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Min-Max $\pi g^*\beta$-continuous and Max-Min $\pi g^*\beta$-continuous functions in topological spaces

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Abstract
The aim of this paper is to study the notions of minimal $\pi g^*\beta$-closed set, maximal $\pi g^*\beta$-open set, minimal $\pi g^*\beta$-open set, maximal $\pi g^*\beta$-closed set, minimal $\pi g^*\beta$-continuous, maximal $\pi g^*\beta$-continuous, minimal $\pi g^*\beta$-irresolute, maximal $\pi g^*\beta$-irresolute , minimal-maximal $\pi g^*\beta$-continuous and maximal-minimal $\pi g^*\beta$-continuous and their basic properties are studied.

Keywords
minimal $\pi g^*\beta$-closed set, maximal $\pi g^*\beta$-open set, minimal $\pi g^*\beta$-open set, maximal $\pi g^*\beta$-closed set, minimal $\pi g^*\beta$-continuous, maximal $\pi g^*\beta$-continuous, minimal $\pi g^*\beta$-irresolute, maximal $\pi g^*\beta$-irresolute , min-max $\pi g^*\beta$-continuous and max-min $\pi g^*\beta$-continuous.

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Contents

1. Introduction

Norman Levine [3] introduced the concept of generalized closed sets in topological spaces. Recently, the concept of π generalized regular star beta closed sets in topological spaces was introduced by Devika et. al.,[2]. Nakaoka and Oda [4,5,6] have introduced minimal open sets and maximal open sets, which are subclasses of open sets. Later on many authors concentrated in this direction and defined many different types of minimal and maximal open sets. Inspired with these developments, we further study a new type of closed and open sets namely minimal $\pi g^*\beta$-closed sets, maximal $\pi g^*\beta$-open sets, minimal $\pi g^*\beta$-open sets, maximal $\pi g^*\beta$-closed sets and their respective continuous and irresolute functions.

Throughout this paper a space X means a topological space (X,τ). The class of $\pi g^*\beta$ -closed sets is denoted by $\Pi G^*\beta\ C(X)$. For any subset A of X its complement, interior, closure, $\pi g^*\beta$ -interior, $\pi g^*\beta$ -closure are denoted respectively by the symbols A^c, int(A), cl(A), $\pi g^*\beta$-int(A), $\pi g^*\beta$ -cl(A).

2. Preliminaries

Definition 2.1. *A subset A of a topological space (X,τ), is called*

1. *a generalized closed set (briefly, g-closed) [3] if $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X.*
2. *a generalized regular star closed set (briefly $\pi g^*\beta$ -closed) [2] if $\beta cl(A)\subseteq U$ whenever $A\subseteq U$ and U is πg-open subset of X.*

Definition 2.2. [1] *A proper nonempty open subset U of X is said to be a minimal open set if any open set contained in U is ϕ or U.*

Definition 2.3. [1] *A proper nonempty open subset U of X is said to be a maximal open set if any open set containing in U is X or U.*

Definition 2.4. [1] *A proper nonempty closed subset F of X is said to be a minimal closed set if any closed set contained in F is ϕ or F.*

Definition 2.5. [1] *A proper nonempty closed subset F of X is said to be a maximal closed set if any closed set contained in F is X or F.*

Theorem 2.6. [1] *Let X be a topological space and $F \subset X$. F is a minimal closed set iff $X - F$ is a maximal open set.*

Theorem 2.7. [1] *Let X be a topological space and $U \subset X$. U is a minimal open set iff $X - U$ is a maximal closed set.*

Definition 2.8. [1] *Let X and Y be the topological spaces. A function $f : X \longrightarrow Y$ is called*

1. *minimal continuous (briefly, min-continuous) if $f^{-1}(A)$ is an open set in X for every minimal open set A in Y.*
2. *maximal continuous (briefly, max-continuous) if $f^{-1}(A)$ is an open set in X for every maximal open set A in Y.*
3. *minimal irresolute (briefly, min-irresolute) if $f^{-1}(A)$ is minimal open set in X for every minimal open set A in Y.*
4. *maximal irresolute (briefly, max-irresolute) if $f^{-1}(A)$ is maximal open set in X for every maximal open set A in Y.*
5. *minimal-maximal continuous (briefly, min-max-continuous) if $f^{-1}(A)$ is maximal open set in X for every minimal open set A in Y.*
6. *maximal-minimal continuous (briefly, max-min-continuous) if $f^{-1}(A)$ is minimal open set in X for every maximal open set A in Y.*

We now introduce minimal $\pi g^*\beta$-open sets and maximal $\pi g^*\beta$-closed sets in topological spaces as follows.

