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Solving linear integral equations with Fibonacci polynomials

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Abstract
The goal of this work is to seek an approximate solution of Fredholm and Volterra integral equations using Fibonacci polynomials with hat basis functions, in order to obtain a variational problem and reduce this one to a linear system, where its solution is to find the Fibonacci coefficients of the unknown function and thereafter the solution of the equation. The convergence of this method is assured and the high accuracy of the error estimation is compared with other numerical methods.

Keywords
Linear integral equations, Fibonacci polynomials, collocation methods, Sloan approximation.

AMS Subject Classification
45A05, 45B05, 45D05.

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Contents

1. Introduction

Integral equations appear in mathematical modeling of different disciplines such a biology, chemistry, physics, engineering. In recent years, many numerical methods for approximating the solution of Fredholm and Volterra integral equations are used, such a triangular orthogonal functions where the authors use a complementary pair of orthogonal triangular functions derived from the well-known block pulse functions set for solving Fredholm integral equations [1], the radial basis function (RBF) interpolation is applied to approximate the numerical solution of both Fredholm and Volterra functional integral equations [4]. The Taylor-series expansion method for a class of Volterra integral equations of second kind with smooth or weakly singular kernels where the authors transform integral equation to linear differential equation [5]. The application of the four Chebyshev polynomials, Legendre wavelets, Bernoulli series, Euler series, Legendre series and Hermite series with hat basis functions for solving linear integral equations [6–12].

In this paper, we present the Fibonacci polynomials crossed with test functions for solving numerically Fredholm and Volterra integral equations given by

$$\varphi(x) - \int_{\Omega} k(x,y)\varphi(y)dy = f(x), \qquad x \in \Omega \tag{1.1}$$

where $k(x,t)$ is given and assumed to be complex-valued and continuous on the square $\Omega \times \Omega$,The free term $f(x)$ is assumed to be complex-valued and continuous on Ω.The unknown function $\varphi(x)$ is to be determined as continuous function in Ω. Depending on the domain $\Omega = [a,x]$ or $[a,b]$ the equation (1) describes the Volterra integral equation or Fredholm integral equation, respectively.

For the solution of the equation (1.1) in the complete function spaces $L^2(\Omega)$, we multiply equation (1.1) by a test function $\psi(x)$ and integrating, we obtain the weak formulation of (1.1)

$$\langle \varphi(x), \psi(x) \rangle - \left\langle \int_{\Omega} k(x,y)\varphi(y)dy, \psi(x) \right\rangle = \langle f(x), \psi(x) \rangle, \qquad \forall \psi \in L^2(\Omega). \tag{1.2}$$

Due to the equivalence between the problems (1.1) and (1.2) we solve the second one (1.2) to define an approximation to φ. Choosing a sequence of finite dimensional subspaces $V_n,\ n \geq 1$, having n basis functions $\{F_1,\ F_2,\ \ldots, F_n\}$ with dimension of $V_n = n$, the approximate function $\varphi_n \in V_n$ of the function φ is given by

$$\varphi_n(x) = \sum_{j=1}^{n} \alpha_j F_j(x), \tag{1.3}$$

where the expression (1.3) describes the truncated Fibonacci series of the solution of the equation (1.2), with the functions $\{F_k\}_{1\leq j\leq n}$ represent the Fibonacci polynomials and $\{\alpha_k\}_{1\leq j\leq n}$ the coefficients to be determined. So, we write

$$\langle \varphi_n(x), \psi(x)\rangle - \left\langle \int_{\Omega} k(x,y)\varphi_n(y)dy, \psi(x)\right\rangle = \langle f(x), \psi(x)\rangle . \tag{1.4}$$

For the solution of the equation (1.4), we construct a variational form using the Galerkin-Petrov method. That is to say, we seek to determine a function $\psi \in L^2(\Omega)$ solves the equation (1.4).

