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Existence of solutions of a coupled system of functional integro-differential equations of arbitrary (fractional) orders

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Abstract
In this paper, we study the existence of solutions for a coupled system of functional integro-differential equations.

Keywords
Fractional order differential equations, functional equations, existence of solutions, integral solutions, continuous solutions, fractional calculus.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

During the past few years, some investigators have established a lot of useful and interesting coupled system of functional equations, in order to achieve various goals; see [1–6], and the references cited therein.
In this paper, we are concerned with the coupled system of functional integro-differential equations

$$\begin{cases} \frac{dx}{dt} = f_1(t, D^\alpha y(t), D^\beta x(t)), & t \in (0,T], \\ \frac{dy}{dt} = f_2(t, D^\alpha y(t), D^\beta x(t), & t \in (0,T], \end{cases} \tag{1.1}$$

with the nonlocal condition

$$\begin{aligned} x(0) + \sum_{k=1}^{n} a_k x(\tau_k) &= x_0, \quad a_k > 0, \quad \tau_k(0,T), \\ y(0) + \sum_{j=1}^{n} b_j y(\eta_j) &= y_0, \quad b_j > 0, \quad \eta_k(0,T). \end{aligned} \tag{1.2}$$

The existence of at least one solution (x,y), $x,y \in C^1[0,T]$ is proved, under certain conditions.
Also, under other assumptions, The existence of at least one solution (x,y), $x,y \in AC[0,T]$ of the problem (1.1)-(1.2) we be proved.

2. Preliminaries

In this section, we giving the following definitions and theorems which used in our results:

Definition 2.1. *[7] The fractional order integral of order α of $f \in L^1$ is defined by*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds.$$

Definition 2.2. *[7] The Caputo fractional-order derivative D_a^α of order $\alpha \in (0,1]$ of the absolutely continuous function*

$f(t)$ is given by

$$D_a^{\alpha} f(t) = I^{1-\alpha}\frac{d}{dt}f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{ds}f(s)ds.$$

Definition 2.3. *[8] A mapping $H : E \to E$ is called completely continuous (compact) if H is continuous and $H(Y)$ is relatively compact for every bounded subset of Y.*

Theorem 2.4. *[8, Arzela-Ascoli Theorem] Let E be a compact metric space and $C(E)$ be the Banach space of real or complex valued continuous function normed by*

$$\|y\| = \sup_{t\in E}|y(t)|.$$

If $A = \{y_n\}$ is a sequence in $C(E)$ such that $\{y_n\}$ is uniformly bounded and equi-continuous, then $\bar{A}$ is compact.

Theorem 2.5. *[9, Kolomogorov Comoactness Criterion Theorem] Let $\Omega \subseteq L_p(0,1)$, $1 \le p < \infty$. If*

- *Ω is bounded in $L_p(0,1)$,*
- *$x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$,*

then Ω is relatively compact in $L_p(0,1)$, where

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)ds.$$

Theorem 2.6. *[10, Schauder Fixed Point Theorem] Let E be a Banach space and Q be a convex subset of E, and $T : Q \to Q$ is compact, continuous map, Then T has at least one fixed point in Q.*

Next section, we present our main results by proving the existence of at least one continuous and integrable solutions.

3. Main Results

Consider the initial value problem of the coupled system of functional integro-differential equations (1.1) with the condition (1.2).

- Let $D^{\beta}x(t) = u(t)$, $D^{\alpha}y(t) = v(t)$ in (1.1), we obtain

$$\begin{cases} u(t) = \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s, v(s), u(s))ds, & t \in (0,T], \\ v(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, v(s), u(s))ds, & t \in (0,T], \end{cases} \tag{3.1}$$

where

$$\begin{cases} x(t) = x(0) + I^{1-\beta}u(t), & t \in (0,T], \\ y(t) = y(0) + I^{1-\alpha}v(t), & t \in (0,T], \end{cases} \tag{3.2}$$

- Let $\frac{dx}{dt} = \eta(t)$, $\frac{dy}{dt} = \omega(t)$ in (1.1), we obtain

$$\begin{cases} \eta(t) = f_1(t, I^{1-\alpha}\omega(t), I^{1-\beta}\eta(t)), & t \in (0,T], \\ \omega(t) = f_2(t, I^{1-\alpha}\omega(t), I^{1-\beta}\eta(t)), & t \in (0,T], \end{cases} \tag{3.3}$$

where

$$\begin{cases} x(t) = x(0) + \int_0^t \eta(s)ds, & t \in (0,T], \\ y(t) = y(0) + \int_0^t \omega(s)ds, & t \in (0,T]. \end{cases} \tag{3.4}$$