Definition 2.9. *A proper nonempty $\pi g^*\beta$-open subset U of X is said to be a minimal $\pi g^*\beta$-open set if any $\pi g^*\beta$-open set contained in U is φ or U.*

Remark 2.10. *Every minimal open set is minimal $\pi g^*\beta$-open but not converserly.*

Example 2.11. *Let $X = \{a,b,c\}$ and $\tau = \{\varphi, \{c\}, \{a,b\}, X\}$. Since $\{a\}$ is minimal $\pi g^*\beta$-open set but not minimal open set.*

Remark 2.12. *Every minimal $\pi g^*\beta$-open set is minimal $\pi g-$open but not conversely.*

Example 2.13. *Let $X = \{a,b,c\}$ and $\tau = \{\varphi, \{c\}, \{b,c\}, X\}$. Since $\{a\}$ is minimal πg-open set but not a minimal $\pi g^*\beta$-open set.*

Theorem 2.14. *Every minimal open set is $\pi g^*\beta$-open set but not converserly.*

Example 2.15. *$X = \{a,b,c\}$ and $\tau = \{\varphi, \{c\}, \{a,c\}, X\}$ Then the subset $\{a,b\}$ is $\pi g^*\beta$-open set but not minimal open set.*

Theorem 2.16. *Every minimal πg-open set is $\pi g^*\beta$-open set but not conversely.*

Example 2.17. *In Example 2.15, the subset $\{a,b\}$ is $\pi g^*\beta$-open set but not minimal πg-open set.*

Theorem 2.18.
1. *Let U be a minimal $\pi g^*\beta$-open set and W be a $\pi g^*\beta$-open set. Then $U \cap W = \varphi$ or $U \subset W$.*
2. *Let U and V be minimal $\pi g^*\beta$-open sets. Then $U \cap V = \varphi$ or $U = V$.*

Proof:

1. *Let U be a minimal $\pi g^*\beta$-open set and W be a $\pi g^*\beta$-open set. If $U \cap W = \varphi$, then there is nothing to prove. If $U \cap W \neq \varphi$. Then $U \cap W \subset U$. Since U is a minimal $\pi g^*\beta$-open set, we have $U \cap W = U$. Therefore $U \subset W$.*
2. *Let U and V be minimal $\pi g^*\beta$-open set. If $U \cap V \neq \varphi$, then $U \subset V$ and $V \subset U$ by (i). Therefore $U = V$.*

Theorem 2.19. *Let U be a minimal $\pi g^*\beta$-open set. If $x \in U$, then $U \subset W$ for any regular open neighborhood W of x.*
Proof: *Let U be a minimal $\pi g^*\beta$-open set and x be an element of U. Suppose there exists a regular open neighborhood W of x such that $U \not\subset W$. Then $U \cap W$ is a $\pi g^*\beta$-open set such that $U \cap W \subset U$ and $U \cap W \neq \varphi$. Since U is a minimal $\pi g^*\beta$-open set, we have $U \cap W = U$. That is $U \subset W$, which is a contradiction for $U \not\subset W$. Therefore $U \subset W$ for any regular open neighborhood W of x.*

Theorem 2.20. *Let U be a minimal $\pi g^*\beta$-open set. If $x \in U$, then $U \subset W$ for any $\pi g^*\beta$-open set W containing x.*

Theorem 2.21. *Let U be a minimal $\pi g^*\beta$-open set. Then $U = \cap\{W : W \in \Pi G^*\beta O(X,x)\}$ for any element x of U.*
Proof: *By Theorems[2.18-2.20] and U is $\pi g^*\beta$-open set containing x, we have $U \subset \cap\{W : W \in \Pi G^*\beta O(X,x)\} \subset U$*

Theorem 2.22. *Let U be a nonempty $\pi g^*\beta$-open set. Then the following three conditions are equivalent.*

1. *U is a minimal $\pi g^*\beta$-open set*
2. *$U \subset \pi g^*\beta$-$cl(S)$ for any nonempty subset S of U*
3. *$\pi g^*\beta$-$cl(U) = \pi g^*\beta$-$cl(S)$ for any nonempty subset S of U.*

