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An accurate five-step trigonometrically-fitted numerical scheme for approximating solutions of second order ordinary differential equations with oscillatory solutions

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Abstract
In this paper, class of second order ordinary differential equation with oscillatory solutions is considered. By employing the trigonometric basis function, a continuous five-step scheme known as five-step trigonometrically fitted scheme is derived to approximate solutions to the class of considered equation. Consistency and zero stability of the developed method were proved. Stability and convergence properties of this new scheme were also established. The scheme so obtained is used to solve standard initial value problems with oscillatory solutions. From the numerical results obtained, it was revealed that the proposed method performs better than some of the existing methods in the literature.

Keywords
Continuous schemes, Multistep collocation, Trigonometrically fitted method, Initial value problem.

AMS Subject Classification
Primary 65L05, Secondary 65L06

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Contents

1. Introduction

Differential equations arising from the modeling of physical phenomena, often do not have exact solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary, to the extent that several numerical methods such as finite difference methods, finite element methods and finite volume methods, among others, have been developed based on the nature and type of the differential equation to be solved.

Here, we are concerned with solutions of second order initial value problem of the form

$$y'' = f(x,y,y'), \quad y(a) = \eta_0, \quad y'(a) = \eta_1 \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, and $y, y_0, y' \in \mathbb{R}$ are given real constants.

Many scholars such as Henrici[11], Jeltsch[13], Twizel and Khaliq [16], Awoyemi[4], Simos[15], Yusuph and Onumanyi[18], Adeniran, Odejide and Ogundare[2] have devoted lots of attention to the development of various methods for solving directly (1.1) without reducing to system of first order equations. Hairer and Wanner[10] developed Nystrom-type methods for (1.1) in which conditions for the determination of the parameters of the method were listed. Gear[8], Hairer[11], Chawla and Sharma[6], independently developed explicit and implicit Runge-Kutta Nystrom type methods. Dormand and Prince[7] also developed two classes of embedded Runge-Kutta- Nystrom methods for the direct solution of (1.1).

Several numerical methods based on the use of polynomial functions (Power series, Legendre, Chebyshev, e.t.c) have been used as basis function to develop numerical methods for direct solution of (1.1) using interpolation and collocation procedure. However, it is well known that polynomial of high degree tend to oscillate strongly and in many cases they are liable to produce very poor approximations. Psihoyious and Simos[14] developed a trigonometrically fitted predictor-corrector method for numerical solution of IVPs with oscillating solutions. Numerical experiment showed that the method is efficient. Vigo-Aguiar and Ramos[17] in their paper titled " On the choice of the frequency in trigonometrically-fitted method use the trigonometrically-fitted method to obtain an approximate solution to some nonlinear oscillators and also presented a strategy for the choice of frequency in trigonometrically-fitted methods.

The main focus of this article is to employ trigonometric function as basis function to develop a new five-step numerical methods using interpolation and collocation procedure for the direct solution of (1.1).

2. Development of the method

The main objective in this section is to construct a continuous five-step trigonometrically fitted method. The method has the form

$$y_{n+5} = \alpha_0 y_n + \alpha_1 y_{n+1} + h^2 \sum_{j=0}^{5} \beta_j(u) f_{n+j} \tag{2.1}$$

where $u = wh$, $\beta_j(u)$, $j = 0, 1 \cdots 5$ are the coefficients that depend on the step-size and frequency. In order to derive (2.1), we proceed by seeking to approximate the exact solution $y(x)$ on the interval $[x_n, x_{n+h}]$ by interpolation function $U(x)$ of the form

$$U(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 \sin(wx) + a_7 \cos(wx) \tag{2.2}$$

where $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 are coefficients that must be uniquely determined. We then impose that interpolating function (2.2) coincides with the exact solution at the end point x_n and x_{n+1} to obtain the equations

$$U(x_n) = y_n \quad \text{and} \quad U(x_{n+1}) = y_{n+1}. \tag{2.3}$$

It is also demanded that the function (2.2) satisfies the differential equation (1) at points x_{n+j}, $j = 0, 1, \cdots, 5$ to obtain the following set of six equations:

$$U''(x_{n+j}) = f_{n+j}, \quad j = 0, 1, \cdots, 5. \tag{2.4}$$

Equations (2.3) and (2.4) leads to a system of eight equations which is solved by any linear system solvers such as Crammer's rule to obtain a_j, $j = 0, 1, \ldots, 7$. The a_j's obtained are then substituted into (2.2) to obtain the continuous form of the method

$$U(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + h^2 \sum_{j=0}^{5} \beta_j(u) f_{n+j} \tag{2.5}$$

where w is the frequency, α_j and β_j are continuous coefficients. The continuous method (2.5) is used to generate the main method of the form (2.1). That is, we evaluate at $x = x_{n+5}$ and letting $u = wh$, we obtain our main method

$$y_{n+5} = \alpha_0 y_n + \alpha_1 y_{n+1} + h^2 \sum_{j=0}^{5} \beta_j(u) f_{n+j} \tag{2.6}$$

with coefficients

$$\alpha_0 = \frac{-48 + 96q - 48q^2}{12(q^2 - 2q + 1)};$$
$$\alpha_1 = \frac{60 + 60q^2 - 120q}{12(q^2 - 2q + 1)};$$
$$\beta_0 = \frac{-29u^2 q - 48q^2 + 36q + 12 - u^2}{12(q^2 - 2q + 1)};$$
$$\beta_1 = \frac{-6u^2 q + 58u^2 q^2 + 180q^2 - 120q - 60}{12(q^2 - 2q + 1)} + \frac{68u^2}{12(q^2 - 2q + 1)};$$
$$\beta_2 = \frac{16u^2 q^2 - 182u^2 q - 240q^2 + 120q}{12(q^2 - 2q + 1)} + \frac{120 - 14u^2}{12(q^2 - 2q + 1)};$$
$$\beta_3 = \frac{12u^2 q + 34u^2 q^2 + 120q^2 - 120 + 74u^2}{12(q^2 - 2q + 1)};$$
$$\beta_4 = \frac{12u^2 q^2 - 29u^2 cos(u) - 60q + 60 - 13u}{12(q^2 - 2q + 1)};$$
$$\beta_5 = \frac{-6u^2 q + 24q - 12 + 6u^2 - 12q^2}{12(q^2 - 2q + 1)}$$

where variable $q = \cos u$.

We remark that by evaluating (2.5) at other points $x = x_{n+4}$, $x = x_{n+3}$, $x = x_{n+2}$, additional methods are derived namely: evaluating at $x = x_{n+4}$ gives

$$y_{n+4} = \hat{\alpha}_0 y_n + \hat{\alpha}_1 y_{n+1} + h^2 \sum_{j=0}^{5} \hat{\beta}_j(u) f_{n+j} \tag{2.7}$$

with

$$\hat{\alpha}_0 = \frac{-36 + 72q - 36q^2}{12(q^2 - 2q + 1)},$$
$$\hat{\alpha}_1 = \frac{48 - 96q + 48q^2}{12(q^2 - 2q + 1)},$$

$$
\begin{aligned}
\hat{\beta}_0 &= \frac{12-36q^2+24q-22u^2q-2u^2}{12(q^2-2q+1)},\\
\hat{\beta}_1 &= \frac{-60-84q+144q^2+44u^2q^2}{12(q^2-2q+1)},\\
&\quad + \frac{u^2q+57u^2}{12(q^2-2q+1)}\\
\hat{\beta}_2 &= \frac{120-216q^2+96q-148u^2q}{12(q^2-2q+1)},\\
&\quad + \frac{6u^2q^2-26u^2}{12(q^2-2q+1)}\\
\hat{\beta}_3 &= \frac{-120-24q+144q^2+46u^2q}{12(q^2-2q+1)},\\
&\quad + \frac{24u^2q^2+62u^2}{12(q^2-2q+1)}\\
\hat{\beta}_4 &= \frac{60-24q-36q^2-2u^2q^2}{12(q^2-2q+1)},\\
&\quad - \frac{22u^2q+24u^2}{12(q^2-2q+1)}\\
\hat{\beta}_5 &= \frac{-12+12q+u^2q+5u^2}{12(q^2-2q+1)};
\end{aligned}
$$

