Malaya Journal of Matematik, Vol. 6, No. 4, 744-750, 2018

https://doi.org/10.26637/MJM0604/0007

On quadratic integral equations of Volterra type in Fréchet spaces

Latifa Benhamouche[1,3] and Smaïl Djebali[2,3]*

Abstract

In this work, we investigate the existence of solutions to a quadratic integral equation of Volterra type. By using the Schauder Tychonoff fixed point theorem in $C(\Omega,\mathbb{R})$, the Fréchet Space of real continuous functions on unbounded open subset $\Omega\subset\mathbb{R}^n$, we establish the existence of at least one solution.

Keywords

Quadratic integral equation, Schauder-Tychonoff fixed point theorem, Volterra operator, Fréchet space.

AMS Subject Classification

45D05, 47H10.

[1] *Department of Mathematics, Faculty of Sciences, BP 270. Blida, 09000. Algeria.*

[2] *Department of Mathematics, Faculty of Sciences, Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Saad Dahlab University, route de Soumaa.*

[3] *Laboratoire "Théorie du Point Fixe et Applications", ENS, BP 92 Kouba. Algiers, 16006. Algeria.*

***Corresponding author**: [2] djebali@hotmail.com; [1] h_betrouni@yahoo.com

Article History: Received **11** April **2018**; Accepted **29** September **2018**

Contents

1. Introduction

The aim of this paper is to study the existence of continuous solutions to a class of general quadratic integral equations of Volterra type with deviated argument of the form:

$$u(x)=f(x,u(x))+g(x,u(x))\int\limits_{\Lambda(x)} h(x,y,u(\xi(y))dy, \quad (1.1)$$

for $x\in\Omega$. Here Ω is an unbounded open set of $\mathbb{R}^n$, f, g, h, ξ are given functions and u is the unknown.

Over the last decade, the solvability of nonlinear integral equations in Banach spaces has been subject of several works (see, e.g., [2]). In [6], the existence of integrable solutions to the nonlinear equation

$$x(t)=u(t,x(t))+g(t,x(t))\int_0^{\phi(t)} k(t,s)f(s,x(s))ds,$$

for $0\le t\le 1$ is obtained using the technique of measure of weak noncompactness combined with the concept of weak-strong compact operator. Also in [11], a quadratic equation of the form:

$$\begin{aligned}x(t)&=f(t,x(\varphi_1(t)))\\&+g(t,x(\varphi_2(t))\psi\left(\int_0^{\alpha(t)} u(t,s,x(\varphi_3(t)))ds\right),\end{aligned}$$

for $t\in[0,T]$ was studied and the existence of at least one positive L^1-solution was obtained.

In [15], the authors discussed the existence of nondecreasing continuous solutions for the following quadratic integral equation of Volterra type on a bounded interval:

$$x(t)=g(t,x(t))\left(h(t)+\int_0^t k(t,s)f(s,x(\lambda s))ds\right), \quad 0\le t\le 1.$$

Moreover, several authors have been interested in the existence of continuous solutions to different types of integral equations on unbounded intervals; most of their results have been obtained in the setting of the Banach space of bounded and continuous functions on the nonnegative real half-axis and have employed the fixed point theory and some properties of measures of non-compactness.

Indeed, the use of measures of non-compactness turns out to be a strong technique allowing not only authors to obtain the existence of solutions but also to derive some characterizations of the solutions. The efficiency of this technique is shown in, e.g., [3], where the existence of asymptotically stable and ultimately nondecreasing solutions to a quadratic functional integral equation of Hammerstein-Volterra type of the form:

$$x(t) = m(t) + f(t,x(t)) \int_0^t g(t,\tau)h(\tau,x(\tau))d\tau,$$

for $t \geq 0$, is proved.

In [8], the authors investigated the questions of existence and asymptotic behavior of solutions of the quadratic Urysohn integral equation:

$$x(t) = a(t) + f(t,x(t)) \int_0^\infty u(t,s,x(s))ds,$$

for $t \geq 0$. However, some restrictive conditions may appear when dealing with such measures of noncompactness, especially when these measures are not regular (see, e.g., [2, (3.5), p.72]).

Another approach to discuss the solvability of integral equations is given in [7], where the authors obtained the existence of a unique solution to a quadratic integral equation of Urysohn type in a Fréchet space by using the nonlinear alternative of Leray-Schauder for contractive mappings.

Recently, in [9, 12], the authors defined a sequence of measures of noncompactness and obtained existence results in the Fréchet space of continuous functions on the real half-axis.

More recently, the authors in [14] developed some fixed point theorems in locally convex spaces and obtained an existence result in the space $C(\mathbb{R}^+,\mathbb{R}^d)$, without appealing to the technique of measures of noncompactness.