Proof: (1) $\Rightarrow$ (2) *Let $x \in U$; U be minimal $\pi g^*\beta$-open set and $S(\neq \varphi) \subset U$. By Theorem[2.18-2.20], for any $\pi g^*\beta$-open set W containing x, $S \subset U \subset W \Rightarrow S \subset W$. Now $S = S \cap U \subset S \cap W$. Since $S \neq \varphi$, $S \cap W \neq \varphi$. Since W is any $\pi g^*\beta$-open set containing x, then $x \in \pi g^*\beta$-$cl(S)$. That is $x \in U \Rightarrow x \in \pi g^*\beta$-$cl(S) \Rightarrow U \subset \pi g^*\beta$-$cl(S)$ for any nonempty subset Sof U*
(2) $\Rightarrow$ (3) *Let S be a nonempty subset of U. That is $S \subset U \Rightarrow \pi g^*\beta$-$cl(S) \subset \pi g^*\beta$-$cl(U) \longrightarrow (i)$. Again from (2) $U \subset \pi g^*\beta$-$cl(S)$ for any $S(\neq \varphi) \subset U \Rightarrow \pi g^*\beta$-$cl(U) \subset \pi g^*\beta$-$cl(\pi g^*\beta$-$cl(S)) = \pi g^*\beta$-$cl(S)$. That is $\pi g^*\beta$-$cl(U) \subset \pi g^*\beta$-$cl(s) \longrightarrow (ii)$. From (i) and (ii), we have $\pi g^*\beta$-$cl(U) = \pi g^*\beta$-$cl(S)$ for any nonempty subset S of U.*
(3) $\Rightarrow$ (1) *From (3) we have $\pi g^*\beta$-$cl(U) = \pi g^*\beta$-$cl(S)$ for any nonempty subset S of U. Suppose U is not a minimal*

$\pi g^*\beta$-open set. Then $\exists$ a nonempty $\pi g^*\beta$ -open set V such that $V \subset U$ and $V \neq U$. Now $\exists$ an element a in U such that $a \notin V \Rightarrow a \in V^c$. That is $\pi g^*\beta$-$cl(\{a\}) \subset \pi g^*\beta$-$cl(V^c) = V^c$, as V^c is $\pi g^*\beta$-closed set in X. It follows that $\pi g^*\beta$ -$cl(\{a\}) \neq \pi g^*\beta$-$cl(U)$. This is a contradiction for $\pi g^*\beta$-$cl(\{a\}) = \pi g^*\beta$- $cl(U)$ for any $\{a\}(\neq \phi) \subset U$. Therefore U is minimal $\pi g^*\beta$-open set.

Theorem 2.23. *Let V be a nonempty finite $\pi g^*\beta$-open set. Then there exists at least one (finite) minimal $\pi g^*\beta$-open set U such that $U \subset V$.*
Proof: *Let V be a nonempty finite $\pi g^*\beta$-open set. If V is a minimal $\pi g^*\beta$-open set, we may set $U = V$. If V is not a minimal $\pi g^*\beta$-open set, then $\exists$ (finite) $\pi g^*\beta$-open set V_1 such that $\varphi \neq V_1 \subset V$. If V_1 is a minimal $\pi g^*\beta$-open set, we may set $U = V_1$. If V_1 is not a minimal $\pi g^*\beta$-open set, then there exists (finite) $\pi g^*\beta$-open set V_2 such that $\varphi \neq V_2 \subset V_1$.Continuing this process, we have a sequence of $\pi g^*\beta$-open set $V \supset V_1 \supset V_2 \supset V_3 \supset .... \supset V_k \supset ....$ Since V is a finite set, this process repeats only finitely. Then finally we get a minimal $\pi g^*\beta$-open set $U = V_n$ for some positive integer n.*
A topological space X is said to be locally finite space if each of its elements is contained in a finite open set.

Corollary 2.24. *Let X be a locally finite space and V be a nonempty $\pi g^*\beta$-open set. Then $\exists$ at least one (finite) minimal $\pi g^*\beta$-open set U such that $U \subset V$.*
Proof: *Let X be a locally finite space and V be a nonempty $\pi g^*\beta$-open set. Let x in V. Since X is locally finite space, we have a finite open set V_x such that x in V_x. Then $V \cap V_x$ is a finite $\pi g^*\beta$-open set. By Theorem 2.22 there exists at least one (finite) minimal $\pi g^*\beta$-open set U such that $U \subset V \cap V_x$. That is $U \subset V \cap V_x \subset V$. Hence there exists at least one (finite) minimal $\pi g^*\beta$-open set U such that $U \subset V$.*