and evaluation of (2.5) at $x = x_{n+3}$ gives

$$
y_{n+3} = \check{\alpha}_0 y_n + \check{\alpha}_1 y_{n+1} + h^2 \sum_{j=0}^{5} \check{\beta}_j(u) f_{n+j} \tag{2.8}
$$

with

$$
\begin{aligned}
\check{\alpha}_0 &= \frac{96q-48q^2-48}{24(q^2-2q+1)},\\
\check{\alpha}_1 &= \frac{72q^2-144q+72}{24(q^2-2q+1)},\\
\check{\beta}_0 &= \frac{12-u^2-48q^2+36q-29u^2q}{24(q^2-2q+1)},\\
\check{\beta}_1 &= \frac{-72+72u^2+192q^2-120q}{24(q^2-2q+1)},\\
&\quad + \frac{58u^2q^2+2u^2q}{24(q^2-2q+1)}\\
\check{\beta}_2 &= \frac{-46u^2+168+120q-288q^2}{24(q^2-2q+1)},\\
&\quad - \frac{182u^2q}{24(q^2+2q+1)}\\
\check{\beta}_3 &= \frac{-192+82u^2+192q^2+92u^2q}{24(q^2-2q+1)},\\
&\quad + \frac{18u^2q^2}{24(q^2-2q+1)}\\
\check{\beta}_4 &= \frac{108-45u^2-48q^2-60q-4u^2q^2}{24(q^2-2q+1)},\\
&\quad - \frac{29u^2q}{24(q^2-2q+1)}
\end{aligned}
$$

and

$$
\check{\beta}_5 = \frac{-24+10u^2+24q+2u^2q}{24(q^2-2q+1)}.
$$

Again, evaluating (2.5) at $x = x_{n+2}$ gives

$$
y_{n+2} = \acute{\alpha}_0 y_n + \acute{\alpha}_1 y_{n+1} + h^2 \sum_{j=0}^{5} \acute{\beta}_j(u) f_{n+j} \tag{2.9}
$$

with

$$
\begin{aligned}
\acute{\alpha}_0 &= \frac{48q-24q^2-24}{24(q^2-2q+1)},\\
\acute{\alpha}_1 &= \frac{48q^2-96q+48}{24(q^2-2q+1)},\\
\acute{\beta}_0 &= \frac{2u^2+24q-24q^2-14u^2q}{(q^2-2q+1)},\\
\acute{\beta}_1 &= \frac{25u^2-12+96q^2-84q+u^2q}{24(q^2-2q+1)},\\
&\quad + \frac{28u^2q^2}{24(q^2-2q+1)}\\
\acute{\beta}_2 &= \frac{-18u^2-144q^2+96q+48}{24(q^2-2q+1)},\\
&\quad - \frac{68u^2q+10u^2q^2}{24(q^2-2q+1)}\\
\acute{\beta}_3 &= \frac{30u^2-72+96q^2-24q+8u^2q^2}{24(q^2-2q+1)},\\
&\quad + \frac{46u^2q}{24(q^2-2q+1)}\\
\acute{\beta}_4 &= \frac{-20u^2-24q^2-24q+48}{24(q^2-2q+1)},\\
&\quad - \frac{14u^2q+2u^2q^2}{24(q^2-2q+1)}\\
\acute{\beta}_5 &= \frac{5u^2+12q-12+u^2q}{24(q^2-2q+1)}.
\end{aligned}
$$

In order to incorporate the second initial condition of (1.1) in the derived methods, we differentiate (2.5) and evaluate at point $x = x_n$, $x = x_{n+1}$ and $x = x_{n+5}$ to have:

$$
\begin{aligned}
y'_n = \grave{\alpha}_0 y_n + \grave{\alpha}_1 y_{n+1} + h^2[\grave{\beta}_0(u) f_n + \grave{\beta}_1(u) f_{n+1} \qquad (2.10)\\
+ \grave{\beta}_2(u) f_{n+2} + \grave{\beta}_3(u) f_{n+3} + \grave{\beta}_4(u) f_{n+4}\\
+ \grave{\beta}_5(u) f_{n+5}]
\end{aligned}
$$

