introduction

In this article, we investigate the following wave equation with dynamic boundary conditions and delay term:

$$\begin{cases} u_{tt}-\Delta u-\displaystyle\int_0^t g(t-s)\Delta u(s)ds-\delta\Delta u_t=|u|^{p-1}u, \\ in \quad \Omega\times(0,+\infty), \\ u=0, \quad on \quad \Gamma_0\times(0,+\infty), \\ \\ u_{tt}=-a[\frac{\partial u}{\partial \upsilon}(x,t)+\delta\frac{\partial u_t}{\partial \upsilon}(x,t)+\mu_1(t)u_t(x,t) \\ +\mu_2(t)u_t(x,t-\tau)], \quad on \quad \Gamma_1\times(0,+\infty), \\ \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x\in\Omega, \\ u_t(x,t-\tau)=f_0(x,t-\tau), \quad on \quad \Gamma_1\times(0,+\infty), \end{cases} \tag{1.1}$$

where $u=u(x,t)$, $t\geq 0$, $x\in\Omega$ and Δ denotes the Laplacian operator with respect to the x variable. Ω is a regular and bounded domain of $\mathbb{R}^N$, $(N\geq 1)$, $\partial\Omega=\Gamma_1\cup\Gamma_0$, $\Gamma_1\cap\Gamma_0=\emptyset$ and $\frac{\partial}{\partial\nu}$ denotes the unit outer normal derivative, μ_1 and μ_2 are functions depend on t. Moreover, $\tau>0$ represents the delay and u_0, u_1, f_0 are given functions belonging to suitable spaces that will be specified later. This type of problems arises (for example) in modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term Δu_t, indicates that the stress is proportional not only to the strain, but also to the strain rate (See [5]).

This type of problem without delay (i.e $\mu_i=0$), has been considered by many authors during the past decades and many results have been obtained (see [4], [6], [7], [13]) and the references therein.

Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1.1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass (see [4], [1], [6] for more details), which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also some of them as in problem (1.1) appear when we assume that is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [2] for more details). This type of dynamic boundary conditions is known as acoustic boundary conditions. Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-Van Dalsen ([7], [8], [9]) in which the author has made contributions to this field. And In [7] the author has introduced a model which describes the damped longitudinal vibrations of a homogeneous flexible horizontal rod of length L when the end $x = 0$ is rigidly fixed while the other end $x = L$ is free to move with an attached load. This yields to a system of two second order equations of the form:

$$\begin{cases} u_{tt} - u_{xx} - u_{txx} = 0, \quad x \in (0,L), \quad t > 0, \\ u = 0, \qquad\qquad t > 0, \\ u_{tt}(L,t) = -[u_x + u_{tx}](L,t),\ t > 0, \\ u_t(x,0) = u_1(x), u_t(L,0) = \mu,\ x \in (0,L), \\ u(L,0) = \eta, \quad u_t(L,0) = \mu. \end{cases} \tag{1.2}$$

By rewriting problem (1.2) within the framework of the abstract theories of the so-called B-evolution theory, the existence of a unique solution in the strong sense has been shown and an exponential decay result was also proved in [8] for a problem related to (1.2), which describes the weakly damped vibrations of an extensible beam (See [8] for more details). Subsequently, Zang and Hu [30], considered the problem

$$u_{tt} - p(u_x)_{xt} - q(u_x)_x = 0, \quad x \in (0,1), \quad t > 0,$$

$$p(u_x)_t + q(u_x)(1,t) + ku_{tt}(1,t) = 0, \quad u(0,t) = 0, \quad t \geq 0.$$

By using the Nakao's inequality and under appropriate conditions on p and q, they established both exponential and polynomial decay rates for the energy depending on the form of the terms p and q. It is clear that in the absence of the delay term and for $\mu_1 = 0$, problem (1.2) is the one dimensional model of (1.1). Similarly, and always in the absence of the delay term, Pellicer and Sola-Morales [22] considered the one dimensional problem of (1.1) as an alternative model for the classical spring-mass damper system and by using the dominant eigenvalues method, they proved that their system has the classical second order differential equation

$$m_1 u''(t) + d_1 u'(t) + k_1 u(t),$$

as a limit, where the parameters m_1, d_1 and k_1 are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence.(See also [21], [22]) for related results.

It is widely known that delay effects, which arise in many practical problems are source of some instabilities. In this way Datko and al [20] showed that a small delay in a boundary control turns to be a well-behaved hyperbolic system into a wide one which in turn, becomes a source of instability. Nicaise and al [12] studied the following system of a wave equation with a linear boundary term:

$$\begin{cases} u_{tt} - \triangle u(x,t) = 0, \quad in \quad \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad on \quad \Gamma_0 \times (0,+\infty), \\ \frac{\partial u}{\partial \upsilon} = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\sigma), on\ \Gamma_1 \times \mathbb{R}_+^*, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega, \quad t \in (0,\tau), \end{cases} \tag{1.3}$$

and proved that the energy is exponentially stable, where υ is the unit outward normal to $\partial\Omega$, under the condition

$$\mu_2 < \mu_1. \tag{1.4}$$

On the contrary, if (1.4) doesn't hold, there is a sequence of delays for which the corresponding solution of (1.3) will be instable.

The problem (1.3) with time varying delay term has been studied by Nicaise and al. We refer the readers to ([11], [12]).