Corollary 2.25. *Let V be a finite minimal open set. Then there exists at least one (finite) minimal $\pi g^*\beta$-open set U such that $U \subset V$.*
Proof: *Let V be a finite minimal open set. Then V is a nonempty finite $\pi g^*\beta$-open set. By Theorem 2.23, there exists at least one (finite) minimal $\pi g^*\beta$-open set U such that $U \subset V$.*

Theorem 2.26. *Let U and U_λ be minimal $\pi g^*\beta$-open sets for any element $\lambda \in \Gamma$. If $U \subset \cup_{\lambda\in\Gamma} U_\lambda$, then there exists an element $\lambda \in \Gamma$ such that $U = U_\lambda$.*
Proof: *Let $U \subset \cup_{\lambda\in\Gamma}U_\lambda$. Then $U \cap (\cup_{\lambda\in\Gamma}U_\lambda) = U$. That is $\cup_{\lambda\in\Gamma}(U \cap U_\lambda) = U$. Also by Theorem [2.22] (2), $U \cap U_\lambda = \varphi$ or $U = U_\lambda$ for any $\lambda \in \Gamma$. It follows that $\exists$ an element $\lambda \in \Gamma$ such that $U = U_\lambda$.*

Theorem 2.27. *Let U and U_λ be minimal $\pi g^*\beta$-open sets for any $\lambda \in \Gamma$. If $U = U_\lambda$ for any $\lambda \in \Gamma$, then $(\cup_{\lambda\in\Gamma}U_\lambda) \cap U = \varphi$.*
Proof: *Suppose $(\cup_{\lambda\in\Gamma}U_\lambda) \cap U \neq \varphi$. That is $\cup_{\lambda\in\Gamma}(U_\lambda \cap U) \neq \varphi$. Then there exists an element $\lambda \in \Gamma$ such that $U \cap U_\lambda \neq \varphi$. By Theorem 2.22(2), we have $U = U_\lambda$, which contradicts the fact that $U \neq U_\lambda$ for any $\lambda \in \Gamma$. Hence $(\cup_{\lambda\in\Gamma}U_\lambda) \cap U = \varphi$.*

We now introduce Maximal $\pi g^*\beta$-closed sets in topological spaces as follows.

Definition 2.28. *A proper nonempty $\pi g^*\beta$-closed $F \subset X$ is said to be maximal $\pi g^*\beta$-closed set if any $\pi g^*\beta$-closed set containing F is either X or F.*

Remark 2.29. *Every maximal $\pi g^*\beta$-closed set is maximal πg-closed but not conversely.*

Theorem 2.30. *A proper nonempty subset F of X is maximal $\pi g^*\beta$-closed set iff $X - F$ is a minimal $\pi g^*\beta$-open set.*
Proof: *Let F be a maximal $\pi g^*\beta$-closed set. Suppose $X - F$ is not a minimal $\pi g^*\beta$-open set. Then there exists $\pi g^*\beta$-open set $U \neq X - F$ such that $\varphi \neq U \subset X - F$. That is $F \subset X - U$ and $X - U$ is a $\pi g^*\beta$-closed set which is a contradiction for F is a maximal $\pi g^*\beta$-closed set.*
Conversely let $X - F$ be a minimal $\pi g^*\beta$-open set. Suppose F is not a maximal $\pi g^*\beta$-closed set, then there exists $\pi g^*\beta$-closed set $E \neq F$ such that $F \subset E \neq X$. That is $\varphi \neq X - E \subset X - F$ and $X - E$ is a $\pi g^*\beta$-open set which is a contradiction for $X - F$ is a minimal $\pi g^*\beta$-open set. Therefore F is a maximal $\pi g^*\beta$-closed set.

Theorem 2.31. *Let F be a maximal $\pi g^*\beta$-closed set. If x is an element of F, then for any $\pi g^*\beta$-closed set S containing x, $F \cup S = X$ or $S \subset F$.*
Proof: *Let F be a maximal $\pi g^*\beta$-closed set and x is an element of F. Suppose there exists $\pi g^*\beta$-closed set S containing x such that $F \cup S \neq X$. Then $F \subset F \cup S$ and let $F \cup S$ is a $\pi g^*\beta$-closed set. Since F is a $\pi g^*\beta$-closed set, we $F \cup S = F$. Therefore $S \subset F$.*