(2.11)

where

$$
\begin{aligned}
\grave{\alpha}_0 &= \frac{1440qp-720q^2p-720p}{720(q^2-2q+1)hp},\\
\grave{\alpha}_1 &= \frac{720q^2p-1440qp+720p}{720(q^2-2q+1)hp},
\end{aligned}
$$

$$\grave{\beta}_0 = \frac{502u^2qp+180p+720uq^3u-720q^2p}{720(q^2-2q+1)hp},$$
$$+\frac{360pq}{720(q^2-2q+1)hp}$$
$$\frac{-540uq-262u^2p}{720(q^2-2q+1)hp}$$
$$\grave{\beta}_1 = \frac{-9u^2p-1004u^2q^2p-97u^2qp-900p}{720(q^2-2q+1)hp}$$
$$\frac{-1080pq+2880q^2p-360uq^2}{720(q^2-2q+1)hp},$$
$$\frac{2160uq-2880uq^3+180u}{720(q^2-2q+1)hp}$$
$$\grave{\beta}_2 = \frac{-118u^2p+916u^2qp+1242u^2q^2p}{720(q^2-2q+1)hp},$$
$$+\frac{1800p+720pq-4320q^2p+3240uq}{720(q^2-2q+1)hp},$$
$$+\frac{1440uq^2+4320uq^3-720u}{720(q^2-2q+1)hp}$$
$$\grave{\beta}_3 = \frac{-62u^2p-792u^2q^2p-1006u^2qp}{720(q^2-2q+1)hp},$$
$$+\frac{-1800p+720pq+2880q^2p-2160uq^2}{720(q^2-2q+1)hp}$$
$$+\frac{-2880uq^3+2160uq+1080u}{720(q^2-2q+1)hp},$$
$$\grave{\beta}_4 = \frac{144u^2p+502u^2qp+194u^2q^2p}{720(q^2-2q+1)hp},$$
$$+\frac{900p+720uq^3+1080pq720q^2p}{720(q^2-2q+1)hp}$$
$$-\frac{1440uq^2-540uq-720u}{720(q^2-2q+1)hp},$$
$$\grave{\beta}_5 = \frac{-97u^2qp-180p+360pq-360uq^2}{720(q^2-2q+1)hp}$$
$$+\frac{180u-53u^2p}{720(q^2-2q+1)hp}$$

where in the above, $p=\sin u$. Again, differentiating and evaluating at $x=x_{n+1}$, we obtain

$$y'_{n+1} = \tilde{\alpha}_0 y_n + \tilde{\alpha}_1 y_{n+1} + h^2[\tilde{\beta}_0(u)f_n + \tilde{\beta}_1(u)f_{n+1} + \tilde{\beta}_2(u)f_{n+2} + \tilde{\beta}_3(u)f_{n+3} + \tilde{\beta}_4(u)f_{n+4} + \tilde{\beta}_5(u)f_{n+5}] \qquad (2.12)$$

where

$$\tilde{\alpha}_0 = \frac{-720u^2p-720q^2u^2p+1440qu^2p}{720(q^2-2q+1)hp},$$
$$\tilde{\alpha}_1 = \frac{720u^2p+720q^2u^2p-1440u^2qp}{720(q^2-2q+1)hp},$$