2. Solution with collocation methods on one-dimensional space [0,1]

Choose a selection of distinct points $x_0,\ x_1,\ x_2, \ldots.. x_{n+1}$ of the interval $[0,1]$ such that

$$0 = x_0 < x_1 < x_2 < \ldots. < x_{n+1} = 1,$$

and create a basis from the hat function

$$\psi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{if } x_{i-1} \leq x \leq x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{if } x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

Remark that, $W_n = span\{\psi_1,\ \psi_2,\ \ldots, \psi_n\}$ is a subspace of $L^2(\Omega)$ and that W_n has finite dimension n. For the determination of the approximation $\varphi_n \in V_n$ to φ, one uses the Galerkin-Petrov method. Say, for all $\psi_i(x) \in W_n$, and for all $i, j = 1, 2, \ldots, n$ we write

$$\sum_{j=1}^{n} \alpha_j \langle F_j(x), \psi_i(x)\rangle - \sum_{j=1}^{n} \alpha_j \left\langle \int_0^1 k(x,t)F_j(t)dt, \psi_i(x)\right\rangle = \langle f(s), \psi_i(x)\rangle ,$$

or still

$$\sum_{j=1}^{n} \alpha_j \left(\langle F_j(x), \psi_i(x)\rangle - \left\langle \int_0^1 k(x,y)F_j(y)dy, \psi_i(x)\right\rangle \right) = \langle f(x), \psi_i(x)\rangle . \tag{2.1}$$

The equation (2.1) leads us to determine the coefficients $\{\ \alpha_1,\ \alpha_2,\ \ldots, \alpha_n\}$ solution of the linear system

$$\sum_{j=1}^{n} \alpha_j \int_0^1 F_j(x)\psi_i(x)dx - \int_0^1 \left(\int_0^1 k(x,y)F_j(y)dy\right)\psi_i(x)dx = \int_0^1 f(x)\psi_i(x)dx. \tag{2.2}$$

Define the matrices

$$F = (F_{ij}) = \int_0^1 \psi_i(x)F_j(x)dx$$

and

$$K = (K_{ij}) = \int_0^1 \psi_i(x)\left(\int_0^1 k(x,y)F_j(y)dy\right)dx.$$

If the $\det(F - K) \neq 0$, we can ensure that, there exists a solution of the linear system (2.2) and consequently the approximate solution $\varphi_n(x)$ as a linear combination

$$\varphi_n(x) = \sum_{j=1}^{n} \alpha_j F_j(x),$$

In fact, the linear system may be written in matrix

$$(F - K)\alpha = B, \tag{2.3}$$

where $\alpha = (\alpha_1,\ \alpha_2,\ \ldots, \alpha_n)^T$ and $B = \left(\int_0^1 f(x)\psi_1(x)dx, \int_0^1 f(x)\psi_2(x)dx\ , \ldots, \int_0^1 f(x)\psi_n(x)dx\right)^T$. For the determinant of the system (2.3) is different from zero $\det(F - K) \neq 0$, then it has a unique solution

$$\alpha = (\alpha_1,\ \alpha_2,\ \ldots, \alpha_n)^T = (F - K)^{-1}B.$$

The corresponding approximate solution

$$\varphi_n(x) = \sum_{k=1}^{n} \alpha_k F_k(x),$$

3. Fibonacci polynomials

Let us consider the Fibonacci polynomials $F_n(x)$ defined as

$$F_n(x) = x^1 F_{n-1}(x) + x^0 F_{n-2}(x),$$

where x is an indeterminate and $F_1(x) = 0$ and $F_2(x) = 1$.This polynomials can be expressed by means of the Binet form [3]

$$F_n(x) = \frac{(\alpha^n - \beta^n)}{\Delta},$$

where

$$\begin{aligned} \Delta &= \sqrt{x^2 + 4} \\ \alpha &= \frac{x + \Delta}{2} \\ \beta &= \frac{x + \Delta}{2}. \end{aligned}$$