Recently, inspired by the works of Al and Nicaise [12], Sthéphane Gherbi and Said El Houari [15] considered the following problem in bounded domain:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = 0, \quad in \quad \Omega \times (0,+\infty), \\ u = 0, \quad on \quad \Gamma_0 \times (0,+\infty), \\ u_{tt} = -a[\frac{\partial u}{\partial \upsilon}(x,t) + \alpha \frac{\partial u_t}{\partial \upsilon}(x,t) + \mu_1 u_t(x,t) \\ +\mu_2 u_t(x,t-\tau)], \quad on \quad \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau(t)) = f_0(x,t-\tau), on\ \Gamma_1 \times (0,+\infty), \end{cases} \tag{1.5}$$

and obtained several results concerning global existence and exponential decay rates for various sign of μ_1, μ_2.

Motivated by the previous works, in the present paper we investigate problem (1.1) in which we generalize the results obtained in [31] by supposing new conditions with which the global existence and stability results are assured. The stable set is used to prove the existence result and Nakao's technique to establish energy decay rates.

The content of this paper is organized as follows: In Section 2, we provide assumptions that will be used later. In Section 3, we state and prove the global existence result. In Section 4, the stability result given in Theorem 4.1 will be proved.

2. Preliminary Results

In this section, we present some material in the proof of our main result. We assume

(A_1) $\mu_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left| \frac{\mu_1'(t)}{\mu_1(t)} \right| \leq M, \tag{2.1}$$

(A_2) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta \mu_1(t), \tag{2.2}$$

$$|\mu_2'(t)| \leq \bar{M} \mu_1(t), \tag{2.3}$$

for some $0 < \beta < 1$ and $\bar{M} > 0$. For the relaxation function g we assume

(A_3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l < 1.$$

(A_4) There exists a nonincreasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \leq -\eta(t)g(t).$$

(A_5) We suppose therefore

$$2 \leq p \leq \frac{2n-2}{n-2} \quad if \quad n \geq 3; \quad p > 2, \quad if \quad n = 1,2. \tag{2.4}$$

Now we choose $\tilde{\zeta}$ such that

$$\tau\beta < \tilde{\zeta} < \tau(2-\beta). \tag{2.5}$$

Lemma 2.1. *(Sobolev-Poincaré's inequality). Let $2 \leq m \leq \frac{2n}{n-2}$. The inequality*

$$\|u\|_m \leq c_s \|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega),$$

holds with some positive constant c_s.

Lemma 2.2. *[2] For any $g \in C^1$ and $\phi \in H^1(0,T)$, we have*

$$\int_0^t \int_\Omega g(t-s)\varphi\varphi_t dxds = -\frac{1}{2}\frac{d}{dt}\left((go\varphi)(t) + \int_0^t g(s)ds\|\varphi\|_2^2\right) - g(t)\|\varphi\|_2^2 + (g'o\varphi)(t),$$

where

$$(go\varphi)(t) = \int_0^t g(t-s)\int_\Omega |\varphi(s) - \varphi(t)|^2 dxds.$$

Lemma 2.3. *[2] For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx \leq (1-l)c_s^2 \times (go\nabla u)(t), \tag{2.6}$$

where c_s^2 is the Poincaré's constant and l is given in (A_3) and

$$(go\nabla u)(t) = \int_0^t g(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2 dxds.$$

Lemma 2.4. *[32] Let ϕ be a nonincreasing and nonnegative function on $[0,T]$, $T > 1$, such that*

$$\phi(t)^{1+r} \leq \omega_0(\phi(t) - \phi(t+1)), \quad on \quad [0,T],$$

where $\omega_0 > 1$ and $r \geq 0$. Then we have, for all $t \in [0,T]$
(i) if $r = 0$, then

$$\phi(t) \leq \phi(0)e^{-\omega_1[t-1]^+},$$

(ii) if $r > 0$, then

$$\phi(t) \leq \left(\phi(0)^{-r} + \omega_0^{-1} r[t-1]^+\right)^{\frac{-1}{r}},$$

where $\omega_1 = \ln\left(\frac{\omega_0}{\omega_0 - 1}\right)$ and $[t-1]^+ = max(t-1,0)$.

3. Global existence and energy decay

We introduce the new variable z as in [12],

$$z(x,k,t) = u_t(x,t-\tau k), \quad x \in \Gamma_1, \quad k \in (0,1),$$

which implies that

$$\tau z_t(x,k,t) + z_k(x,k,t) = 0, \quad in \quad \Gamma_1 \times (0,1) \times (0,\infty).$$

Therefore, problem (1.1) can be transformed as follows

$$\begin{cases} u_{tt} - \Delta u - \delta\Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-1}u, \\ \quad in \quad \Omega \times (0,\infty), \\ u_{tt} = -a\left[\frac{\partial u}{\partial \upsilon}(x,t) + \alpha\frac{\partial u_t}{\partial \upsilon}(x,t) + \mu_1(t)u_t(x,t) \right. \\ \left. +\mu_2(t)z_k(x,1,t)\right], \ on \quad \Gamma_1 \times (0,+\infty), \\ \tau z_t(x,k,t) + z_k(x,k,t) = 0, \ in \ \Gamma_1 \times (0,1) \times \mathbb{R}_+^*, \\ z(x,k,0) = f_0(x,-\tau k), \quad x \in \Gamma_1, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \Gamma_0, \quad t \geq 0. \end{cases} \tag{3.1}$$