3.1 Continuous solution

We discuss the existence of at least one continuous solution (u,v), $u,v \in C[0,T]$ of the coupled system of functional integral equations (3.1), under the following assumptions:

1. $f_i : [0,T] \times \mathbb{R}^2 \to \mathbb{R}, i = 1,2$ satisfy Caratheodory condition i. e f_i are measurable in t for any $u,v \in \mathbb{R}$ and continuous in u,v for almost all $t \in [0,T]$. There exist two functions $m_i(t) \in L^1[0,T]$, such that
$$|f_i(t,v,u)| \le m_i(t).$$
2. $$\sup_{t\in[0,T]} I^{1-\beta}m_1(t) \le M_1, \qquad \sup_{t\in[0,T]} I^{1-\alpha}m_2(t) \le M_2.$$

Let X be the Banach space of all order pairs (u,v) with the norm

$$\|(u,v)\|_X = \|u\|_C + \|v\|_C = \sup_{t\in[0,T]}|u(t)| + \sup_{t\in[0,T]}|v(t)|.$$

Definition 3.1. *By a solution of the coupled system of functional integral equations (3.1), we mean a function (u,v), $u,v \in C[0,T]$ that satisfies (3.1).*

Theorem 3.2. *Let the assumptions (1)-(2) be satisfied, then the coupled system of functional integral equations (3.1) has at least one continuous solution (u,v), $u,v \in C[0,T]$.*

Proof. Define the operator A associated with the coupled system of functional integral equations (3.1) by

$$A(u,v) = \left(\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds, \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds\right).$$

Let $Q_r = \{(u,v) \in \mathbb{R}^2 : ||u|| \le r_1, ||v|| \le r_2, ||(u,v)|| \le r_1 + r_2 = r\}$, where $r = M_1 + M_2$.
Then we have, for $(u,v) \in Q_r$

$$A(u,v) = \left(\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds, \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds\right).$$

But

$$|\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds| \leq$$
$$\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} |f_1(s,v(s),u(s))|ds$$
$$\leq \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} m_1(s)ds$$
$$\leq M_1.$$

Hence

$$||\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds||_C \leq M_1. \tag{3.5}$$

And

$$|\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds| \leq$$
$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} |f_2(s,v(s),u(s))|ds$$
$$\leq \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} m_2(s)ds$$
$$\leq M_2.$$

Hence

$$||\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds||_C \leq M_2. \tag{3.6}$$

From (3.5) and (3.6), we get

$$||A(u,v)||_X \leq M_1 + M_2.$$

This prove That $A: Q_r \to Q_r$, and the class of function $\{A_n(u,v)\}$ is uniformly bounded in Q_r.
Let $t_1, t_2 \in (0,T]$ such that $|t_2 - t_1| < \delta$, then

$$A(u(t_2),v(t_2)) - A(u(t_1),v(t_1)) =$$
$$\Big(\int_0^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds,$$
$$\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds\Big)$$
$$-\Big(\int_0^{t_1} \frac{(t_1-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds,$$
$$\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds\Big)$$
$$= \Big(\int_0^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds,$$
$$\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s)ds$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds\Big)$$

But

$$|\int_0^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds|$$
$$\leq |\int_0^{t_1} \frac{(t_1-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds$$
$$+\int_{t_1}^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds|$$
$$\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} |f_1(s,v(s),u(s))|ds$$
$$\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} m_1(s)ds.$$

Hence

$$||\int_0^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s,v(s),u(s))ds||_C$$
$$\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)} m_1(s)ds. \tag{3.7}$$

And

$$|\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s)ds$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds|$$
$$\leq |\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s)ds$$
$$+\int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds|$$
$$\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} |f_2(s,v(s),u(s))|ds$$
$$\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} m_2(s)ds.$$

Hence

$$||\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s)ds \tag{3.8}$$
$$-\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s,v(s),u(s))ds||_C$$
$$\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} m_2(s)ds. \tag{3.9}$$