and

$$\tilde{\beta}_0 = \frac{-180u+113u^2p+360uq^2-720pq^2}{720(q^2-2q+1)hp}$$
$$+\frac{360pq-323u^2qp+180p}{720(q^2-2q+1)hp}$$
$$\tilde{\beta}_1 = \frac{276u^2p-1440uq^2-180uq+720u}{720(q^2-2q+1)hp}$$
$$-\frac{1080pq}{720(q^2-2q+1)hp}$$
$$\frac{+2880pq^2+646u^2q^2p+38w^2qp-900p}{720(q^2-2q+1)hp},$$
$$\tilde{\beta}_2 = \frac{-1080u+720uq-178u^2p+2160uq^2}{720(q^2-2q+1)hp}$$
$$+\frac{720pq-4320pq^2}{720(q^2-2q+1)hp},$$
$$\frac{-1034u^2qp-528u^2q^2p+1800p}{720(q^2-2q+1)hp},$$
$$\tilde{\beta}_3 = \frac{720u-1080uq+358u^2p-1440uq^2}{720(q^2-2q+1)hp}$$
$$+\frac{720pq}{720(q^2-2q+1)hp}$$
$$\frac{+2880pq^2+318u^2q^2p+884u^2qu^2p}{720(q^2-2q+1)hp},$$
$$+\frac{-1800p}{720(q^2-2q+1)hp}$$
$$\tilde{\beta}_4 = \frac{-180u+720uq-291u^2p+360uq^2}{720(q^2-2q+1)hp}$$
$$+\frac{-720pq^2-1080pq}{720(q^2-2q+1)hp}$$
$$\frac{-323u^2qu^2p-76u^2q^2p+900p}{720(q^2-2q+1)hp},$$
$$\tilde{\beta}_5 = \frac{-180uq+82u^2p+360pq+38u^2qp}{720(q^2-2q+1)hp}$$
$$+\frac{-180p}{720(q^2-2q+1)hp}.$$

Finally, differentiating and evaluating at $x=x_{n+5}$ gives

$$y'_{n+5} = \ddot{\alpha}_0 y_n + \ddot{\alpha}_1 y_{n+1} + h^2[\ddot{\beta}_0(u)f_n + \ddot{\beta}_1(u)f_{n+1} + \ddot{\beta}_2(u)f_{n+2} + \ddot{\beta}_3(u)f_{n+3} + \ddot{\beta}_4(u)f_{n+4} + \ddot{\beta}_5(u)f_{n+5}] \qquad (2.13)$$

with

$$\ddot{\alpha}_0 = -\frac{1}{720}\left(\frac{720q^2p-1440qp+720p}{(q^2-2q+1)hp}\right),$$
$$\ddot{\alpha}_1 = -\frac{1}{720}\left(\frac{1440qp-720q^2p-720p}{(q^2-2q+1)hp}\right),$$

$$\ddot{\beta}_0 = -\frac{1}{720}\left(\frac{-360uq^2+720pq^2-360pq}{(q^2-2q+1)hp} + \frac{323u^2qp+180u-113u^2p}{(q^2-2q+1)hp}\right),$$

$$\ddot{\beta}_1 = -\frac{1}{720}\left(\frac{720uq^3-540uq+1440uq^2}{(q^2-2q+1)hp} - \frac{2880pq^2}{(q^2-2q+1)hp} + \frac{1080pq+900p-646u^2q^2p}{(q^2-2q+1)hp} \frac{-180p+922u^2qp-720u-516u^2p}{(q^2-2q+1)hp}\right),$$

$$\ddot{\beta}_2 = -\frac{1}{720}\left(\frac{-2160uq^2-2880uq^3+2160uq}{(q^2-2q+1)hp} - \frac{720pq}{(q^2-2q+1)hp} + \frac{+4320pq^2-1800p-1392u^2q^2p+}{(q^2-2q+1)hp} + \frac{1034u^2qp+1080u-782u^2p}{(q^2-2q+1)hp}\right),$$

$$\ddot{\beta}_3 = -\frac{1}{720}\left(\frac{4320uq^3-3240uq+1440uq^2}{(q^2-2q+1)hp} - \frac{720pq-2880pq^2}{(q^2-2q+1)hp} + \frac{1800p+642u^2q^2p+2956u^2q}{(q^2-2q+1)hp} + \frac{u^2p-720u-838u^2p}{(q^2-2q+1)hp}\right),$$

$$\ddot{\beta}_4 = -\frac{1}{720}\left(\frac{-2880uq^3+2160uq-360uq^2}{(q^2-2q+1)hp} + \frac{720pq^2}{(q^2-2q+1)hp} + \frac{1080pq+323u^2qp-900p}{(q^2-2q+1)hp} - \frac{1844u^2q^2p+180u-669u^2p}{(q^2-2q+1)hp}\right),$$

$$\ddot{\beta}_5 = -\frac{1}{720}\left(\frac{720uq^3-540uq-360pq+180p}{(q^2-2q+1)hp} + \frac{922u^2qp-322u^2p}{(q^2-2q+1)hp}\right).$$

The methods derived in equations (2.6) - (2.12) above will be combined and implemented as a block in solving numerical examples.