Remark 3.1. *For seeking of simplicity, we take $a = 1$ in (3.1).*

Now inspired by [25] and [31], we define the energy functional related with problem (3.1) by

$$\begin{aligned} E(t) = &\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2.\Gamma_1}^2 + \frac{1}{2}(go\nabla u)(t) \\ &+ \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ &- \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\zeta(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,k,s)dkd\gamma, \end{aligned} \tag{3.2}$$

where

$$\zeta(t) = \tilde{\zeta}\mu_1(t).$$

Lemma 3.2. *Let (u,z) be the solution of (3.1) then, the energy equation satisfies*

$$
\begin{aligned}
E'(t) \leq & \frac{1}{2}(g' o \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 \\
& - \mu_1(t)\left(1 - \frac{\tilde{\zeta}}{2\tau} - \frac{\beta}{2}\right) \|u_t(t)\|_{2.\Gamma_1}^2 \\
& - \mu_1(t)\left(\frac{\tilde{\zeta}}{2\tau} - \frac{\beta}{2}\right) \|z(x,1,t)\|_{2.\Gamma_1}^2 \\
& - \delta \|\nabla u_t(t)\|_2^2 \leq 0
\end{aligned} \tag{3.3}
$$

Proof. By multiplying the first and second equation in (3.1) by $u_t(t)$, integrating the first equation over Ω and the second equation over Γ_1, using Green's formula, we get

$$
\begin{aligned}
& \frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 \right. \\
& \left. - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}\right] - \delta \|\nabla u_t\|_2^2 + \mu_1(t)\|u_t(t)\|_{2.\Gamma_1}^2 \\
& - \int_\Omega \int_0^t g(t-s) \nabla u(s) \nabla u_t(t)\, ds\, dx \\
& + \int_{\Gamma_1} \mu_2(t) z(\gamma,1,t) u_t(t)\, d\gamma = 0.
\end{aligned} \tag{3.4}
$$

As in [?] we multiply the third equation in (3.1) by $\zeta(t)z$ and integrate over $\Gamma_1 \times (0,1)$ to obtain

$$
\begin{aligned}
& \zeta(t)\tau \int_{\Gamma_1}\int_0^1 z_t z(\gamma,k,t)\, dk\, d\gamma \\
& + \zeta(t) \int_{\Gamma_1}\int_0^1 z_k z(x,k,t)\, dk\, d\gamma = 0,
\end{aligned} \tag{3.5}
$$

this yields

$$
\begin{aligned}
& \frac{\zeta(t)\tau}{2}\frac{d}{dt}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)\, dk\, d\gamma \\
& + \frac{\zeta(t)}{2}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial k} z^2(\gamma,k,t)\, dk\, d\gamma = 0,
\end{aligned} \tag{3.6}
$$

then

$$
\begin{aligned}
& \frac{\tau}{2}\left[\frac{d}{dt}\left(\zeta(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)\, dk\, d\gamma\right) \right. \\
& \left. - \zeta'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)\, dk\, d\gamma\right] \\
& + \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(\gamma,1,t)\, d\gamma - \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(x,0,t)\, d\gamma = 0,
\end{aligned} \tag{3.7}
$$

consequently

$$
\begin{aligned}
& \frac{\tau}{2}\frac{d}{dt}\left(\zeta(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)\, dk\, d\gamma\right) = \\
& - \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(\gamma,1,t)\, d\gamma + \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(x,0,t)\, d\gamma \\
& + \frac{\tau}{2}\zeta'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)\, dk\, d\gamma,
\end{aligned} \tag{3.8}
$$

finally from (3.4) and (3.7), we get

$$
\begin{aligned}
& E(t) + \left(\mu_1(t) - \frac{\zeta(t)}{2\tau}\right)\|u_t(t)\|_{2.\Gamma_1}^2 \\
& + \mu_2(t)\int_{\Gamma_1} z^2(\gamma,1,t) u_t(\gamma,t)\, d\gamma \\
& - \frac{\zeta'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)\, dk\, d\gamma + \frac{\zeta(t)}{2\tau}\int_{\Gamma_1} z^2(\gamma,1,t)\, d\gamma \\
& - \frac{1}{2}\int_0^t (g' o \nabla u)(s)\, ds + \frac{1}{2}\int_0^t g(s)\|\nabla u(s)\|_2^2\, ds = 0.
\end{aligned} \tag{3.9}
$$

Due to Young's inequality, we have

$$
\begin{aligned}
\int_{\Gamma_1} z(\gamma,1,t) u_t(\gamma,t)\, d\gamma \leq & \frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2 \\
& + \frac{1}{2}\int_{\Gamma_1} z^2(\gamma,1,t)\, d\gamma.
\end{aligned} \tag{3.10}
$$

Noting that $\zeta'(t) \leq 0$. Inserting (3.10) into (3.9) and deriving it, we get the desired result. ☐

Now we are in position to state the local existence result to problem (3.1), which can be established by combining arguments of ([19],[20],[22]).