Besides, the expression of $F_n(x)$ is given by

$$F_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-j}{j} x^{n-1-2j},$$

where $[n]$ denotes the greatest integer not exceeding n and $n \geq 2$. Noting that, the Fibonacci polynomial $F_n(x)$ is polynomials with rational coefficients

$$\begin{aligned} F_1(x) &= 0 \\ F_2(x) &= 1 \\ F_3(x) &= x \\ F_4(x) &= x^2+1 \\ F_5(x) &= x^3+2x \end{aligned}$$

4. Sloan iterate convergence procedure

we define the operator projection P_nA as

$$P_nA\varphi_n(x) = A_n\varphi_n = \int_\Omega k_n(t,x)\varphi_n(t)dt,$$

so, with the solution φ_n of the equation $\varphi_n - A_n\varphi_n = f_n$, we construct the Sloan approximation as

$$\widetilde{\varphi}_n = f - A\varphi_n \tag{4.1}$$

where it is easy to see that $\widetilde{\varphi}_n$ is the projection of the approximate solution φ_n into V_n. Noting that if the equation (4.1) verified the Banach theorem with the application the expression (4.1) we can give the error bound

$$\|\varphi - \widetilde{\varphi}_n\| \leq \|A\| \, \|\varphi - \varphi_n\| \, .$$

This shows that, the convergence of $\widetilde{\varphi}_n$ to the exact solution φ is faster than φ_n to φ.

5. Illustrating Examples

Example 1. Consider the linear integral equation of Fredholm

$$\varphi(x) - \int_{-1}^{1} \exp(-y)\varphi(y)dy = \exp(x) - 2, \qquad 0 \leq x,y \leq 1,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \exp(x).$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Fibonacci series method.

Table 1. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [12].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [12]
-1.000000	3.678794e-001	3.678794e-001	0.00e-000	1.7 E-08
-0.600000	5.488116e-001	5.488116e-001	2.22e-016	5.8 E-09
-0.200000	8.187308e-001	8.187308e-001	3.33e-016	5.9 E-09
0.200000	1.221403e+000	1.221403e+000	0.00e-000	6.0 E-09
0.600000	1.822119e+000	1.822119e+000	0.00e-000	6.0 E-09
1.000000	2.718282e+000	2.718291e+000	0.00e-000	3.3 E-08

Example 2. Consider the linear integral equation of Fredholm

$$\varphi(x) + \frac{1}{(1+x)}\int_0^1 (x-y)\varphi(y)dy = \frac{1}{(1+x)}\left(x^3+x^2+\frac{1}{3}x-\frac{1}{4}\right),$$

$0 \leq x,y \leq 1$ where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = x^2.$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Fibonacci series method.

Table 2. We present the exact and the approximate solutions of the equation in the example 3 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [4].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [4]
0.000000	0.000000e+000	0.000000e+000	1.06e-016	1.13e-005
0.200000	4.000000e-002	4.000000e-002	7.91e-016	1.13e-005
0.400000	1.600000e-001	1.600000e-001	6.38e-016	1.13e-005
0.600000	3.600000e-001	3.600000e-001	2.22e-016	1.13e-005
0.600000	6.400000e-001	6.400000e-001	0.00e+000	1.13e-005
1.000000	1.000000e+000	1.000000e+000	4.44e-016	1.13e-005

Example 3. Consider the linear integral equation of Fredholm

$$\varphi(x) + \int_0^\pi (\cos y + \cos x)\varphi(y)dy = \sin x, \quad 0 \leq x,y \leq \pi,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \sin x + \frac{4}{2-\pi^2}\cos x + \frac{2\pi}{2-\pi^2}.$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Fibonacci series method.

