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The relationship between the solutions according to the noniterative method and the generalized differentiability of the fuzzy boundary value problem

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Abstract
In this paper is studied the fuzzy boundary value problem according to the noniterative method and according to the generalized differentiability. Comparison result and relationship between them are given. It is seen on the example.

Keywords
Two-point fuzzy boundary value problems, Fuzzy differential equation.

AMS Subject Classification
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Contents

1. Introduction

Fuzzy differential equations is investigated by many researchers [1–3, 5–7, 14, 15]. Because fuzzy differential equations are a suitable tool for modeling science and engineering problems in which uncertainties. Fuzzy differential equations can be studied by different approach. The first approach is Hukuhara differentiability. The second approach is generalized differentiability [17]. Two-point boundary value problems have four different solution using the generalized differentiabilty. The third approach is to generate the fuzzy solution from the crips solution [7, 8, 11, 13].

In this paper is studied the fuzzy boundary value problem according to the noniterative method and according to the generalized differentiability. Comparison result and relationship between them are given. It is seen on the example.

2. Preliminaries

Definition 2.1. *[18] A fuzzy number is a function $u:\mathbb{R}\rightarrow [0,1]$ satisfying the properties normal, convex fuzzy set, upper semi-continuous on $\mathbb{R}$ and $cl\{x\varepsilon\mathbb{R} \mid u(x)>0\}$ is compact,where cl denotes the closure of a subset.*

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2. *[17] Let $u\varepsilon\mathbb{R}_F$. The α-level set of u, denoted , $[u]^{\alpha}$, $0<\alpha\leq 1$, is*

$$[u]^{\alpha}=\{x\varepsilon\mathbb{R} \mid u(x)\geq 0\}.$$

The notation, denotes explicitly the α-level set of u. The notation, $[u]^{\alpha}=[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ denotes explicitly the α-level set of u.We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

Remark 2.3. *[10, 17] The sufficient and necessary conditions for $[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ to define the parametric form of a fuzzy number as follows:*

$\underline{u}_{\alpha}$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,
$\overline{u}_{\alpha}$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,
$\underline{u}_{\alpha}\leq\overline{u}_{\alpha}$, $0\leq\alpha\leq 1$.

Definition 2.4. *[17] For $u,v\varepsilon\mathbb{R}_F$ and $\lambda\in\mathbb{R}$, the sum $u+v$ and the product λu are defined by $[u+v]^{\alpha}=[u]^{\alpha}+[v]^{\alpha}$, where means the usual addition of two intervals (subsets) of*

$\mathbb{R}$ and $\lambda [u]^{\alpha}$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}_{\alpha} - \underline{v}_{\alpha}|, |\overline{u}_{\alpha} - \overline{v}_{\alpha}|\}.$$

Definition 2.5. *[18] If A is a symmetric triangular numbers with supports $[\underline{a},\overline{a}]$, the $\alpha-$level sets of $[A]^{\alpha}$ is*

$$[A]^{\alpha} = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right].$$

Definition 2.6. *[12, 17, 19] Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of fuzzy numbers u and v, and it is denoted by $w = u \ominus v$.*

Definition 2.7. *[4, 12, 17] Let $f : [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$. We say that f is Hukuhara differentiable at t_0, if there exists an element $f^{'}(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0+h) \ominus f(t_0),\ f(t_0) \ominus f(t_0-h)$ and the limits*

$$\lim_{h\to 0} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to 0} \frac{f(t_0)\ominus f(t_0-h)}{h} = f^{'}(t_0).$$

Definition 2.8. *[17] Let $f : [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$. We say that f is (1)-differentiable at t_0, if there exists an element $f^{'}(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small near to 0, exist $f(t_0+h)\ominus f(t_0),\ f(t_0)\ominus f(t_0-h)$ and the limits*

$$\lim_{h\to 0} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to 0} \frac{f(t_0)\ominus f(t_0-h)}{h} = f^{'}(t_0),$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small near to 0, exist $f(t_0)\ominus f(t_0+h),\ f(t_0-h)\ominus f(t_0)$ and the limits

$$\lim_{h\to 0} \frac{f(t_0)\ominus f(t_0+h)}{-h} = \lim_{h\to 0} \frac{f(t_0-h)\ominus f(t_0)}{-h} = f^{'}(t_0).$$

Theorem 2.9. *[16] Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^{\alpha} = \left[\underline{f}_{\alpha}(t), \overline{f}_{\alpha}(t)\right]$, for each $\alpha \in [0,1]$.*