Theorem 3.3. *Let $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ and $f_0 \in L^2(\Gamma_1 \times (0,1))$ be given. Suppose that $(A_1) - (A_5)$ hold. Then the problem (3.1) admits a unique weak solution (u,z) satisfying*

$$
u \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)), \quad u_t \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)),
$$
$$
u_{tt} \in L^\infty((0,T); L^2(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)).
$$

Now we will prove that the solution obtained above is global and bounded in time. For this purpose let us define

$$
\begin{aligned}
I(t) = & \left(1 - \int_0^t g(s)\, ds\right)\|\nabla u\|_2^2 + (g o \nabla u)(t) \\
& - \|u\|_{p+1}^{p+1} + \zeta(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,s)\, dk\, d\gamma,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
J(t) = & \frac{1}{2}(g o \nabla u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\|\nabla u\|_2^2 \\
& - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\zeta(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,s)\, dk\, d\gamma,
\end{aligned} \tag{3.12}
$$

and

$$
E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2.\Gamma_1}^2. \tag{3.13}
$$

Lemma 3.4. *Suppose that assumptions $(A_3) - (A_4)$ are fulfilled. Let (u,z) be the solution of the problem (12). Assume further that $I(0) > 0$ and*

$$
\alpha = \left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{3.14}
$$

then $I(t) > 0$ for all $t \geq 0$.

Proof. Since $I(0)>0$, then there exists (by continuity of $u(t)$) $T^*<T$ such that for all $t\in[0,T^*]$, we have

$$I(t)\geq 0. \tag{3.15}$$

From (3.11) and (3.12) we obtain

$$\begin{aligned} J(t) &\geq \frac{p-1}{2(p+1)}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \right.\\ &\left.+go\nabla u(t)\right]+\frac{1}{p+1}I(t)\\ &+\frac{(p-1)}{2(p+1)}\left[\zeta(t)\int_0^1\int_{\Gamma_1} z^2(x,k,t)dkd\gamma\right]\\ &\geq \frac{p-1}{2(p+1)}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right.\\ &\left.+(go\nabla u)(t)\right]\\ &+\frac{(p-1)}{2(p+1)}\left[\zeta(t)\int_0^1\int_{\Gamma_1} z^2(\gamma,k,t)dkd\gamma\right]\\ &\geq \frac{p-1}{2(p+1)}l\|\nabla u\|_2^2. \end{aligned} \tag{3.16}$$

Thus by using (3.13) and (3.14) we deduce $\forall t\in[0,T^*]$

$$\begin{aligned} l\|\nabla u\|_2^2 &\leq \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\\ &\leq \frac{2(p+1)}{(p-1)}E(t)\leq\frac{2(p+1)}{(p-1))}E(0). \end{aligned} \tag{3.17}$$

Exploiting Lemma 2.1 and formula (3.17), we obtain

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq c_s^{p+1}\|\nabla u\|_2^{p+1}\leq\frac{c_s^{p+1}}{l}\|\nabla u\|_2^{p-1}\|\nabla u\|_2^2\\ &\leq\frac{c_s^{p+1}}{l}\left(\frac{2(p+1)}{(p-1)l}E(0)\right)^{\frac{p-1}{2}}l\|\nabla u\|_2^2=\alpha l\|\nabla u\|_2^2\\ &<\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2,\ \forall t\in[0,T^*]. \end{aligned} \tag{3.18}$$

Hence for all $t\in[0,T^*]$, we have

$$\begin{aligned} I(t) &= \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+(go\nabla u)(t)-\|u\|_{p+1}^{p+1}\\ &+\zeta(t)\int_0^1\int_{\Gamma_1} z^2(\gamma,k,t)dkd\gamma>0. \end{aligned}$$

Repeating this procedure and using the fact that

$$\lim_{t\to T^*}\frac{c_s^{p+1}}{l}\left(\frac{2(p+1)}{2l(p-1)}E(u(t))\right)^{\frac{p-1}{2}}\leq\alpha<1,$$

we can take $T^*=T$. This completes the proof. □

4. Stability result

Theorem 4.1. *Assume that $(A_3)-(A_5)$ hold. Let $u_0\in H^1_{\Gamma_0}(\Omega)$, $u_1\in L^2(\Omega)$ and $f_0\in L^2(\Gamma_1\times(0,1))$ be given. Then the solution of the problem (3.1) is global and bounded in time. Furthermore, there exists $\theta>0$, such that*

$$\theta>\frac{4-3l-l^2}{2l}, \tag{4.1}$$

and we have the following decay estimate:

$$E(t)\leq E(0)e^{-\sigma t},\quad \forall t\geq 0,\quad \sigma=\ln\left(\frac{c_{12}}{c_{12}-1}\right),$$

where c_{12} is a positive constant.

Proof. First, we prove $T=\infty$. It is sufficient to show that $l\|\nabla u\|_2^2$ is bounded independently of t. From (24) we have

$$E(0)\geq E(t)=\frac{1}{2}\|u_t\|_2^2+\|u_t\|_{2.\Gamma_1}^2+J(t)u\|_2^2\geq l\|\nabla u\|_2^2.$$

Therefore $l\|\nabla u\|_2^2\leq\rho E(0)$, where ρ is a positive constant which depends only on p, thus we obtain the global existence result. From now and on, we focus our attention to the decay rate of the solutions to problem (3.1). In order to do so, we will derive the decay rate of the energy function for problem (3.1) by Nakao's method, as in [32]. For this aim, we have to show that the energy function defined by (3.13) satisfies the hypotheses of Lemma 2.4. By integrating (3.3) over $[t,t+1]$, we have

$$E(t)-E(t+1)=D(t)^2, \tag{4.2}$$

where

$$\begin{aligned} D(t)^2 &= c_1\int_t^{t+1}\mu_1(s)\|u_t\|_{2.\Gamma_1}^2ds\\ &+c_2\int_t^{t+1}\mu_2(s)\int_{\Gamma_1}z^2(\gamma,1,s)d\gamma ds\\ &-\int_t^{t+1}\frac{1}{2}(g'o\nabla u)(s)ds+\int_t^{t+1}\frac{1}{2}g(s)\|\nabla u(s)\|_2^2ds\\ &+c_3\int_t^{t+1}\|\nabla u_t\|_2^2ds. \end{aligned} \tag{4.3}$$