3. Error and Stability Analysis of the Scheme

3.1 Local truncation error

Following Simos [15], Taylor series technique shall be used to estimate the local truncation error of the derived five-step method. Thus, for small values of the parameter u, the coefficients in equation (2.6) can be expressed as

$$\alpha_0 = -\frac{48}{12}$$

$$\alpha_1 = \frac{60}{12}$$

$$\beta_0 = \frac{7}{24} + \frac{349}{30240}u^2 + \frac{53}{181440}u^4 + \frac{71}{26611200}u^6 - \frac{4087}{18681062400}u^8 - \frac{142231}{7846046208000}u^{10} - \frac{16727}{17784371404800}u^{12} - \frac{69525593}{1703031405723648000}u^{14} - \frac{28465181}{17747379912278016000}u^{16} - \frac{34602653}{586213531493990400000}u^{18} + \cdots$$

$$\beta_1 = \frac{15}{4} - \frac{127}{3024}u^2 - \frac{13}{12960}u^4 - \frac{13}{2661120}u^6 + \frac{9929}{9340531200}u^8 + \frac{8779}{112086374400}u^{10} + \frac{3889}{988020633600}u^{12} + \cdots$$

$$\beta_2 = \frac{11}{4} + \frac{53}{1008}u^2 + \frac{11}{10080}u^4 - \frac{19}{2661120}u^6 - \frac{6427}{3113510400}u^8 - \frac{11509}{87178291200}u^{10} - \frac{56369}{8892185702400}u^{12} + \cdots$$

$$\beta_3 = \frac{13}{6} - \frac{4}{189}u^2 - \frac{1}{5670}u^4 + \frac{1}{41580}u^6 + \frac{167}{83397600}u^8 + \frac{2633}{24518894400}u^{10} + \frac{2671}{555761606400}u^{12} + \cdots$$

$$\beta_4 = \frac{23}{24} - \frac{31}{6048}u^2 - \frac{67}{181440}u^4 - \frac{109}{5322240}u^6 - \frac{18127}{18681062400}u^8 - \frac{64931}{1569209241600}u^{10} - \frac{9701}{5928123801600}u^{12}$$

$$\beta_5 = \frac{1}{12} + \frac{1}{240}u^2 + \frac{1}{6048}u^4 + \frac{1}{172800}u^6 + \frac{1}{5322240}u^8 + \frac{691}{118879488000}u^{10} + \frac{1}{5748019200}u^{12} + \cdots$$

The Local Truncation Error (LTE) for the five-step method described by equation (2.6) is obtained as

$$\text{LTE}((2.6)) = -\frac{95}{6048}h^5\left(w^2y^{(3)}(x_n)+y^{(5)}(x_n)\right) \quad (3.1)$$

where $y^{(i)}$ denotes the ith derivative of y with respect to the independent variable x.

3.2 Zero stability

The block method is zero stable if the roots z_s, $s = 1,2$ of the first characteristic polynomial $\rho(z)$ which is defined by

$$\bar{\rho}(z) = \det[zI_n - \bar{E}] \quad (3.2)$$

satisfies $|z_s| \leq 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$. where I is the identity matrix, and