Our aim in this work is twofold: first we extend the results obtained in [9, 12, 14] to the case of unbounded open subsets of $\mathbb{R}^n$ for a quite general quadratic Volterra type equation of the form (1.1) and secondly we slightly relax the assumptions in [9]. This is the content of Section 3, where the functions f and g satisfy some Lipschitz conditions while h obeys a general growth condition in its third argument. Two examples of applications are provided in Section 4 to illustrate the main existence result. Comparison with recent results are given in the last Section 5 to show how Equation (1.1) encompasses some particular equations already discussed in the very recent literature. The next section is devoted to presenting some auxiliary results.

2. Preliminaries

We introduce some notations, definitions, and theorems which are used throughout this paper. The first one follows from simple topological arguments (see, e.g., [13]):

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open subset. Then there exists a sequence $(K_j)_{j=1}^\infty$ of compact subsets of Ω with the following properties:*

(a) $\Omega = \cup_{j=1}^\infty K_j$,

(b) $K_j \subset \mathring{K}_{j+1}$, *for all* $j \in \{1,2,\dots\}$.

Owing to Lemma 2.1, we can introduce a metric on the space $C(\Omega)$ of continuous real functions defined on an open subset Ω of $\mathbb{R}^n$. Indeed, for $u, v \in C(\Omega)$, we first define a family of semi-norms by:

$$p_j(u) = \sup\{|u(x)| : x \in K_j\}, \quad j \in \{1,2,\dots\}$$

and then the distance:

$$d(u,v) = \sum_{j=1}^\infty \frac{1}{2^j} \frac{p_j(u-v)}{1+p_j(u-v)}.$$

Endowed with this metric, $C(\Omega)$ may be considered as a Fréchet space satisfying the following property (see, e.g., [13]):

Lemma 2.2. *Let $\mathscr{K}$ be the collection of all compact subsets of Ω. Then the topology on $C(\Omega)$ induced by d coincides with $\tau_{\mathscr{K}}$, the topology of uniform convergence on compact subsets.*

Definition 2.3. *A set $E \subset C(\Omega)$ is said to be bounded if for all $j = 1,2,\dots$, there are numbers $M_j < \infty$ such that*

$$p_j(u) \leq M_j, \quad \forall u \in E,$$

i.e., $|u(x)| \leq M_j$, for all $u \in E$ and all $x \in K_j$.

Also we have the following characterization of convergence:

Lemma 2.4. *A sequence $(u_k)_{k\geq 1}$ converges to some limit u in the space $C(\Omega)$ if and only if for all $j \geq 1$, we have $p_j(u_k - u) \to 0$, as $k \to \infty$.*

Finally by the Ascoli-Arzela Lemma and Lemma 2.2, the characterization of compact subsets with respect to the topology of $C(\Omega)$ reads as follows:

Lemma 2.5. *A set $M \subset C(\Omega)$ is relatively compact if and only if for each j, the restriction of all functions from $M_{|K_j}$ forms an equicontinuous and uniformly bounded subset in the Banach space $C(K_j)$.*

Let X be a Hausdorff locally convex space with a topology generated by a family of semi-norms $\mathscr{P}$. As in Definition 2.3, a natural definition of the contraction is given by:

Definition 2.6. *Let $C \subset X$ and $p \in \mathscr{P}$. A mapping $A : C \to C$ is said to be a p-contraction if there exists α_p with $0 \leq \alpha_p < 1$ such that*

$$p(Ax - Ay) \leq \alpha_p p(x-y), \ \forall x, y \in C.$$

We have (see, e.g., [10])

Lemma 2.7. *Suppose C is a sequentially complete subset of X and the mapping $A : C \to C$ is a p-contraction, for every $p \in \mathscr{P}$. Then A has a unique fixed point $\bar{x} \in C$ and $A^k x \to \bar{x}$, for every $x \in C$, where A^k is the k-th iterate of the mapping A.*

The following lemma is then a direct consequence of Lemma 2.7.

Corollary 2.8. *Let $M \subset C(\Omega)$ be a closed set and $T : M \to M$ an operator with the property*

$$\forall j \geq 1,\ \exists \alpha_j \in [0,1),\ \forall u_1, u_2 \in M,$$
$$|Tu_1(x) - Tu_2(x)| \leq \alpha_j |u_1(x) - u_2(x)|,\ \forall x \in K_j.$$

Then T admits a unique fixed point in M.

We will also make use of the Schauder-Tychonoff fixed point theorem (we refer, e.g., to [1, p. 96]):

Lemma 2.9. *Let E be a Hausdorff locally convex linear topological space, C a convex subset of E, and $F : C \to E$ a continuous mapping such that*

$$F(C) \subset A \subset C,$$

where A is compact. Then F has at least one fixed point.