By virtue of (4.3) and Holder's inequality, we observe that

$$\begin{aligned} &\int_t^{t+1}\int_{\Gamma_1}\mu_1(s)|u_t|^2d\gamma ds\\ &+\int_t^{t+1}\mu_2(s)\int_{\Gamma_1}|z(\gamma,1,s)|^2d\gamma ds+\int_t^{t+1}\|\nabla u_t\|_2^2ds\\ &\leq c(\Gamma_1)D(t)^2, \end{aligned} \tag{4.4}$$

where $c(\Gamma_1)=vol(\Gamma_1)$. Applying the mean value, then there exist $t_1\in[t,t+\frac{1}{4}]$ and $t_2\in[t+\frac{3}{4},t+1]$ such that for $i=1,2$, we get

$$\begin{aligned} &\mu_1(t_i)\|u_t(t_i)\|_{2.\Gamma_1}^2+\mu_2(t_i)\|z(\gamma,1,t_i)\|_{2.\Gamma_1}^2+\|\nabla u_t(t_i)\|_2^2\\ &\leq c(\Gamma_1)D(t)^2. \end{aligned} \tag{4.5}$$

Multiplying the first equation in (3.1) by u and integrating over $\Omega\times[t_1,t_2]$, multiplying the second equation in (3.1) by u and integrating over $\Gamma_1\times[t_1,t_2]$, adding and subtracting the following term $\int_0^1\int_{\Gamma_1}\zeta(t)z^2(\gamma,k,t)dkd\gamma$, we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2} I(t)dt \le \sum_{i=1}^{2}\|u_t(t_i)\|_2\|u(t_i)\|_2+\int_{t_1}^{t_2}\|u_t\|_2^2dt\\
&+\int_{t_1}^{t_2}\int_\Omega \nabla u_t\nabla u dt dx+\int_{t_1}^{t_2}(go\nabla u)(t)dt\\
&+\int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)\nabla u(t)(\nabla u(s)-\nabla u(t))dsdxdt\\
&+\int_{t_1}^{t_2}\zeta(t)\int_0^1\int_{\Gamma_1} z^2(x,k,t)dkd\gamma dt\\
&-\int_{\Gamma_1}\int_{t_1}^{t_2}\mu_2(t)z(x,1,t)udtd\gamma-\int_{t_1}^{t_2}\mu_1(t)\int_{\Gamma_1}u_t u d\gamma dt.
\end{aligned}\tag{4.6}$$

Since

$$\begin{aligned}
&\int_\Omega\int_0^t g(t-s)\nabla u(t)(\nabla u(s)-\nabla u(t))dsdx\\
&=\frac{1}{2}\left[\int_0^t g(t-s)(\|\nabla u(s)\|_2^2+\|\nabla u(t)\|_2^2)ds\right.\\
&\left.-\int_0^t g(t-s)(\|\nabla u(s)\|_2^2+\|\nabla u(t)\|_2^2)ds\right]\\
&-\int_\Omega\int_0^t g(s)|\nabla u(t)|^2dsdx=\\
&-\frac{1}{2}\int_\Omega\int_0^t g(s)|\nabla u(s)|^2dsdx\\
&+\frac{1}{2}\int_0^t g(t-s)ds\|\nabla u(s)\|_2^2ds-\frac{1}{2}(go\nabla u)(t).
\end{aligned}\tag{4.7}$$

Then (4.6) takes the form

$$\begin{aligned}
&\int_{t_1}^{t_2} I(t)dt \le \sum_{i=1}^{2}\|u_t(t_i)\|_2\|u(t_i)\|_2+\int_{t_1}^{t_2}\|u_t\|_2^2dt\\
&+\frac{1}{2}\int_{t_1}^{t_2}(go\nabla u)(t)dt+\int_{t_1}^{t_2}\int_\Omega\nabla u_t\nabla u dt dx\\
&-\frac{1}{2}\int_\Omega\int_0^t g(s)|\nabla u(s)|^2dsdx\\
&+\frac{1}{2}\int_0^t g(t-s)ds\|\nabla u(s)\|_2^2ds\\
&+\int_{t_1}^{t_2}\int_0^1\zeta(t)\int_{\Gamma_1}z^2(x,k,t)dkd\gamma dt\\
&-\int_{\Gamma_1}\int_{t_1}^{t_2}\mu_2(t)z(x,1,t)udtd\gamma-\int_{t_1}^{t_2}\mu_1(t)\int_{\Gamma_1}u_t u d\gamma dt.
\end{aligned}\tag{4.8}$$

Now we will estimate the right hand side of (4.8). First by (4.7) and Lemma 2.1, we have

$$\begin{aligned}
&\|u_t(t_i)\|_2\|u(t_i)\|_2\le c_s c(\Gamma)^{\frac{1}{2}}D(t)\sup_{t_1\le s\le t_2}\|\nabla u(s)\|_2\\
&\le D(t)c_s c(\Gamma)^{\frac{1}{2}}\left(\frac{2(p+1)}{p-1}\right)^{\frac{1}{2}}\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2}}\\
&\le D(t)c_s c(\Gamma)^{\frac{1}{2}}\left(\frac{2(p+1)}{p-1}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}.
\end{aligned}\tag{4.9}$$