From (3.7) and (3.8), we get

$$\begin{aligned}&\|A(u(t_2),v(t_2))-A(u(t_1),v(t_1))\|_X\\&\le\int_{t_1}^{t_2}\frac{(t_2-s)^{-\beta}}{\Gamma(1-\beta)}m_1(s)ds\\&+\int_{t_1}^{t_2}\frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}m_2(s)ds.\end{aligned}$$

This mean that the class of functions {A(u,v)} is equi-continuous in Q_r.
Let $v_n \to v$ and $u_n \to u$, then from continuity of the two functions f_i, we obtain $f_1(t,v_n(t),u_n(t)) \to f_1(t,v(t),u(t))$ and $f_2(t,v_n(t),u_n(t)) \to f_2(t,v(t),u(t))$ as $n\to\infty$

$$\begin{aligned}\lim_{n\to\infty}A(u_n,v_n) &= \lim_{n\to\infty}\Big(\int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}f_1(s,v_n(s),u_n(s))ds,\\&\quad\int_0^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f_2(s,v_n(s),u_n(s))ds\Big)\\&= \Big(\lim_{n\to\infty}\int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}f_1(s,v_n(s),u_n(s))ds,\\&\quad\lim_{n\to\infty}\int_0^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f_2(s,v_n(s),u_n(s))ds\Big).\end{aligned}$$

Using assumptions (1)-(2) and Lebesgue dominated convergence Theorem [8] we obtain

$$\begin{aligned}\lim_{n\to\infty}A(u_n,v_n) &= \Big(\int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}f_1(s,v(s),u(s))ds,\\&\quad\int_0^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}f_2(s,v(s),u(s))ds\Big)\\&= A(u,v).\end{aligned}$$

Then $v_n\to v, u_n\to u \Rightarrow A(u_n,v_n)\to A(u,v)$ as $n\to\infty$. This mean that the operator A is continuous.
Then by Schauder fixed point Theorem 2.6 there exist at least one solution (u,v), $u,v\in C[0,T]$ of the coupled system of functional integral equations (3.1). The proof is completed. □

3.2 Integrable solution

We discuss the existence of at least one integrable solution (η,ω), $\eta,\omega\in L^1[0,T]$ of the coupled system of functional integral equations (3.3), under the following assumptions:

I $f_i:[0,T]\times\mathbb{R}^2\to\mathbb{R}, i=1,2$ satisfy Caratheodory condition i. e f_i are measurable in t for any $\eta,\omega\in\mathbb{R}$ and continuous in η,ω for almost all $t\in[0,T]$. There exist two functions $a_i(t)\in L^1[0,T]$ and two positive constants $b_i>0$, such that

$$|f_i(t,\eta,\omega)|\le a_i(t)+b_i(|\eta|+|\omega|).$$

II $(b_2+b_1)k<1$.

Let Y be the Banach space of all order pairs (η,ω) with the norm

$$\|(\eta,\omega)\|_Y=\|\eta\|_{L^1}+\|\omega\|_{L^1}=\int_0^T|\eta(s)|ds+\int_0^T|\omega(s)|ds$$

Definition 3.3. *By a solution of the coupled system of functional integral equations (3.3), we mean a function (η,ω), $\eta,\omega\in L^1[0,T]$ that satisfies (3.3).*

Theorem 3.4. *Let the assumptions (I)-(II) be satisfied, then the coupled system of functional integral equations (3.3) has at least one integrable solution (η,ω), $\eta,\omega\in L^1[0,T]$.*

Proof. Define the operator A associated with the coupled system of functional integral equations (3.3) by

$$A(\eta,\omega)=\Big(f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)),f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))\Big).$$

Let $Q_p=\{(\eta,\omega)\in\mathbb{R}^2:||\eta||\le p_1,||\omega||\le p_2,||(\eta,\omega)||\le p_1+p_2=p\}$, where $p=\frac{\|a_1\|_{L^1}+\|a_2\|_{L^1}}{1-(b_2k+b_1k)}$.
Let $(u,v)\in Q_p$, then

$$A(\eta,\omega)=\Big(f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)),f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))\Big).$$