3. Main Results

In this section, we present an existence result of continuous solutions for Equation (1.1). The following definition can be found in [4]:

Definition 3.1. *Let Ω be an unbounded subset of $\mathbb{R}^n$. A mapping $\Lambda : \Omega \to \mathfrak{M}_{\mathbb{R}^n}$ is said to be a continuous function if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in \Omega$:*

$$|x - y| < \delta \Longrightarrow \mu(\Lambda(x) \vartriangle \Lambda(y)) < \varepsilon,$$

where μ is the Lebesgue measure on $\mathbb{R}^n$ and $\vartriangle$ denotes the symmetric difference of sets in the euclidian space $\mathbb{R}^n$.

Equation (1.1) will be studied under the following assumptions:

(H_0) The mappings
$\Lambda : \Omega \to \mathfrak{M}_{\mathbb{R}^n}$, $\xi : \bigcup_{x\in\Omega} \Lambda(x) \to \Omega$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$, and $h : \Omega \times \bigcup\limits_{x\in\Omega} \Lambda(x) \times \mathbb{R} \to \mathbb{R}$ are continuous.

Here $\mathfrak{M}_{\mathbb{R}^n}$ refers to the family of nonempty, bounded, and measurable subsets of $\mathbb{R}^n$.

(H_1) For each K_j, the set $\bigcup\limits_{x\in K_j} \Lambda(x)$ is bounded.

(H_2) There exists a nonnegative function $a \in C(\Omega)$ such that

$$|f(x,u) - f(x,v)| \leq a(x)|u - v|,$$

for all $x \in \Omega$ and all $u, v \in \mathbb{R}$.

(H_3) There exists a nonnegative function $b \in C(\Omega)$ such that

$$|g(x,u) - g(x,v)| \leq b(x)|u - v|,$$

for all $x \in \Omega$ and all $u, v \in \mathbb{R}$.

(H_4) There exists a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and a continuous function $k : \Omega \times \bigcup\limits_{x\in\Omega} \Lambda(x) \to \mathbb{R}^+$ such that, for all $x \in \Omega$, $y \in \bigcup\limits_{x\in\Omega} \Lambda(x)$, and $u \in \mathbb{R}$, we have

$$|h(x,y,u)| \leq k(x,y)\varphi(|u|). \tag{3.1}$$

(H_5) There exists a continuous function $r : \Omega \to \mathbb{R}^+$ such that for all $x \in \Omega$

$$a(x)r(x) + b(x)r(x)\int_{\Lambda(x)} k(x,y)\varphi(r(\xi(y)))dy \tag{3.2}$$
$$+c(x)\int_{\Lambda(x)} k(x,y)\varphi(r(\xi(y)))dy + d(x) \leq r(x)$$

and for all $j \geq 1$

$$\sup_{x\in K_j}\left\{a(x) + b(x)\int_{\Lambda(x)} k(x,y)\varphi(r(\xi(y)))dy\right\} < 1, \tag{3.3}$$

where $d(x) = |f(x,0)|$ and $c(x) = |g(x,0)|$.

Define the nonempty, bounded, closed, and convex subset of $C(\Omega)$:

$$M = \{u \in C(\Omega) : |u(x)| \leq r(x), \quad \forall x \in \Omega\},$$

where the function r is as defined in (H_5).

Remark 3.2. *The existence of functions r satisfying (H_5) is discussed in the examples, Section 4 and in the second remark, Section 5.*

Before stating the main existence result, we need to prove two technical lemmas:

Lemma 3.3. *Under $(H_0) - (H_1)$ and $(H_4) - (H_5)$, the Volterra operator $H : M \longrightarrow C(\Omega)$ defined by*

$$H[u](x) = \int_{\Lambda(x)} h(x,y,u(\xi(y)))dy, \text{ for all } x \in \Omega \tag{3.4}$$

is continuous and $H(M)$ is relatively compact in $C(\Omega)$.

Proof.

Claim 1. $H(M)$ is relatively compact in $C(\Omega)$. Given $u \in M$, it is easy to check that Hu is continuous on Ω for, by (H_0), the functions ξ, Λ, and h are continuous. The restriction of all functions of $H(M)$ forms a uniformly bounded set on K_j. In fact, for all $x \in K_j$, we have by (H_1) and $(H_4) - (H_5)$:

$$\begin{aligned} |H[u](x)| &\leq \int_{\Lambda(x)} |h(x,y,u(\xi(y)))|dy \\ &\leq \int_{\Lambda(x)} |k(x,y)\varphi(r(\xi(y)))|dy \\ &\leq \bar{k}_j \varphi(\bar{R}_j)\Lambda^j, \end{aligned}$$

where $\widetilde{K}_j$ is the image under the continuous function ξ of the set $\overline{\bigcup_{x\in K_j}\Lambda(x)}$ which is bounded and closed in $\mathbb{R}^n$, and where we have set

$$\begin{array}{rcl}\bar{R}_j &=& \sup\{r(x), x\in\widetilde{K}_j\}\\ \Lambda^j &=& \sup\{m(\Lambda(x)), x\in K_j\}\\ \bar{k}_j &=& \sup\{k(x,y), x\in K_j, y\in\overline{\bigcup_{x\in K_j}\Lambda(x)}\}.\end{array}$$