As in [33], by employing Young's inequality for convolution $\|\varphi*\phi\|\le\|\varphi\|\|\phi\|$ and noting that

$$l\|\nabla u(t)\|_2^2\le\frac{1}{\theta}I(t),\tag{4.10}$$

then we have

$$\begin{aligned}
&\int_0^t\int_{t_1}^{t_2} g(t-s)\|\nabla u(s)\|_2^2dsdt\\
&\le\int_{t_1}^{t_2}g(t)dt\int_{t_1}^{t_2}\|\nabla u(t)\|_2^2dt\\
&\le(1-l)\int_{t_1}^{t_2}\|\nabla u(t)\|_2^2dt\le\frac{1-l}{l\theta}\int_{t_1}^{t_2}I(t)dt.
\end{aligned}\tag{4.11}$$

By exploiting (4.7), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{t_1}^{t_2}(go\nabla u)(t)dt=\\
&\frac{1}{2}\int_0^t\int_{t_1}^{t_2}g(t-s)\|\nabla u(s)-\nabla u(t)\|dsdt\\
&\le\int_0^t g(t)dt\int_{t_1}^{t_2}\|\nabla u(t)\|dt\\
&\le\int_0^t g(t)dt\int_{t_1}^{t_2}l\|\nabla u(t)\|dt\le\frac{(1-l)}{2\theta}\int_{t_1}^{t_2}I(t)dt.
\end{aligned}\tag{4.12}$$

Multiplying the second equation in (3.1) by ζz and integrating the result over $\Gamma_1\times(0,1)$, we get

$$\begin{aligned}
&\frac{\tau}{2}\frac{d}{dt}\left(\zeta(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\right)=\\
&\frac{\tau}{2}\zeta'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\\
&-\frac{\zeta(t)}{2}\int_{\Gamma_1}z^2(\gamma,1,t)d\gamma+\frac{\zeta(t)}{2}\int_{\Gamma_1}z^2(x,0,t)d\gamma.
\end{aligned}\tag{4.13}$$

Recalling that $\zeta'(t) \leq 0$, we have

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\Gamma_1}\zeta(t)\int_0^1 z^2(x,k,t)dkd\gamma dt \leq \\ &\int_{t_1}^{t_2}\int_{t_1}^{t_2}\frac{\tilde{\zeta}\mu_1(s)}{2\tau}\|u_t(s)\|^2_{2.\Gamma_1}dsd\nu \\ \leq &c\left(\int_{t_1}^{t_2}d\nu\right)\left(\int_{t_1}^{t_2}\mu_1(s)\|u_t(s)\|^2_{2.\Gamma_1}ds\right) \\ &\leq c(\Gamma_1)(t_2-t_1)D(t)^2.\end{aligned} \tag{4.14}$$

Using Sobolev's inequality, also we get

$$\begin{aligned}&\left|\int_{t_1}^{t_2}\mu_2(s)z(x,1,t)udxdt\right| \leq \\ &\int_{t_1}^{t_2}\mu_2(s)\|z(x,1,t)\|_2\|u\|_2dt \leq c_s\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}} \\ &\times\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\mu_2(s)\|z(x,1,t)\|_2dt \\ &\leq c_sc(\Gamma_1)^{\frac{1}{2}}\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D(t),\end{aligned} \tag{4.15}$$

and

$$\int_{t_1}^{t_2}\|u_t\|_2^2dt \leq c_s^2\int_{t_1}^{t_2}\|\nabla u_t\|_2^2dt \leq c_s^2c(\Gamma_1)D(t)^2, \tag{4.16}$$

$$\begin{aligned}&\left|\int_{t_1}^{t_2}\int_\Omega \nabla u_t\nabla udtdx\right| \leq \int_{t_1}^{t_2}\|\nabla u_t\|_2\|\nabla u\|_2dt \\ &\leq \left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|\nabla u_t\|_2 \\ &\leq c(\Gamma_1)\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D(t),\end{aligned} \tag{4.17}$$

also we have

$$\begin{aligned}&\left|\int_{t_1}^{t_2}\int_{\Gamma_1}\mu_1(s)u_tud\gamma dt\right| \leq \\ &\int_{t_1}^{t_2}\mu_1(s)\|u_t\|_{2.\Gamma_1}\|\nabla u\|_{2.\Gamma_1}dt \leq c_sB\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}} \\ &\times\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\mu_1(s)\|u_t\|_{2.\Gamma_1}dt \\ &\leq c_sc(\Gamma_1)B\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D(t),\end{aligned} \tag{4.18}$$

therefore, from (4.9) – (4.18) we deduce

$$\begin{aligned}\int_{t_1}^{t_2}I(t)dt &\leq c(\Gamma_1)^{\frac{1}{2}}\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}} \\ &\times[c_s(B+2)+1]E(t)^{\frac{1}{2}}D(t) \\ &+\left(2\frac{(1-l)}{l\theta}+\frac{1-l}{\theta}\right)\int_{t_1}^{t_2}I(t)dt \\ &+\left(\frac{3}{4}c(\Gamma_1)+c_s^2c(\Gamma_1)\right)D(t)^2.\end{aligned} \tag{4.19}$$