But

$$\begin{aligned}&|f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))|\\&\le|a_1(t)|+b_1\int_0^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\omega(s)ds\\&+b_1\int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\eta(s)ds,\end{aligned}$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$\begin{aligned}&\int_0^T|f_1(s,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))ds|\le\int_0^T|a_1(s)|ds\\&+b_1\int_0^T\int_0^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\omega(s)dsdt\\&+b_1\int_0^T\int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\eta(s)dsdt\\&\le\|a_1\|_{L^1}+b_1\int_0^T\int_s^T\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\omega(s)dtds\\&+b_1\int_0^T\int_s^T\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\eta(s)dtds\\&\le\|a_1\|_{L^1}+b_1\frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)}\|\omega\|_{L^1}+b_1\frac{(T)^{1-\beta}}{\Gamma(2-\beta)}\|\eta\|_{L^1}\end{aligned}$$

Hence

$$\begin{aligned}&||f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))||_{L^1}\\&\le\|a_1\|_{L^1}+b_1\frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)}\|\omega\|_{L^1}+b_1\frac{(T)^{1-\beta}}{\Gamma(2-\beta)}\|\eta\|_{L^1}.\end{aligned}\tag{3.10}$$

And

$$|f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))| \leq |a_2(t)|+b_2\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\omega(s)ds + b_2\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\eta(s)ds,$$

Integrating the above inequality from 0 to T and making the change of variable we have

$$\begin{aligned}&\int_0^T |f_1(s,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))ds| \leq \int_0^T |a_2(s)|ds \\ &+b_2\int_0^T\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\omega(s)dsd \\ &+b_2\int_0^T\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\eta(s)dsdt \\ &\leq \|a_2\|_{L^1}+b_2\int_0^T\int_s^T \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\omega(s)dtds \\ &+b_2\int_0^T\int_s^T \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\eta(s)dtds \\ &\leq \|a_2\|_{L^1}+b_2\frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)}\|\omega\|_{L^1}+b_2\frac{(T)^{1-\beta}}{\Gamma(2-\beta)}\|\eta\|_{L^1}.\end{aligned}$$

Hence

$$||f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))||_{L^1} \leq \|a_2\|_{L^1}+b_2\frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)}\|\omega\|_{L^1}+b_2\frac{(T)^{1-\beta}}{\Gamma(2-\beta)}\|\eta\|_{L^1}. \tag{3.11}$$

From (3.10) and (3.11), we get

$$\begin{aligned}&||A(\eta,\omega)||_Y \\ &\leq \|a_1\|_{L^1}+b_1\frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)}\|\omega\|_{L^1}+b_1\frac{(T)^{1-\beta}}{\Gamma(2-\beta)}\|\eta\|_{L^1} \\ &+\|a_2\|_{L^1}+b_2\frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)}\|\omega\|_{L^1}+b_2\frac{(T)^{1-\beta}}{\Gamma(2-\beta)}\|\eta\|_{L^1} \\ &\leq \|a_1\|_{L^1}+\|a_2\|_{L^1}+(b_1k+b_2k)(p_1+p_2)=p,\end{aligned}$$

where, $k=\max\{\frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)},\frac{(T)^{1-\beta}}{\Gamma(2-\beta)}\}$.
This prove That $A:Q_p\to Q_p$, and the class of function $\{A_n(\eta,\omega)\}$ is uniformly bounded on Q_p.
Let Ω be bounded subset of Q_p and $A:Q_p\to Q_p$. Then $A(\Omega)$ is also bounded on Q_p.

Let $(\eta,\omega)\in\Omega$, then

$$\begin{aligned}&(A(\eta,\omega))_h-A(\eta,\omega)= \\ &\Big(\frac{1}{h}\int_t^{t+h} f_1(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))d\theta, \\ &\frac{1}{h}\int_t^{t+h} f_2(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))d\theta\Big) \\ &-\Big(f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)),f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))\Big) \\ &=\Big(\frac{1}{h}\int_t^{t+h} f_1(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))d\theta \\ &-f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)), \\ &\frac{1}{h}\int_t^{t+h} f_2(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))d\theta \\ &-f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))\Big).\end{aligned}$$

But

$$\begin{aligned}&\int_0^T |\frac{1}{h}\int_t^{t+h}(f_1(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)) \\ &-f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)))d\theta|dt \\ &\leq \int_0^T\frac{1}{h}\int_t^{t+h}|f_1(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)) \\ &-f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))|d\theta dt.\end{aligned}$$

Since $f_1\in L^1[0,T]$, It follows that

$$\begin{aligned}&\frac{1}{h}\int_t^{t+h}|f_1(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)) \\ &-f_1(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))|d\theta\to 0 \quad as \qquad h\to 0,\end{aligned}$$