The next step consists in showing that, for all $j\geq 1$, the restriction of all functions from $H(M)$ to K_j are equi-continuous. For this, let $j\geq 1$ be fixed and $\varepsilon>0$. Since h is uniformly continuous on the compact

$$K_j\times\overline{\bigcup_{x\in K_j}\Lambda(x)}\times[-\bar{R}_j,\bar{R}_j],$$

then there exists $\delta_1>0$, such that for all $u\in M$, $y\in\overline{\bigcup_{x\in K_j}\Lambda(x)}$, and $x_1,x_2\in K_j$ with $|x_1-x_2|\leq\delta_1$, we have

$$|h(x_1,y,u(\xi(y)))-h(x_2,y,u(\xi(y)))|<\frac{\varepsilon}{2\Lambda^j}$$

and there exists $\delta_2>0$ such that for all $x_1,x_2\in K_j$ with $|x_1-x_2|\leq\delta_2$, we have

$$\mu(\Lambda(x_1)\bigtriangleup\Lambda(x_2))<\frac{\varepsilon}{2\bar{k}_j\varphi(\bar{R}_j)}.$$

Taking

$$\delta=\min\{\delta_1,\delta_2\},$$

we find that for all $u\in M$ and $x_1,x_2\in K_j$ with $|x_1-x_2|\leq\delta$, we have the estimates:

$$\begin{array}{l}|H[u](x_1)-H[u](x_2)|=\\ \left|\int\limits_{\Lambda(x_1)}h(x_1,y,u(\xi(y)))dy-\int\limits_{\Lambda(x_2)}h(x_2,y,u(\xi(y)))dy\right|\\ \leq\left|\int\limits_{\Lambda(x_1)}h(x_1,y,u(\xi(y)))-h(x_2,y,u(\xi(y)))dy\right|\\ +\left|\int\limits_{\Lambda(x_1)}h(x_2,y,u(\xi(y)))dy-\int\limits_{\Lambda(x_2)}h(x_2,y,u(\xi(y)))dy\right|\\ <\frac{\varepsilon\mu(\Lambda(x_1))}{2\Lambda^j}+\bar{k}_j\varphi(\bar{R}_j)\mu(\Lambda(x_1)\bigtriangleup\Lambda(x_2))\\ <\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{array}$$

From its uniform boundedness and equicontinuity, we deduce that the set $H(M)$ is relatively compact in $C(\Omega)$.

Claim 2. Continuity of the operator H on M. Since $C(\Omega)$ is a metric space, it is sufficient to show that H is sequentially continuous. Let $(u_n)_n$ be a sequence which converges to some limit $u\in M$, i.e., it converges uniformly on each compact subset of Ω. Due to the continuity of h and the convergence of u_n, we have that for all $j\geq 1$ and $x\in K_j$, there exists $N_\varepsilon\in\mathbb{N}$, such that for all $n\geq N_\varepsilon$ and $y\in\overline{\bigcup_{x\in K_j}\Lambda(x)}$, we have

$$|h(x,y,u_n(\xi(y)))-h(x,y,u(\xi(y)))|<\frac{\varepsilon}{\Lambda^j}.$$

Hence

$$\begin{array}{l}|H[u_n](x)-H[u](x)|=\\ \left|\int\limits_{\Lambda(x)}h(x,y,u_n(\xi(y)))-h(x,y,u(\xi(y)))dy\right|\\ <\frac{\varepsilon}{\Lambda^j}\mu(\Lambda(x))\\ <\frac{\varepsilon}{\Lambda^j}\Lambda^j=\varepsilon.\end{array}$$

Therefore the operator H is continuous on M. □

Lemma 3.4. *Under Assumptions $(H_0)-(H_5)$, for each $v\in M$, there exists a unique $\psi_v\in M$ such that, for all $x\in\Omega$, we have*

$$\psi_v(x)=f(x,\psi_v(x))+g(x,\psi_v(x))\int\limits_{\Lambda(x)}h(x,y,v(\xi(y)))dy.$$

Proof. Let $v\in M$ be fixed and define the operator $\mathscr{T}_v$ by

$$\mathscr{T}_v(u)(x)=f(x,u(x))+g(x,u(x))\int\limits_{\Lambda(x)}h(x,y,v(\xi(y)))dy.$$