Then, rewriting (4.19), we get

$$c_5\int_{t_1}^{t_2}I(t)dt \leq c_4D(t)^2+c_3E(t)^{\frac{1}{2}}D(t), \tag{4.20}$$

with

$$c_5=\left[1-\frac{(2-2l)}{l\theta}-\frac{1-l}{2\theta}\right], c_4=c(\Gamma_1)\frac{3}{4}+c_s^2c(\Gamma_1),$$

and

$$c_3=c(\Gamma_1)^{\frac{1}{2}}\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}[c_s(B+2)+1].$$

From the condition (4.1) and observing that is equivalent to $c_5>0$, thus

$$\int_{t_1}^{t_2}I(t)dt \leq c_7\left[D(t)^2+E(t)^{\frac{1}{2}}D(t)\right], \tag{4.21}$$

where $c_7=\frac{max(c_3,c_4)}{c_5}$. On the other hand, from the definition of $E(t)$ and by (3.11) and (3.13) we have

$$\begin{aligned}&\int_{t_1}^{t_2}E(t)dt \leq \frac{p-1}{2(p+1)} \\ &\times\int_{t_1}^{t_2}\left[(go\nabla u)(t)+\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right] \\ &+\frac{p-1}{2(p+1)}\int_{t_1}^{t_2}\int_{\Gamma_1}\int_0^1 z^2(x,k,s)dkd\gamma dt \\ &+\frac{1}{p+1}\int_{t_1}^{t_2}I(t)dt+\int_{t_1}^{t_2}\frac{1}{2}\|u_t\|_2^2dt \\ &+\int_{t_1}^{t_2}\frac{1}{2}\|u_t\|^2_{2.\Gamma_1}dt \\ &\leq c_7\frac{p-1}{2(p+1)}\left[\frac{1}{\theta}+\frac{1-l}{2\theta}+\frac{1}{p+1}\right] \\ &\times(D(t)^2+E(t)^{\frac{1}{2}}D(t)) \\ &+\left[c_s^2c(\Gamma_1(1+B)+c(\Gamma_1)\frac{3}{4}\right]D(t)^2 \\ &\leq\left[c_8D(t)^2+c_9E(t)^{\frac{1}{2}}D(t)\right] \\ &\leq c_{10}\left[D(t)^2+E(t)^{\frac{1}{2}}D(t)\right],\end{aligned} \tag{4.22}$$

where

$$c_8 = c_7\frac{p-1}{2(p+1)}\left[\frac{1-l}{2\theta}+\frac{1}{\theta}+\frac{1}{p+1} + c_s^2 c(\Gamma_1)(1+B)+\frac{3}{4}c(\Gamma_1)\right],$$

$$c_9 = c_7\frac{p-1}{2(p+1)}\left[\frac{1-l}{2\theta}+\frac{1}{\theta}+\frac{1}{p+1}\right], c_{10} = max(c_8,c_9).$$

Moreover, integrating (3.3) over (t,t_2) and using (4.22) and the fact that $E(t_2) \le 2\int_{t_1}^{t_2} E(t)dt$, due to $t_2 - t_1 \ge \frac{1}{2}$, we obtain

$$\begin{aligned} E(t) &= E(t_2) + \int_t^{t_2} \frac{1}{2}(g' o\nabla u)(t)dt \\ &+ \int_t^{t_2} \frac{1}{2} g(t)\|\nabla u(t)\|_2^2 dt \\ &+ \int_t^{t_2} \mu_1(t)\left(1-\frac{\tilde{\xi}}{2\tau}-\frac{\beta}{2}\right)\|u_t(t)\|_{2.\Gamma_1}^2 dt \\ &+ \int_t^{t_2} \mu_1(t)\left(\frac{\tilde{\xi}}{2\tau}-\frac{\beta}{2}\right)\|z(x,1,t)\|_{2.\Gamma_1}^2 dt \\ &+ \delta\int_t^{t_2} \|\nabla u_t(t)\|_2^2 dt \le 2\int_{t_1}^{t_2} E(t)dt, \end{aligned} \tag{4.23}$$

hence by exploiting (4.22) we arrive at

$$E(t) \le c_{11}\left[D(t)^2 + E(t)^{\frac{1}{2}}D(t)\right]. \tag{4.24}$$

Then a simple application of Young's inequality gives, for all $t \ge 0$

$$E(t) \le c_{12} D(t)^2, \tag{4.25}$$

where c_{11}, c_{12} are some positive constants. Therefore, using the formula from (4.25), we get

$$E(t) \le c_{12}[E(t) - E(t+1)].$$

Here we choose $c_{12} > 1$. Thus by Lemma 2.4, we obtain for $t \ge 0$

$$E(t) \le E(0)e^{-\sigma t} \quad \text{with} \quad \sigma = ln\left(\frac{c_{12}}{c_{12}-1}\right).$$

□

Acknowledgment

The work of the authors is supported by the "Ministry of Higher Education and Scientific Research, Algeria under grant number: CNEPRU-COOL03UN220120150001".

References

[1] K. T. Andrews, K. L. Kuttler and M. Shillor, Second order evolution equations with dynamic boundary conditions, *J. Math. Anal. Appl.*, 197(3) (1996), 781–795.

[2] J. T. Beale, Spectral properties of an acoustic boundary condition, *Indiana Univ. Math. J.*, 25(9) (1976), 895–917.