Theorem 2.32. *Let F be a proper nonempty cofinite $\pi g^*\beta$-closed set. Then there exists (cofinite) maximal $\pi g^*\beta$-closed set E such that $F \subset E$.*
Proof: *If F is maximal $\pi g^*\beta$-closed set, we may set $E = F$. If F is not a maximal $\pi g^*\beta$-closed set, then there exists (cofinite) $\pi g^*\beta$-closed set F_1 such that $F \subset F_1 \neq X$. If F_1 is a maximal $\pi g^*\beta$-closed set, we may set $E = F_1$. If F_1 is not a maximal $\pi g^*\beta$-closed set, then there exists a (cofinite) $\pi g^*\beta$-closed set F_2 such that $F \subset F_1 \subset F_2 \neq X$. Continuing this process, we have a sequence of $\pi g^*\beta$-closed, $F \subset F_1 \subset F_2 \subset ------\subset F_k --------$. Since F is a cofinite set, this process repeats only finitely. Then, finally we get a maximal $\pi g^*\beta$-closed set $E = E_n$ for some positive integer n.*

3. Minimal $\pi g^*\beta$-closed set and maximal $\pi g^*\beta$-open set

We now introduce minimal $\pi g^*\beta$-closed sets and maximal $\pi g^*\beta$-open sets in topological spaces as follows.

Definition 3.1. *A proper nonempty $\pi g^*\beta$-closed subset F of X is said to be a minimal $\pi g^*\beta$-closed set if any $\pi g^*\beta$-closed set contained in F is φ or F.*

Remark 3.2. *Every minimal closed set is minimal $\pi g^*\beta$-closed but not conversely.*

Definition 3.3. *A proper nonempty $\pi g^*\beta$-open subset $U \subset X$ is said to be a maximal $\pi g^*\beta$-open set if any $\pi g^*\beta$-open set containing U is either X or U.*

Remark 3.4. *Every maximal open set is maximal $\pi g^*\beta$-open but not conversely.*

Theorem 3.5. *A proper nonempty subset U of X is maximal $\pi g^*\beta$-open set iff $X-U$ is a minimal $\pi g^*\beta$-closed set.*
Proof: *Let U be a maximal $\pi g^*\beta$-open set. Suppose $X-U$ is not a minimal $\pi g^*\beta$-closed set. Then there exists $\pi g^*\beta$-closed set $V \neq X-U$ such that $\varphi \neq V \subset X-U$. That is $U \subset X-V$ and $X-V$ is a $\pi g^*\beta$-open set which is a contradiction for U is a minimal $\pi g^*\beta$-closed set. Conversely let $X-U$ be a minimal $\pi g^*\beta$-closed set. Suppose U is not a maximal $\pi g^*\beta$-open set. Then there exists a $\pi g^*\beta$-open set $E \neq U$ such that $U \subset E \neq X$. That is $\varphi \neq X-E \subset X-U$ and $X-E$ is a $\pi g^*\beta$-closed set which is a contradiction for $X-U$ is a minimal $\pi g^*\beta$-closed set. Therefore U is a maximal $\pi g^*\beta$-closed set.*

Theorem 3.6. *(i) Let F be a maximal $\pi g^*\beta$-open set and W be a $\pi g^*\beta$-open set. Then $F \cup W = X$ or $W \cup F$.*
(ii) Let F and S be maximal $\pi g^*\beta$-open set. Then $F \cup S = X$ or $F=S$.
proof: *(i) Let F be a maximal $\pi g^*\beta$-open set and W be a $\pi g^*\beta$-open set. If $F \cup W=X$, then there is nothing to prove. Suppose $F \cup W \neq X$. Then $F \subseteq F \cup W$. Therefore $F \cap W=F$ as F is a maximal $\pi g^*\beta$-open set in X. Hence $W \cup F$.*
(ii) Let F and S be maximal $\pi g^*\beta$-open sets. If $F \cup S \neq X$, then we have $F \subseteq S$ and $S \subseteq F$ by (i). Therefore $F=S$.