Making use of $(H_2)-(H_4)$, we find that, for each $u_1,u_2\in M$ and for all $x\in\Omega$, we have

$$\begin{array}{l}|\mathscr{T}_v(u_1)(x)-\mathscr{T}_v(u_2)(x)|\\ \leq|f(x,u_1(x))-f(x,u_2(x))|\\ +|g(x,u_1(x))-g(x,u_2(x))|\int\limits_{\Lambda(x)}|h(x,y,v(\xi(y)))|dy\\ \leq\left(a(x)+b(x)\int\limits_{\Lambda(x)}|h(x,y,v(\xi(y)))|dy\right)|u_1(x)-u_2(x)|\\ \leq\left(a(x)+b(x)\int\limits_{\Lambda(x)}k(x,y)\varphi(r(\xi(y)))dy\right)|u_1(x)-u_2(x)|.\end{array}$$

Writing this for $x\in K_j$, passing to the supremum over K_j, and appealing to (H_5) shows that $\mathscr{T}_v$ is a contraction. Then Corollary 2.8 guarantees that the operator $\mathscr{T}_v:M\to M$ admits a unique fixed point ψ_v in M. □

We are now in position to state and prove the main existence result of this paper:

Theorem 3.5. *Under assumptions $(H_0)-(H_5)$, Equation (1.1) has at least one solution in the Fréchet space $C(\Omega)$.*

Proof. The proof of this theorem is based on an application of the Schauder-Tychonov fixed point theorem, namely Lemma 2.9. Using Lemma 3.4, we first define the mapping $\mathscr{A}:M\to M$ which assigns to each v the image $\mathscr{A}v$ solution of the following nonlinear equation:

$$\mathscr{A}v(x)=f(x,\mathscr{A}v(x))+g(x,\mathscr{A}v(x))H[v](x),\quad x\in\Omega,$$

where H is defined by (3.4). It is clear that a fixed point of the mapping $\mathscr{A}$ is a solution to Equation (1.1).

Claim 1. $\mathscr{A}$ is a continuous operator on M. It is sufficient to show that it is sequentially continuous.

Let $(v_n)_n$ be a sequence which converges to $v \in M$, let $j \geq 1$ be fixed, and $x \in K_j$. Using $(H_0)-(H_4)$, we get the estimates:

$$\begin{aligned}
&|\mathscr{A}v_n(x)-\mathscr{A}v(x)| \\
&\leq |f(x,\mathscr{A}v_n(x))-f(x,\mathscr{A}v(x))| \\
&+\Bigg| g(x,\mathscr{A}v_n(x))\int_{\Lambda(x)} h(x,y,v_n(\xi(y)))dy \\
&-g(x,\mathscr{A}v(x))\int_{\Lambda(x)} h(x,y,v_n(\xi(y)))dy\Bigg| \\
&+|g(x,\mathscr{A}v(x))||H[v_n](x)-H[v](x)| \\
&\leq \alpha_j|\mathscr{A}v_n(x)-\mathscr{A}v(x)|+\bar{g}^j|H[v_n](x)-H[v](x)|,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_j &= \sup_{x\in K_j}\{a(x)+b(x)\int_{\Lambda(x)} k(x,y)\varphi(r(\xi(y)))dy\}, \\
\bar{g}^j &= \sup\left\{|g(x,u)|,\, x\in K_j,\, u\in[-\bar{r}^j,\bar{r}^j]\right\}, \\
\bar{r}^j &= \sup\{r(x),\, x\in K_j\}.
\end{aligned}$$

Hence

$$|\mathscr{A}v_n(x)-\mathscr{A}v(x)| \leq \frac{\bar{g}^j}{1-\alpha_j}|H[v]_n(x)-H[v](x)|.$$

Since H is continuous, from the last estimate we deduce the continuity of the operator $\mathscr{A}$ on M.

Claim 2. $\mathscr{A}(M)$ is equi-continuous. Given $j\geq 1$, for all $x_1,x_2\in K_j$ and $v\in M$, we have the estimate:

$$\begin{aligned}
&|\mathscr{A}v(x_1)-\mathscr{A}v(x_2)| \\
&\leq |f(x_1,\mathscr{A}v(x_1))-f(x_2,\mathscr{A}v(x_1))| \\
&+|f(x_2,\mathscr{A}v(x_1))-f(x_2,\mathscr{A}v(x_2))| \\
&+|g(x_1,\mathscr{A}v(x_1))-g(x_2,\mathscr{A}v(x_1))||Hv(x_1)| \\
&+|g(x_2,\mathscr{A}v(x_1))-g(x_2,\mathscr{A}v(x_2))||Hv(x_1)| \\
&+|g(x_2,\mathscr{A}v(x_2))||H[v](x_1)-H[v](x_2)|.
\end{aligned}$$