$$E = \begin{pmatrix} \alpha_0 & \alpha_1 & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \hat{\alpha}_0 & \hat{\alpha}_1 & \hat{\beta}_0 & \hat{\beta}_1 & \hat{\beta}_2 & \hat{\beta}_3 & \hat{\beta}_4 & \hat{\beta}_5 \\ \check{\alpha}_0 & \check{\alpha}_1 & \check{\beta}_0 & \check{\beta}_1 & \check{\beta}_2 & \check{\beta}_3 & \check{\beta}_4 & \check{\beta}_5 \\ \acute{\alpha}_0 & \acute{\alpha}_1 & \acute{\beta}_0 & \acute{\beta}_1 & \acute{\beta}_2 & \acute{\beta}_3 & \acute{\beta}_4 & \acute{\beta}_5 \\ \grave{\alpha}_0 & \grave{\alpha}_1 & \grave{\beta}_0 & \grave{\beta}_1 & \grave{\beta}_2 & \grave{\beta}_3 & \grave{\beta}_4 & \grave{\beta}_5 \\ \tilde{\alpha}_0 & \tilde{\alpha}_1 & \tilde{\beta}_0 & \tilde{\beta}_1 & \tilde{\beta}_2 & \tilde{\beta}_3 & \tilde{\beta}_4 & \tilde{\beta}_5 \\ \ddot{\alpha}_0 & \ddot{\alpha}_1 & \ddot{\beta}_0 & \ddot{\beta}_1 & \ddot{\beta}_2 & \ddot{\beta}_3 & \ddot{\beta}_4 & \ddot{\beta}_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\bar{E} = \lim_{h\to 0}(E)$$

$$\bar{E} = \begin{pmatrix} -4 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\bar{\rho}(z) = \det[zI - \bar{E}] = z^6(-1+z^2)$$

solving for z in

$$z^6(-1+z^2) = 0$$

gives $z = 0$ or $z = 1$. Hence the block method is zero stable.

4. Convergence of the method

According to Gurjinder et.al (2013),

Definition 4.1. *A block method is said to be consistent if it has an order of convergence p, with* $p \geq 1$.

The block method derived in this article are all consistent as all the methods are of order $p > 1$.

It is a standard result that *The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable, see Dahlquist [Henrici[11]].* Thus the block methods derived in this article are convergent.

5. Implementation of the Scheme

The strategy adopted for the implementation of the methods is such that all the discrete methods obtained from the continuous method as well as their derivatives, which have the same order of accuracy, with very low error constants for fixed h, are combined as simultaneous integrators. The absolute errors calculated in the code are defined as

$$\text{Error} = |\text{yexact} - \text{ycomputed}|$$

where yexact is the exact solution, ycomputed is the computed result and Error is the absolute error. The value of w that produce an optimal solution in terms of accuracy are considered. All computations were carried out using Maple 17

6. Numerical Examples

Example1
We consider the following problem:

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \; y'(0) = 11 \quad (6.1)$$

whose exact solution is given by $y(x) = \cos(10x) + \sin(10x) + \sin(x)$.

Example 2
We consider a highly oscillatory test problem

$$y'' + \lambda^2 y = 0, \quad y(0) = 1, \; y'(0) = 2. \quad (6.2)$$

For $\lambda = 2$, the exact solution is known to be $y(x) = \cos(2x) + \sin(2x)$.

Example 3
We consider the non-linear initial value problem

$$y''(x) = \frac{(y')^2}{2y} - 2y, \; y\left(\frac{\pi}{6}\right) = \frac{1}{4} \text{ and } y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad (6.3)$$

whose exact solution is given by $y = \sin^2 x$.

7. Conclusion

We have proposed a five-step block trigonometrically fitted methods for the direct solution of general second order initial value problems with oscillatory solutions. The method has the advantage of being self starting, having good accuracy with order 4, consistent and zero stable. The methods are implemented without the need for the development of predictors nor requiring any other method to generate starting values. Implementation of the method with numerical examples on linear, non-linear and problems with oscillatory solution showed that the methods are superior to most of the existing multistep methods available for approximating similar class of problems.

Table 1. The exact solutions, computed results and error from the proposed methods for Example 1 with $h = \frac{1}{320}$, $w = 5$.

x	yExact	yComputed	Error	Error in Adeniran et.al[2]
$\frac{1}{320}$	1.03388166738420	1.03388166738422	2.00×10^{-14}	9.170×10^{-11}
$\frac{2}{320}$	1.06675678785246	1.06675678785252	6.00×10^{-14}	-
$\frac{3}{320}$	1.09859628036501	1.09859628036514	1.30×10^{-13}	3.0905×10^{-10}
$\frac{4}{320}$	1.12937207509627	1.12937207509648	2.10×10^{-13}	-
$\frac{5}{320}$	1.15905714081491	1.15905714081527	3.60×10^{-13}	-
$\frac{6}{320}$	1.18762550988244	1.18762551125056	5.40×10^{-13}	4.8987×10^{-10}
$\frac{7}{320}$	1.21505230844501	1.21505231041790	7.40×10^{-13}	-
$\frac{8}{320}$	1.24131377434580	1.24131377688084	9.60×10^{-13}	-
$\frac{9}{320}$	1.26638728387076	1.26638728692404	1.24×10^{-12}	-
$\frac{10}{320}$	1.29025137290459	1.29025137661543	1.55×10^{-12}	-