Theorem 3.7. *Let $F_\alpha, F_\beta, F_\delta$ be maximal $\pi g^*\beta$-open sets such that $F_\alpha \neq F_\beta$. If $F_\alpha \cap F_\beta \subset F_\delta$, then either $F_\alpha = F_\delta$ or $F_\beta = F_\delta$*
Proof: *Given that $F_\alpha \cap F_\beta \subset F_\delta$. If $F_\alpha = F_\delta$ then there is nothing to prove.*
If $F_\alpha \neq F_\delta$ then we have to prove $F_\beta = F_\delta$. Now $F_\beta \cap F_\delta = F_\beta \cap (F_\delta \cap X) = F_\beta \cap (F_\delta \cap (F_\alpha \cup F_\beta)(by\, Theorem 3.6(ii)) = F_\beta \cap ((F_\delta \cap F_\alpha) \cup (F_\delta \cap F_\beta)) = (F_\beta \cap F_\delta \cap F_\alpha) \cup (F_\beta \cap F_\delta \cap F_\beta) = (F_\alpha \cap F_\beta) \cup (F_\delta \cap F_\beta)(by\, F_\alpha \cap F_\beta \subset F_\delta) = (F_\alpha \cup F_\delta) \cap F_\beta = X \cap F_\beta$ (Since F_α and F_δ are maximal $\pi g^*\beta$-open sets by Theorem[3.6](ii), $F_\alpha \cup F_\delta = X$) = F_β. That is $F_\beta \cap F_\delta = F_\beta \Rightarrow F_\beta \subset F_\delta$. Since F_β and F_δ are maximal $\pi g^*\beta$-open sets, we have $F_\beta = F_\delta$. Therefore $F_\beta = F_\delta$.

Theorem 3.8. *Let F_α, F_β and F_δ be different maximal $\pi g^*\beta$-open sets to each other. Then $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.*
Proof: *Let $(F_\alpha \cap F_\beta) \subset (F_\alpha \cap F_\delta) \Rightarrow (F_\alpha \cap F_\beta) \cup (F_\delta \cap F_\beta) \subset (F_\alpha \cap F_\delta) \cup (F_\delta \cap F_\beta) \Rightarrow (F_\alpha \cup F_\delta) \cap F_\beta \subset F_\delta \cap (F_\alpha \cup F_\beta)$. Since by Theorem 3.6(ii), $F_\alpha \cup F_\delta = X$ and $F_\alpha \cup F_\beta = X \Rightarrow X \cap F_\beta \subset F_\delta \cap X \Rightarrow F_\beta \subset F_\delta$ From the definition of maximal $\pi g^*\beta$-open set it follows that $F_\beta = F_\delta$, which is a contradiction to the fact that F_α, F_β and F_δ are different to each other. Therefore $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.*

Theorem 3.9. *Let F be a maximal $\pi g^*\beta$-open set and x be an element of F. Then $F = \cup\{ S : S$ is a $\pi g^*\beta$-open set containing x such that $F \cup S \neq X\}$.*

Proof: *By Theorem 3.7 and fact that F is a $\pi g^*\beta$-open set containing x, we have $F \subset \{S : S$ is a $\pi g^*\beta$-open set containing x such that $F \cup S \neq X\} \subset F$. Therefore we have the result.*

Theorem 3.10. *Let F be a proper nonempty cofinite $\pi g^*\beta$-open set. Then there exists (cofinite) maximal $\pi g^*\beta$-open set E such that $F \subset E$.*
Proof: *If F is maximal $\pi g^*\beta$-open set, we may set $E = F$. If F is not a maximal $\pi g^*\beta$-open set, then $\exists$ (cofinite) $\pi g^*\beta$-open set F_1 such that $F \subset F_1 \neq X$. If F_1 is a maximal $\pi g^*\beta$-open set, we may set $E = F_1$. If F_1 is not a maximal $\pi g^*\beta$-open set, then $\exists$ a (cofinite) $\pi g^*\beta$-open set F_2 such that $F \subset F_1 \subset F_2 \neq X$. Continuing this process, we have a sequence of $\pi g^*\beta$-open, $F \subset F_1 \subset F_2 \subset ....F_k...........$ Since F is a cofinite set, this process repeats only finitely. Then, finally we get a maximal $\pi g^*\beta$-open set $E = E_n$ for some positive integer n.*

4. Minimal $\pi g^*\beta$-continuous functions and maximal $\pi g^*\beta$-continuous functions

Definition 4.1. *Let X and Y be the topological spaces. A function $f : X \longrightarrow Y$ is called*