Hence

$$\begin{aligned}
&|\mathscr{A}v(x_1)-\mathscr{A}v(x_2)| \\
&\leq |f(x_1,\mathscr{A}v(x_1))-f(x_2,\mathscr{A}v(x_1))| \\
&+|g(x_1,\mathscr{A}v(x_1))-g(x_2,\mathscr{A}v(x_1))||H[v](x_1)| \\
&+\alpha^j|\mathscr{A}v(x_1)-\mathscr{A}v(x_2)|+\bar{g}^j|H[v](x_1)-H[v](x_2)|.
\end{aligned}$$

By the uniform continuity of functions f and g on the compact set $K_j\times[-\bar{r}^j,\bar{r}^j]$ and the equicontinuity of $H(M)$, for all $\varepsilon>0$, there exists $\delta>0$, such that for all $x_1,x_2\in K_j$ with $|x_1-x_2|<\delta$, we have

$$|f(x_1,\mathscr{A}v(x_1))-f(x_2,\mathscr{A}v(x_1))| < \frac{(1-\alpha^j)\varepsilon}{3},$$

$$|g(x_1,\mathscr{A}v(x_1))-g(x_2,\mathscr{A}v(x_1))| < \frac{(1-\alpha^j)\varepsilon}{3\bar{k}_j\varphi(\bar{R}_j)\Lambda^j},$$

and

$$|H[v](x_1)-H[v](x_2)| < \frac{(1-\alpha^j)\varepsilon}{3\bar{g}^j}.$$

Therefore

$$|\mathscr{A}v(x_1)-\mathscr{A}v(x_2)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,$$

which means that the restriction of all functions from $\mathscr{A}(M)$ forms an equi-continuous set, for each $j\geq 1$. As a consequence, the set $\mathscr{A}(M)$ is relatively compact by Lemma 2.5. Now applying the Schauder-Tychonov fixed point theorem, i.e., Lemma 2.9, we conclude that $\mathscr{A}$ has a fixed point in M, that is a solution to Equation (1.1). □

4. Examples

In this section, we illustrate the applicability of Theorem 3.5 by giving two concrete examples.

Example 4.1. *Consider the following integral equation posed on* $(0,+\infty)\times(0,+\infty)$*:*

$$\begin{aligned}
&u(x_1,x_2) \\
&=(x_1+x_2)\exp^{x_1+x_2} \\
&+\arctan\left(\tfrac{u(x_1,x_2)}{3+x_1^4+x_2^4}\right)\int_0^{x_2}\int_0^{x_1}\tfrac{4x_1^2x_2^2y_1y_2}{(1+x_1^4+x_2^4)}\tfrac{u^2(y_1,y_2)}{(1+u^2(y_1,y_2))}dy_1dy_2.
\end{aligned} \tag{4.1}$$

Let

$$\Lambda(x_1,x_2)=(0,x_1)\times(0,x_2),\quad \xi(x_1,x_2)=(x_1,x_2),$$

$$f(x_1,x_2,u)=(x_1+x_2)\exp^{(x_1+x_2)},$$

$$g(x_1,x_2,u)=\arctan\left(\frac{u}{3+x_1^4+x_2^4}\right),$$

$$h(x_1,x_2,y_1,y_2,u)=\frac{4x_1^2x_2^2}{(1+x_1^4+x_2^4)}\frac{u^2}{(1+u^2)}y_1y_2.$$

We have for all $x_1,x_2,y_1,y_2\in(0,+\infty)$ *and* $u,v\in\mathbb{R}$

$$|f(x_1,x_2,u)-f(x_1,x_2,v)|=0,$$

$$|g(x_1,x_2,u)-g(x_1,x_2,v)|\leq\frac{1}{(3+x_1^4+x_2^4)}|u-v|,$$

$$|h(x_1,x_2,y_1,y_2,u)|\leq\frac{4x_1^2x_2^2}{(1+x_1^4+x_2^4)}y_1y_2.$$

So we can take $a(x)=c(x)=0$, $b(x)=\frac{1}{3+x_1^4+x_2^4}$, $\varphi(u)=1$, *and*

$$k(x_1,x_2,y_1,y_2)=\frac{4x_1^2x_2^2}{1+x_1^4+x_2^4}y_1y_2,$$

such that Assumptions $(H_1)-(H_5)$ *are satisfied. To check* (H_5)*, we choose*

$$r(x)=\frac{(x_1+x_2)\exp^{(x_1+x_2)}}{1-\frac{x_1^4x_2^4}{(3+x_1^4+x_2^4)(1+x_1^4+x_2^4)}}$$

which is a continuous function from $(0,+\infty)\times(0,+\infty)\to\mathbb{R}^+$ *that further satisfies* (H_5) *for*

$$\frac{1}{(3+x_1^4+x_2^4)}\int_0^{x_2}\int_0^{x_1}\frac{4x_1^2x_2^2y_1y_2}{(1+x_1^4+x_2^4)}dy_1dy_2 = \frac{x_1^4x_2^4}{(3+x_1^4+x_2^4)(1+x_1^4+x_2^4)} < 1.$$

By Theorem 3.5, Equation (4.1) has at least one solution in the space $C((0,+\infty)\times(0,+\infty))$.