Table 2. The exact solutions, computed results and error from the proposed methods for Example 2, $h = 0.01$, $w = 2$.

x	yExact	yComputed	Error
0.01	1.01979867335991	1.01979867335991	0
0.02	1.03918944084761	1.03918944084382	3.79×10^{-12}
0.03	1.05816454641465	1.05816454641776	3.11×10^{-12}
0.04	1.07671640027179	1.07671640026785	3.94×10^{-12}
0.05	1.09483758192485	1.09483758192041	4.44×10^{-12}
0.06	1.11252084314278	1.11252084313810	4.68×10^{-12}
0.07	1.12975911085687	1.12975911084779	9.08×10^{-12}
0.08	1.14654548998987	1.14654548998810	1.77×10^{-12}
0.09	1.16287326621394	1.16287326620421	9.73×10^{-12}
0.10	1.17873590863630	1.17873590862580	1.05×10^{-12}

Table 3. Comparison of errors for Example 2. Abhulimen and Okunuga (2008)-ABHUK

x	Proposed method	Adeniran et.al [2]	ABHUK[1]	Awoyemi et. al.[5]
	$p = 4, k = 5$	$p = 3, k = 1$	$p = 6$	$p = 4$ $k = 3$
0.01	0.00	4.00×10^{-11}	2.66×10^{-11}	-
0.02	3.79×10^{-12}	1.03×10^{-10}	2.60×10^{-06}	8.48×10^{-10}
0.03	3.11×10^{-12}	1.88×10^{-10}	4.00×10^{-06}	6.41×10^{-09}
0.04	3.94×10^{-12}	2.95×10^{-10}	5.30×10^{-06}	6.71×10^{-09}
0.05	4.44×10^{-12}	4.24×10^{-10}	6.60×10^{-06}	7.12×10^{-09}
0.06	4.68×10^{-12}	5.74×10^{-10}	7.90×10^{-06}	7.65×10^{-09}
0.07	9.08×10^{-12}	7.44×10^{-10}	9.30×10^{-06}	8.36×10^{-09}
0.08	1.77×10^{-12}	9.34×10^{-10}	1.10×10^{-05}	9.06×10^{-09}
0.09	9.73×10^{-12}	1.14×10^{-09}	1.20×10^{-05}	9.92×10^{-09}
0.1	1.05×10^{-12}	1.37×10^{-09}	1.30×10^{-05}	1.09×10^{-08}

Table 4. The exact solutions, computed results and error from the proposed methods for Example 3, $h = 0.1$, $w = 2$.

x	yExact	yComputed	Error	Error in Alabi et.al.[3]
0.1	0.0099667110793792	0.00996671107827122	1.114×10^{-12}	–
0.2	0.0394695029985574	0.0394695029786944	1.990×10^{-11}	–
0.3	0.0873321925451611	0.0873321924526160	9.255×10^{-11}	–
0.4	0.1516466453264171	0.151646645281091	4.533×10^{-11}	–
0.5	0.229848847065930	0.229848847017083	4.884×10^{-11}	–
0.6	0.318821122761663	0.318821122673452	8.821×10^{-11}	1.013×10^{-08}
0.7	0.415016428549879	0.415016428485594	6.429×10^{-11}	4.782×10^{-08}
0.8	0.514599761150645	0.514599761134647	1.599×10^{-11}	1.109×10^{-07}
0.9	0.613601047346543	0.613601047330533	1.601×10^{-10}	1.892×10^{-07}
1.0	0.708073418273572	0.708073418262838	1.071×10^{-10}	1.196×10^{-07}
2.0	0.826821810431807	0.826821783609997	2.682×10^{-08}	1.435×10^{-06}

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