1. *minimal $\pi g^*\beta$-continuous (briefly, min-$\pi g^*\beta$-continuous) if $f^{-1}(A)$ is $\pi g^*\beta$- open set in X for every minimal open set A in Y.*
2. *maximal $\pi g^*\beta$-continuous (briefly, max-$\pi g^*\beta$-continuous) if $f^{-1}(A)$ is $\pi g^*\beta$- open set in X for every maximal open set A in Y.*
3. *minimal $\pi g^*\beta$-irresolute (briefly, min-$\pi g^*\beta$-irresolute) if $f^{-1}(A)$ is minimal $\pi g^*\beta$-open set in X for every minimal open set A in Y.*
4. *maximal $\pi g^*\beta$-irresolute (briefly, max-$\pi g^*\beta$-irresolute) if $f^{-1}(A)$ is maximal $\pi g^*\beta$-open set in X for every maximal open set A in Y.*
5. *minimal-maximal $\pi g^*\beta$-continuous (briefly, min-max-$\pi g^*\beta$-continuous) if $f^{-1}(A)$ is maximal $\pi g^*\beta$-open set in X for every minimal open set A in Y.*
6. *maximal-minimal $\pi g^*\beta$-continuous (briefly, max-min-$\pi g^*\beta$-continuous) if $f^{-1}(A)$ is minimal $\pi g^*\beta$-open set in X for every maximal open set A in Y.*

Theorem 4.2. *Every continuous function is minimal $\pi g^*\beta$-continuous function but not conversely.*
Proof: *Let $f : X \longrightarrow Y$ be a continuous function. To prove that f is minimal $\pi g^*\beta$-continuous. Let N be any minimal open set in Y. Since every minimal open set is an open set and every open set is $\pi g^*\beta$-open set, N is a $\pi g^*\beta$-open set in Y. Since f is continuous, $f^{-1}(N)$ is a $\pi g^*\beta$-open set in Y. Hence f is a minimal $\pi g^*\beta$-continuous.*

Example 4.3. *Let $X = Y = \{a,b,c\}$ be with $\tau = \{\varphi,\{a\},\{a,b\},X\}$ and $\mu = \{\varphi,\{b\},\{b,c\},Y\}$. Let $f : X \longrightarrow Y$ be an identity function. Then f is a minimal $\pi g^*\beta$-continuous function but it is not a continuous function, since for the open set $\{b,c\}$ in Y, $f^{-1}(\{b,c\}) = \{b,c\}$ which is not a $\pi g^*\beta$-open set in X.*

Theorem 4.4. *Let X and Y be the topological spaces. A function $f : X \longrightarrow Y$ is minimal $\pi g^*\beta$-continuous if and only if the inverse image of each maximal closed set in Y is a $\pi g^*\beta$-closed set in X.*
Proof: *The proof follows from the definition and fact that the complement of minimal $\pi g^*\beta$-open set is maximal $\pi g^*\beta$-closed set.*

Theorem 4.5. *If $f : X \longrightarrow Y$ is continuous function and $g : Y \longrightarrow Z$ is minimal $\pi g^*\beta$-continuous functions. Then $g \circ f : X \longrightarrow Z$ is a minimal $\pi g^*\beta$-continuous.*
Proof: *Let N be any minimal open set in Z. Since g is minimal $\pi g^*\beta$-continuous, $g^{-1}(N)$ is a $\pi g^*\beta$-open set in Y. Again since f is continuous, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is a $\pi g^*\beta$-open set in X. Hence $g \circ f$ is a minimal $\pi g^*\beta$-continuous.*

Theorem 4.6. *Let X and Y be the topological spaces. A function $f : X \longrightarrow Y$ is maximal $\pi g^*\beta$-continuous if and only if the inverse image of each minimal closed set in Y is a $\pi g^*\beta$-closed set in X.*
Proof: *The proof follows from the definition and fact that the complement of maximal $\pi g^*\beta$-open set is minimal $\pi g^*\beta$-closed set.*

Theorem 4.7. *If $f : X \longrightarrow Y$ is continuous function and $g : Y \longrightarrow Z$ is maximal $\pi g^*\beta$-continuous functions, then $g \circ f : X \longrightarrow Z$ is a maximal $\pi g^*\beta$-continuous.*
Proof: *Similar to that of Theorem 4.5.*

Theorem 4.8. *Let X and Y be the topological spaces. A function $f : X \longrightarrow Y$ is minimal $\pi g^*\beta$-irresolute if and only if the inverse image of each maximal $\pi g^*\beta$ - closed set in Y is a maximal $\pi g^*\beta$-closed set in X.*
Proof: *The proof follows from the definition and fact that the complement of minimal $\pi g^*\beta$-open set is maximal $\pi g^*\beta$-closed set.*