Example 4.2. *Consider the following integral equation on* $(0,+\infty)\times(0,+\infty)$

$$\begin{aligned} &u(x_1,x_2)\\ &=\tfrac{1}{2}x_1x_2\cos\left(\tfrac{u(x_1,x_2)}{1+x_1^2x_2^2}\right)\\ &+\tfrac{u(x_1,x_2)}{(1+x_1^2)^2(1+x_2^2)}\int_0^{x_2}\int_{x_1}^{x_1^2+1}\tfrac{2u(y_1,y_2)}{1+|u(y_1,y_2)|}y_1y_2dy_1dy_2. \end{aligned} \tag{4.2}$$

For all $x_1,x_2,y_1,y_2\in(0,+\infty)$ *and* $u,v\in\mathbb{R}$, *let*

$$\Lambda(x_1,x_2)=(x_1,x_1^2+1)\times(0,x_2),\quad \xi(x_1,x_2)=(x_1,x_2),$$

$$f(x_1,x_2,u)=\frac{1}{2}x_1x_2cos(\frac{u}{1+x_1^2x_2^2}),$$

$$g(x_1,x_2,u)=\frac{u}{(1+x_1^2)^2(1+x_2^2)},$$

$$h(x_1,x_2,y_1,y_2,u)=\frac{2u}{1+|u|}y_1y_2.$$

Thus

$$|f(x_1,x_2,u)-f(x_1,x_2,v)|\le\frac{x_1x_2}{2(1+x_1^2x_2^2)}|u-v|,$$

$$|g(x_1,x_2,u)-g(x_1,x_2,v)|\le\frac{1}{(1+x_1^2)^2(1+x_2^2)}|u-v|,$$

$$|h(x_1,x_2,y_1,y_2,u)|\le 2y_1y_2.$$

So we can choose $a(x)=\frac{x_1x_2}{2(1+x_1^2x_2^2)}$, $b(x)=\frac{1}{(1+x_1^2)^2(1+x_2^2)}$, $c(x)=0$, $\varphi(u)=1$, *and*

$$k(x_1,x_2,y_1,y_2)=2y_1y_2,$$

such that assumptions $(H_1)-(H_5)$ *are satisfied. To verify* (H_5), *we consider the function*

$$r(x)=\frac{\frac{1}{2}x_1x_2}{1-\frac{|x_1x_2|}{2(1+x_1^2x_2^2)}-\frac{(x_1^4+x_1^2+1)x_2^2}{2(1+x_1^2)^2(1+x_2^2)}}$$

which is a continuous function from $(0,+\infty)\times(0,+\infty)\to\mathbb{R}^+$ *that satisfies* (H_5) *for*

$$\frac{x_1x_2}{2(1+x_1^2x_2^2)}+\frac{1}{(1+x_1^2)^2(1+x_2^2)}\int_0^{x_2}\int_{x_1}^{x_1^2+1}2y_1y_2dy_1dy_2 < \frac{1}{2}+\frac{1}{2}=1.$$

Applying Theorem 3.5, we conclude that Equation (4.2) has at least one solution in the space $C((0,+\infty)\times(0,+\infty))$.

5. Concluding Remarks

1. Example 1 cannot be covered by the method used in [4] because the function $(x_1+x_2)\exp^{x_1+x_2}$ is not bounded. Example 2 cannot be treated by the method in [9] because $\Omega\subset\mathbb{R}^2$ and f depends on the unknown u.