(i) If f is (1)-differentiable then $\underline{f}_{\alpha}$ and $\overline{f}_{\alpha}$ are differentiable functions and $\left[f^{'}(t)\right]^{\alpha} = \left[\underline{f}^{'}_{\alpha}(t), \overline{f}^{'}_{\alpha}(t)\right]$,

(ii) If f is (2)-differentiable then $\underline{f}_{\alpha}$ and $\overline{f}_{\alpha}$ are differentiable functions and $\left[f^{'}(t)\right]^{\alpha} = \left[\overline{f}^{'}_{\alpha}(t), \underline{f}^{'}_{\alpha}(t)\right]$.

Theorem 2.10. *[16] Let $f^{'} : [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^{\alpha} = \left[\underline{f}_{\alpha}(t), \overline{f}_{\alpha}(t)\right]$, for each $\alpha \in [0,1]$, f is (1)-differentiable or (2)-differentiable.*

(i) If f and $f^{'}$ are (1)-differentiable then $\underline{f}^{'}_{\alpha}$ and $\overline{f}^{'}_{\alpha}$ are differentiable functions and $\left[f^{''}(t)\right]^{\alpha} = \left[\underline{f}^{''}_{\alpha}(t), \overline{f}^{''}_{\alpha}(t)\right]$,

(ii) If f is (1)-differentiable and $f^{'}$ is (2)-differentiable then $\underline{f}^{'}_{\alpha}$ and $\overline{f}^{'}_{\alpha}$ are differentiable functions and $\left[f^{''}(t)\right]^{\alpha} = \left[\overline{f}^{''}_{\alpha}(t), \underline{f}^{''}_{\alpha}(t)\right]$,

(iii) If f is (2)-differentiable and $f^{'}$ is (1)-differentiable then $\underline{f}^{'}_{\alpha}$ and $\overline{f}^{'}_{\alpha}$ are differentiable functions and $\left[f^{''}(t)\right]^{\alpha} = \left[\overline{f}^{''}_{\alpha}(t), \underline{f}^{''}_{\alpha}(t)\right]$,

(iv) If f and $f^{'}$ are (2)-differentiable then $\underline{f}^{'}_{\alpha}$ and $\overline{f}^{'}_{\alpha}$ are differentiable functions and $\left[f^{''}(t)\right]^{\alpha} = \left[\underline{f}^{''}_{\alpha}(t), \overline{f}^{''}_{\alpha}(t)\right]$.

3. Findings and Main Results

Consider the solutions of the following problem

$$L := \tfrac{d^2}{dt^2},$$

$$Lu + u = [g(t)]^{\alpha}, \tag{3.1}$$

$$u(0) = [\varphi]^{\alpha},\ u(1) = [\psi]^{\alpha}, \tag{3.2}$$

where

$$[g(t)]^{\alpha} = \left[\underline{g}_{\alpha}(t), \overline{g}_{\alpha}(t)\right] = [g(t) - 1 + \alpha, g(t) + 1 - \alpha]$$

is fuzzy function,

$$[\varphi]^{\alpha} = \left[\underline{\varphi}_{\alpha}, \overline{\varphi}_{\alpha}\right] \text{ and } [\psi]^{\alpha} = \left[\underline{\psi}_{\alpha}, \overline{\psi}_{\alpha}\right]$$

are symmetric triangular fuzzy numbers.

The Solution According to the Noniterative Method

[9] Let look for the solution of the problem (3.1)-(3.2) as

$$u(t) = u_1(t) + \lambda u_2(t), \tag{3.3}$$

where λ is a constant to be determined.

Substituting the equation (3.3) in (3.1), we have

$$u_1^{''}(t) + u_1(t) = [g(t)]^{\alpha},$$

$$u_2^{''}(t) + u_2(t) = 0.$$

From the first boundary condition, we have

$$u_1(0)+\lambda u_2(0)=[\varphi]^\alpha,$$

$$u_1(0)=[\varphi]^\alpha, u_2(0)=0.$$

If

$$u_1'(0)=0, u_2'(0)=1,$$

the unknown constant λ is identified. Then, from the second boundary condition,

$$u_1(1)+\lambda u_2(1)=[\psi]^\alpha$$

$$\lambda=\frac{[\psi]^\alpha-u_1(1)}{u_2(1)}$$

is obtained. Then, fuzzy initial value problems

$$\begin{cases} u_1''(t)+u_1(t)=[g(t)]^\alpha \\ u_1(0)=[\varphi]^\alpha \\ u_1'(0)=0 \end{cases} \tag{3.4}$$