And

$$\begin{aligned}&\int_0^T |\frac{1}{h}\int_t^{t+h}(f_2(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)) \\ &-f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)))d\theta|dt \\ &\leq \int_0^T\frac{1}{h}\int_t^{t+h}|f_2(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)) \\ &-f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))|d\theta dt.\end{aligned}$$

Since $f_2\in L^1[0,T]$, It follows that

$$\begin{aligned}&\frac{1}{h}\int_t^{t+h}|f_2(\theta,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t)) \\ &-f_2(t,I^{1-\alpha}\omega(t),I^{1-\beta}\eta(t))|d\theta\to 0 \quad as \qquad h\to 0,\end{aligned}$$

then $(A(u,v))_h\to(A(\eta,\omega))$ uniformly. Hence $A(\Omega)$ is relatively compact.
Hence A is compact operator.

Let $\{(\eta_n,\omega_n)\}\subset Q_p$ and $(\eta_n,\omega_n)\to(\eta,\omega)$

$$\lim_{n\to\infty} A(\eta_n,\omega_n) = \lim_{n\to\infty}\Big(f_1(t,I^{1-\alpha}\omega_n(t),I^{1-\beta}\eta_n(t)), f_2(t,I^{1-\alpha}\omega_n(t),I^{1-\beta}\eta_n(t))\Big)=A(\eta,\omega).$$

Then $\omega_n\to\omega,\eta_n\to\eta\Rightarrow A(\eta_n,\omega_n)\to A(\eta,\omega)$ as $n\to\infty$. This mean that the operator A is continuous operator. Then by Schauder fixed point Theorem 2.6 there exist at least one solution $\eta,\omega\in L^1[0,T]$ of the coupled system of functional integral equations (3.3). The proof is completed. □

3.3 The coupled system of functional integro-differential equations

Theorem 3.5. *Let the assumptions of Theorem 3.2 be satisfied, then there exist at last one solution (x,y), $x,y\in C^1[0,T]$ of the initial value problem of the coupled system of functional integro-differential equation (1.1) with the nonlocal condition (1.2).*

Proof. The solution of the coupled system of functional integro-differential equation (1.1) is given by

$$\begin{cases} x(t)=x(0)+\int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}u(s)ds, & t\in(0,T],\\ y(t)=y(0)+\int_0^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}v(s)ds, & t\in(0,T],\end{cases}$$

Using the nonlocal condition(1.2), we get

$$\begin{cases} x(\tau_k)=x(0)+\int_0^{\tau_k}\frac{(\tau_k-s)^{-\beta}}{\Gamma(1-\beta)}u(s)ds,\\ y(\eta_j)=y(0)+\int_0^{\eta_j}\frac{(\eta_j-s)^{-\alpha}}{\Gamma(1-\alpha)}v(s)ds,\end{cases}$$

and

$$\begin{cases} \sum_{k=1}^n a_kx(\tau_k)=x(0)\sum_{k=1}^n a_k+\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{-\beta}}{\Gamma(1-\beta)}u(s)ds,\\ \sum_{k=1}^n b_jy(\eta_j)=y(0)\sum_{k=1}^n b_j+\sum_{k=1}^n b_j\int_0^{\eta_j}\frac{(\eta_j-s)^{-\alpha}}{\Gamma(1-\alpha)}v(s)ds,\end{cases}$$

since,

$$\sum_{k=1}^n a_kx(\tau_k)=x_0-x(0),\qquad \sum_{j=1}^n b_jy(\eta_j)=y_0-y(0),$$

then

$$\begin{cases} x_0-x(0)=x(0)\sum_{k=1}^n a_k+\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{-\beta}}{\Gamma(1-\beta)}u(s)ds,\\ y_0-y(0)=y(0)\sum_{j=1}^n b_j+\sum_{j=1}^n b_j\int_0^{\eta_j}\frac{(\eta_j-s)^{-\alpha}}{\Gamma(1-\alpha)}v(s)ds,\end{cases}$$

and

$$\begin{cases} (1+\sum_{k=1}^n a_k)x(0)=x_0-\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{-\beta}}{\Gamma(1-\beta)}u(s)ds,\\ (1+\sum_{j=1}^n b_j)y(0)=y_0-\sum_{j=1}^n b_j\int_0^{\eta_j}\frac{(\eta_j-s)^{-\alpha}}{\Gamma(1-\alpha)}v(s)ds.\end{cases}$$