2. The nonlinear quadratic Volterra equation:

$$u(x)=a(x)+f(x,u(x))\int_0^x \nu(x,y,u(y)dy,\quad x\ge 0,$$

is solved in [9] via a measure of noncompactness and it is clearly covered by Equation (1.1). From Theorem 3.5, the existence of solution is obtained when $\Lambda(x)=(0,x)$, the nonlinearity f satisfies the Lipschitz condition (H_3) in the second argument, and the function ν is variable-separated dominated as in (H_4) while it is assumed bounded in the third argument in [9, $(H_3)'$, (33)], namely $|\nu(x,y,u)|\le g(x,y)$. Our assumption (H_5) then reduces to [9, $(H_4)'$, (34)]. In this case, notice that if we take $\varphi(s)=1$ in Hypothesis (H_4), then the function r introduced in (H_5) becomes:

$$r(x)=\frac{|f(x,0)|\int_0^x g(x,y)dy+|a(x)|}{1-k_f\int_0^x g(x,y)dy},$$

where k_f is the Lipschitz constant of f and

$$k_f\int_0^x g(x,y)dy<1$$

by [9, $(H_4)'$, (34)], i.e., by Hypothesis (H_5). This shows that a direct application of Schauder-Tychonoff fixed point theorem leads to the same existence result.

3. The nonlinear (non-quadratic) Volterra equation with deviated argument:

$$u(x)=d(x)+\int_0^x \nu(x,y,u(\varphi(y))dy,\ x\ge 0,$$

discussed in [5] is also covered by Equation (1.1). The Schauder fixed point theorem was employed for the nonlinearity ν was assumed to satisfy the sublinear growth condition (see [5, (i), (6)]):

$$|\nu(x,y,u)|\le\eta(x,y)+\alpha(x)\beta(y)|u|,$$

for some positive continuous functions, α,β, and η. This is of course a more restrictive condition than Hypothesis (H_5). Further conditions (ii) and (iii) are imposed in [5, (8)].

4. The usage of MNC in the space $BC(\mathbb{R}^+\times\mathbb{R}^+)$ in [4] has generated several boundedness conditions on the nonlinear functions of the functional-integral equation in consideration (see [4, (i)-(v)]), which did not occur in this work.

5. It may be of interest to relax Lipschitz conditions in $(H_2)-(H_3)$ which are intrinsic to the method used in this work.

References

[1] R.P. Agarwal, M. Meehan, and D.O'Regan, *Fixed Point Theory and Applications,* Cambridge Tracts in Mathematics, 141. Cambridge University Press, Cambridge, 2001.

[2] A. Aghajani, R. Allahyari, and M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, *J. Comput. Appl. Math.,* 260(2014), 68–77.

[3] J. Appell, J. Banaś, and N. Merentes, Measures of noncompactness in the study of asymptotically stable and ultimately nondecreasing solutions of integral equations, *Z. Anal. Anwend.,* 29(3)(2010), 251–273.

[4] R. Arab, R. Allahyari, and A.S. Haghighi, Construction of a measure of noncompactness on $BC(\Omega)$ and its application to Volterra integral equations, *Mediterr. J. Math.,* 13(3)(2016), 1197–1210.

[5] J. Banas, K. Sadarangani, Compactness conditions in the study of functional, differential, and integral equations, *Abstr. Appl. Anal.,* 2013, Art. ID 819315, 14 pp.

[6] A. Bellour, D. O'Regan, and M.A. Taoudi, On the existence of integrable solutions for a nonlinear quadratic integral equation, *J. Appl. Math. Comput.*, 46(1-2)(2014), 67–77.

[7] M. Benchohra, M. A. Darwish, On quadratic integral equations of Urysohn type in Fréchet spaces, *Acta Math. Univ. Comenian.,* 79(1)(2010), 105–110.

[8] M.A. Darwish, J. Banaś, and E.O. Alzahrani, The existence and attractivity of solutions of an Urysohn integral equation on an unbounded interval, *Abstr. Appl. Anal.,* 2013, Art. ID 147409, 9 pp.

[9] S. Dudek, L. Olszzowy, Continuous dependence of the solutions of nonlinear integral quadratic Volterra equation on the parameter, *J. Funct. Spaces*, 2015, Art. ID 471235, 9 pp.

[10] G.L.Jr. Cain, M.Z. Nashed, Fixed points and stability for a sum of two operators in locally convex spaces, *Pac. J. Math.,* 39(1971), 581–592.

[11] H.H.G. Hashem, A.M.A. Elsayed Solvability of nonlinear quadratic functional equations, *App. Math. Inf. Sci.,* 9(5)(2015), 2715-2720.

[12] L. Olszowy, Fixed point theorems in the Fréchet space $C(\mathbb{R}_+)$ and functional integral equations on an unbounded interval, *Appl. Math. Comput.,* 218(18)(2012), 9066–9074.

[13] V. Runde, *A Taste of Topology,* Universitext. Springer, New York, 2005.

[14] F. Wang, H. Zhou, Fixed point theorems in locally convex spaces and a nonlinear integral equation of mixed type, *Fixed Point Theory Appl.,* 2015, 2015:228, 11 pp.

[15] T. Zhu, G. Li, Existence of nondecreasing solutions of a quadratic integral equation of Volterra type, *Appl. Math. Comput.*, 221(2013), 214–220.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