[3] P. Pucci and J. Serrin, Asymptotic stability for nonlinear parabolic systems, Energy Methods in Continuum Mechanics, *Kluwer Acad. Publ. Dordrecht.*, (1996).

[4] B. M. Budak, A. A. Samarskii and A. N. Tikhonov, A collection of problems on mathematical physics, Translated by A. R. M. Robson. *The Macmillan Co, New York.*, 1964.

[5] R. W. Caroll and R. E. Showalter, Singular and Degenerate Cauchy Problems, *Academic Press, New York.*, 1976.

[6] F. Conrad and O. Morgul, Stabilization of a flexible beam with a tip mass, *SIAM J. Control Optim.*, 36(6) (1998), 1962–1986.

[7] M. Grobbelaar-Van Dalsen, On fractional powers of a closed pair of operators and a damped wave equation with dynamic boundary conditions, *Appl. Anal.*, 53(1-2) (1994), 41–54.

[8] M. Grobbelaar-Van Dalsen, On the initial-boundary-value problem for the extensible beam with attached load, *Math. Methods Appl. Sci.*, 19(12) (1996), 943–957.

[9] M. Grobbelaar-Van Dalsen, On the solvability of the boundary-value problem for the elastic beam with attached load,*Math. Models Meth. Appl. Sci. (M3AS).*, 4(1) (1994), 89–105.

[10] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, 45(5) (2006), 1561–1585.

[11] S. Nicaise, J. Valein and E. Fridman, Stabilization of the heat and the wave equations with boundary time-varying delays, *DCDS-S.*, 2(3) (2009), 559–581.

[12] S. Nicaise, C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *DCDS-S.*, 4(3) (2011), 693–722.

[13] S. Gerbi and B. Said-Houari, Asymptotic stability and blow up for a semilinear damped wave equation with dynamic boundary conditions, *Nonlinear analysis.*, 74(2011), 7137–7150.

[14] S. Gerbi and B. Said-Houari, Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions, *Advances in Differential Equations.*, 13(11-12) (2008), 1051–1074.

[15] S. Gerbi and B. Said-Houari, Existence and exponential stability of a damped wave equation with dynamic boundary conditions and a delay term, *arxiv.*,(2012), 1–15.

[16] S.A. Messaoudi, General decay of solutions of a weak viscoelastic equation, *Arab. J. Sci. Eng.*, 36(3),(2011), 1569–1579.

[17] S.H. Park, Decay rate estimates for a weak viscoelastic beam equation with time-varying delay, *A. Math. Letters.*,

31(3) (2014), 46–51.

[18] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Ferreira, Existence and uniform decay of nonlinear viscoelastic equation with strong damping, *Mathematical Methods in Applied Sciences.*, 24 (2001), 1043–1053.

[19] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equation with localized damping, *Electronic J. Diff. Eqns.*, 44 (2002), 1–14.

[20] R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, *SIAM J. Control Optim.*, 26 (1988), 697–713.

[21] M. Pellicer, Large time dynamics of a nonlinear spring-mass-damper model, *Nonlin. Anal.*, 69(1) (2008), 3110–3127.

[22] M. Pellicer and J. Sola-Morales, Analysis of a viscoelastic spring-mass model, *J. Math. Anal. Appl.*, 294(2) (2004), 687–698.

[23] I. H. Suh and Z. Bien, Use of time delay action in the controller design, *IEEE Trans. Automat. Control.*, 25 (1980), 600–603.

[24] Shun-Tang Wu, General decay of solutions for a viscoelastic equation with nonlinear damping and source terms, *Acta Mathematica Scientia.*, 31(B)(4) (2011), 1436–1448.

[25] Shun-Tang Wu, Asymptotic behavior for a viscolastic wave equation with a delay term, *Taiwanese J. Math.*, 364(2) (2013), 765–784.

[26] M. Ferhat and A. Hakem, Global existence and asymptotic behavior for a coupled system of viscoelastic wave equations with a delay term, *Jour. Partial. Diff. Equa.*, (4)(2014), 01–28.

[27] M. Ferhat and A. Hakem, Energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks, *DCDS, Series B.*, Vol 22 (2) (2017), 491–506.

[28] M. Ferhat and A. Hakem, Global existence and energy decay result for a weak viscoelastic wave equations with a dynamic boundary and nonlinear delay term, *Computers and Mathematics with Application.*, Vol 71(3) (2016), 779–804.

[29] M. Ferhat and A. Hakem, Asymptotic behavior for a weak viscoelastic wave equations with a dynamic boundary and time varying delay term, *Journal of Applied Mathematics and Computing.*, Vol 51 (2016), 509–526.

[30] H. Zhang and Q. Hu, Energy decay for a nonlinear viscoelastic rod equations with dynamic boundary conditions, *Math. Methods Appl. Sci.*, 30(3)(2007), 249–256.

[31] A. Benaissa, A. Bengessoum and S. Messaoudi, Energy decay of solution for a wave equation with a costant weak delay and weak internal feedback, *Elec. J. of Qaulitative theory.*, (2014), 1–13.

[32] M. Nakao, A difference inequality and its application to nonlinear evolution equations, *J. Math. Soc. Japan.*, (30)(1978), 747–762.

[33] Shun-Tang Wu, On decay and blow up of solutions for a system of nonlinear wave equations, *J. Math. Anal. Appl.*, 394 (2012), 360–377

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