Theorem 4.9. *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are minimal $\pi g^*\beta$-irresolute functions. Then $g \circ f : X \longrightarrow Z$ is a minimal $\pi g^*\beta$-irresolute function.*
Proof: *Let N be any minimal $\pi g^*\beta$-open set in Z. Since g is minimal $\pi g^*\beta$-irresolute, $g^{-1}(N)$ is a minimal $\pi g^*\beta$-open set in Y. Again since f is minimal $\pi g^*\beta$-irresolute, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is minimal $\pi g^*\beta$-open set in X. Therefore $g \circ f$ is minimal $\pi g^*\beta$-irresolute.*

Theorem 4.10. *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are maximal $\pi g^*\beta$-irresolute functions, then $g \circ f : X \longrightarrow Z$ is a maximal $\pi g^*\beta$-irresolute function.*
Proof: *Similar to that of Theorem 4.9.*

Theorem 4.11. *Every min-max $\pi g^*\beta$-continuous function is minimal $\pi g^*\beta$-continuous function but not conversely.*
Proof: *Let $f : X \longrightarrow Y$ be a min-max $\pi g^*\beta$-continuous function. Let N be any minimal open set in Y. Since f is min-max $\pi g^*\beta$-continuous, $f^{-1}(N)$ is a maximal $\pi g^*\beta$ open set in X. Since every maximal $\pi g^*\beta$ open set is a $\pi g^*\beta$-open set, $f^{-1}(N)$ is a $\pi g^*\beta$-open set in X. Hence f is a minimal $\pi g^*\beta$-continuous.*

Example 4.12. *Let $X = Y = \{a,b,c\}$ be with $\tau = \{\varphi,\{a\},\{a,b\},X\}$ and $\mu = \{\varphi,\{b\},Y\}$. Let $f : X \longrightarrow Y$ be an identity function. Then f is a minimal $\pi g^*\beta$-continuous function but it is not a min-max $\pi g^*\beta$-continuous, since for the minimal open set $\{b\}$ in Y, $f^{-1}(\{b\}) = \{b\}$ which is not a maximal $\pi g^*\beta$-open set in X.*

Theorem 4.13. *Every max-min $\pi g^*\beta$-continuous function is maximal $\pi g^*\beta$-continuous function but not conversely.*
Proof: *Similar to that of Theorem 4.11.*

Example 4.14. *Let $X = Y = \{a,b,c\}$ be with $\tau = \{\varphi,\{b\},\{b,c\},X\}$ and $\mu = \{\varphi,\{a,b\},Y\}$. Let $f : X \longrightarrow Y$ be an identity function. Then f is a maximal $\pi g^*\beta$-continuous function but it is not a max-min $\pi g^*\beta$-continuous, since for the maximal open set $\{a,b\}$ in Y, $f^{-1}(\{a,b\}) = \{a,b\}$ which is not a minimal $\pi g^*\beta$-open set in X.*

Theorem 4.15. *If $f : X \longrightarrow Y$ is maximal $\pi g^*\beta$-irresolute and $g : Y \longrightarrow Z$ is min-max $\pi g^*\beta$-continuous functions, then $g \circ f : X \longrightarrow Z$ is a min-max $\pi g^*\beta$-continuous function.*
Proof: *Let N be any minimal $\pi g^*\beta$-open set in Z. Since g is min-max $\pi g^*\beta$-continuous, $g^{-1}(N)$ is a maximal $\pi g^*\beta$-open set in Y. Again since f is maximal $\pi g^*\beta$-irresolute, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is a maximal $\pi g^*\beta$-open set in X. Hence $g \circ f$ is a min-max $\pi g^*\beta$-continuous.*

Theorem 4.16. *If $f : X \longrightarrow Y$ is maximal $\pi g^*\beta$-continuous (resp. $\pi g^*\beta$-continuous) and $g : Y \longrightarrow Z$ is min-max $\pi g^*\beta$-continuous functions, then $g \circ f : X \longrightarrow Z$ is a minimal $\pi g^*\beta$-continuous.*
Proof: *Let N be any minimal $\pi g^*\beta$-open set in Z. Since g is min-max $\pi g^*\beta$-continuous, $g^{-1}(N)$ is a maximal $\pi g^*\beta$-open set in Y. (resp. since every maximal $\pi g^*\beta$-open set in $\pi g^*\beta$-open, $g^{-1}(N)$ is $\pi g^*\beta$-open set in Y). Again since f is maximal $\pi g^*\beta$-continuous (resp. $\pi g^*\beta$-continuous), $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is a $\pi g^*\beta$-open set in X. Hence $g \circ f$ is a minimal $\pi g^*\beta$-continuous.*

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