Table 3. We present the exact and the approximate solutions of the equation in the example 3 in some arbitrary points, the error for $N = 8$ is calculated and compared with the ones treated in [2] and [5].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [2]	Error [5]
0.000000	-1.306697e+000	-1.306872e+000	1.7e-004	5.1e-001	5.2e-001
0.785398	-4.507167e-001	-4.508725e-001	1.5e-004	2.8e-002	2.1e-002
1.570796	2.015882e-001	2.014807e-001	1.0e-004	3.6e-001	3.6e-001
2.356194	2.681065e-001	2.680475e-001	5.9e-005	1.1e-001	1.1e-001
3.141593	-2.901271e-001	-2.901661e-001	3.9e-005	7.5e-001	7.5e-001

Example 4. Consider the linear integral equation of Volterra

$$\begin{aligned} \varphi(x) &- \int_0^x \left(x+6\,(x-y)-4\,(x-y)^2\right)\varphi(y)dy \\ &= -4x^2 - x - 2 + (3-x)\exp(x), \quad 0 \leq x,y \leq 1, \end{aligned}$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \exp(x).$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Fibonacci series method.

Table 4. We present the exact and the approximate solutions of the equation in the example 4 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [6].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [6]
0.000000	1.000000e+000	1.000000e+000	0.0e+000	3.4e-002
0.200000	1.221403e+000	1.221402e+000	5.5e-007	6.7e-003
0.400000	1.491825e+000	1.491823e+000	1.3e-006	1.7e-002
0.6.00000	1.822119e+000	1.822116e+000	2.6e-006	3.5e-002
0.800000	2.225541e+000	2.225536e+000	4.9e-006	1.9e-002
1.000000	2.718282e+000	2.718273e+000	9.0e-006	1.0e-002

Example 5. Consider the linear integral equation of Volterra

$$\varphi(x) - \int_0^x \left(x^2y^2 - xy\right) \varphi(y)dy = -\frac{3}{4}x^6 + \frac{1}{3}x^5 + x^4$$
$$-\frac{1}{2}x^3 + 3x - 1, \quad 0 \leq x, y \leq 1,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = 3x - 1.$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Fibonacci series method.

Table 5. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [6].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [6]
0.000000	-1.000000e+000	-1.000000e+000	1.1e-016	0.0e-000
0.200000	-4.000000e-001	-4.000000e-001	5.5e-017	4.0e-004
0.400000	2.000000e-001	2.000000e-001	1.6e-016	1.1e-003
0.600000	8.000000e-001	8.000000e-001	3.3e-016	2.9e-003
0.800000	1.400000e+000	1.400000e+000	2.2e-016	3.1e-003
1.000000	2.000000e+000	2.000000e+000	4.4e-016	9.3e-003

Example 6. Consider the linear integral equation of Volterra

$$\varphi(x) - \int_0^x (x - y)\, \varphi(y)dy = 1, \quad 0 \leq x, y \leq 1,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \cos x.$$

The approximate solution $\varphi_n(x)$ of $\varphi(x)$ is obtained by the truncated Fibonacci series method.

Table 6. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [5].

Values of x	Exact solution φ	Approx solution φ_n	Error	Error [5]
0.000000	1.000000e+000	1.000000e+000	0.00e+000	0.00e+000
0.200000	9.800666e-001	9.800666e-001	2.07e-009	1.86e-004
0.400000	9.210610e-001	9.210610e-001	8.11e-009	2.39e-003
0.600000	8.253356e-001	8.253356e-001	1.76e-008	7.80e-003
0.800000	6.967067e-001	6.967067e-001	2.99e-008	1.01e-002
1.000000	5.403023e-001	5.403023e-001	4.38e-008	5.15e-003

6. Conclusion

We introduced a numerical method for solving linear integral equations, based on the Galerkin-Petrov method using the truncated Fibonacci series of the solution. We remark that, the approximate solution $\varphi_n(x)$ is measurably close to the solution $\varphi(x)$ on the entire interval $[0,1]$. The efficiency of this method is tested by solving three examples of Fredholm equations and three examples of volterra equations for which the exact solutions are known. The accuracy of our technical shows very rich comparing with another results treated by another authors [2, 4–6, 12].

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