Hence

$$x(t)=\frac{1}{1+\sum_{k=1}^n a_k}x_0+I^{1-\beta}u(t) -\frac{1}{1+\sum_{k=1}^n a_k}\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{-\beta}}{\Gamma(1-\beta)}u(s)ds,$$

$$y(t)=\frac{1}{1+\sum_{j=1}^n b_j}y_0+I^{1-\alpha}v(t) -\frac{1}{1+\sum_{j=1}^n b_j}\sum_{j=1}^n b_j\int_0^{\eta_j}\frac{(\eta_j-s)^{-\alpha}}{\Gamma(1-\alpha)}v(s)ds.$$

Where $u(t),v(t)$ is defined by the coupled system of functional integral equation (3.1). Then from Theorem 3.2, we can deduce the existence of at least one solution of (x,y), $x,y\in C^1[0,T]$ the problem (1.1) and (1.2). The proof is completed. □

Theorem 3.6. *Let the assumptions of Theorem 3.4 be satisfied, then there exist at last one solution (x,y), $x,y\in AC[0,T]$ of the initial value problem of the coupled system of functional integro-differential equation (1.1) with the nonlocal condition (1.2).*

Proof. The solution of the coupled system of functional integro-differential equation (1.1) is given by

$$\begin{cases} x(t)=x(0)+\int_0^t\eta(s)ds, & t\in(0,T],\\ y(t)=y(0)+\int_0^t\omega(s)ds, & t\in(0,T],\end{cases}$$

Using the nonlocal condition(1.2), we get

$$\begin{cases} x(\tau_k)=x(0)+\int_0^{\tau_k}\eta(s)ds,\\ y(\eta_j)=y(0)+\int_0^{\eta_j}\omega(s)ds,\end{cases}$$

and

$$\begin{cases} \sum_{k=1}^n a_kx(\tau_k)=x(0)\sum_{k=1}^n a_k+\sum_{k=1}^n a_k\int_0^{\tau_k}\eta(s)ds,\\ \sum_{k=1}^n b_jy(\eta_j)=y(0)\sum_{k=1}^n b_j+\sum_{k=1}^n b_j\int_0^{\eta_j}\omega(s)ds,\end{cases}$$

since,

$$\sum_{k=1}^n a_kx(\tau_k)=x_0-x(0),\qquad \sum_{j=1}^n b_jy(\eta_j)=y_0-y(0),$$

then

$$\begin{cases} x_0-x(0)=x(0)\sum_{k=1}^n a_k+\sum_{k=1}^n a_k\int_0^{\tau_k}\eta(s)ds,\\ y_0-y(0)=y(0)\sum_{j=1}^n b_j+\sum_{j=1}^n b_j\int_0^{\eta_j}\omega(s)ds,\end{cases}$$

and

$$\begin{cases} (1+\sum_{k=1}^n a_k)x(0)=x_0-\sum_{k=1}^n a_k\int_0^{\tau_k}\eta(s)ds,\\ (1+\sum_{j=1}^n b_j)y(0)=y_0-\sum_{j=1}^n b_j\int_0^{\eta_j}\omega(s)ds.\end{cases}$$

Hence

$$x(t) = \frac{1}{1+\sum_{k=1}^{n} a_k} x_0 + \int_0^t \eta(s)ds - \frac{1}{1+\sum_{k=1}^{n} a_k} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \eta(s)ds,$$

$$y(t) = \frac{1}{1+\sum_{j=1}^{n} b_j} y_0 + \int_0^t \omega(s)ds - \frac{1}{1+\sum_{j=1}^{n} b_j} \sum_{j=1}^{n} b_j \int_0^{\eta_j} \omega(s)ds,$$

where $\eta(t), \omega(t)$ is defined by the coupled system of functional integral equation (3.3). Then from Theorem 3.4, we can deduce the existence of at least one solution of (x,y), $x, y \in AC[0,T]$ the problem (1.1) and (1.2). The proof is completed. □

4. Conclusion

In this paper, we proved the existence of at least one continuous solution of the coupled system of functional integro-differential equations with the nonlocal condition , under certain conditions.

Also, under other assumptions, we proved the existence of at least one absolutely continuous solution the coupled system of functional integro-differential equations with the nonlocal condition.

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