and

$$\begin{cases} u_2''(t)+u_2(t)=0 \\ u_2(0)=0 \\ u_2'(0)=1 \end{cases} \tag{3.5}$$

is solved. The solution of the problem (3.4) is

$$\underline{u}_{1\alpha}(t)=c_1(\alpha)\cos t+c_2(\alpha)\sin t+\underline{u}_{p\alpha}(t),$$

$$\overline{u}_{1\alpha}(t)=c_3(\alpha)\cos t+c_4(\alpha)\sin t+\overline{u}_{p\alpha}(t),$$

$$[u_1(t)]^\alpha=[\underline{u}_{1\alpha}(t),\overline{u}_{1\alpha}(t)],$$

where

$$[u_p(t)]^\alpha=\left[\underline{u}_{p\alpha}(t),\overline{u}_{p\alpha}(t)\right]$$

is the solution nonhomogeneous fuzzy differential equation in (3.4). Using the initial conditions, the coefficients $c_1(\alpha)$, $c_2(\alpha)$, $c_3(\alpha)$, $c_4(\alpha)$ are obtained as

$$c_1(\alpha)=\underline{\varphi}_\alpha-\underline{u}_{p\alpha}(0),\ c_2(\alpha)=-\underline{u}_{p\alpha}'(0),$$

$$c_3(\alpha)=\overline{\varphi}_\alpha-\overline{u}_{p\alpha}(0),\ c_4(\alpha)=-\overline{u}_{p\alpha}'(0).$$

The solution of the problem (3.5) is

$$\underline{u}_{2\alpha}(t)=a_1(\alpha)\cos t+a_2(\alpha)\sin t,$$

$$\overline{u}_{2\alpha}(t)=a_3(\alpha)\cos t+a_4(\alpha)\sin t,$$

$$[u_2(t)]^\alpha=[\underline{u}_{2\alpha}(t),\overline{u}_{2\alpha}(t)].$$

Using the initial conditions, the coefficients $a_1(\alpha)$, $a_2(\alpha)$, $a_3(\alpha)$, $a_4(\alpha)$ are obtained as

$$a_1(\alpha)=a_3(\alpha)=0,$$

$$a_2(\alpha)=a_4(\alpha)=1.$$

Then,

$$[u_2(t)]^\alpha=[\sin t,\sin t].$$

Since

$$\lambda=\frac{[\psi]^\alpha-u_1(1)}{u_2(1)},$$

$$\underline{\lambda}_\alpha=\frac{\underline{\psi}_\alpha-\left[\left(\underline{\varphi}_\alpha-\underline{u}_{p\alpha}(0)\right)\cos(1)+\underline{u}_{p\alpha}(1)\right]}{\sin(1)}-\underline{u}_{p\alpha}'(0)$$

$$\overline{\lambda}_\alpha=\frac{\overline{\psi}_\alpha-[(\overline{\varphi}_\alpha-\overline{u}_{p\alpha}(0))\cos(1)+\overline{u}_{p\alpha}(1)]}{\sin(1)}-\overline{u}_{p\alpha}'(0)$$

$$[\lambda]^\alpha=\left[\underline{\lambda}_\alpha,\overline{\lambda}_\alpha\right]$$

is found. Since the solution of the problem (3.1)-(3.2)

$$u(t)=u_1(t)+\lambda u_2(t),$$

the lower and the upper solutions the fuzzy problem (3.1)-(3.2) are

$$\begin{aligned}\underline{u}_\alpha(t) &= \left(\underline{\varphi}_\alpha - \underline{u}_{p\alpha}(0)\right)\cos(t) + \left(-\underline{u}'_{p\alpha}(0)\right)\sin(t)\\ &+\underline{u}_{p\alpha}(t) + \underline{\lambda}_\alpha \sin(t),\end{aligned}$$

$$\begin{aligned}\overline{u}_\alpha(t) &= \left(\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0)\right)\cos(t) + \left(-\overline{u}'_{p\alpha}(0)\right)\sin(t)\\ &+\overline{u}_{p\alpha}(t) + \overline{\lambda}_\alpha \sin(t).\end{aligned}$$

Substituting $\underline{\lambda}_\alpha$ and $\overline{\lambda}_\alpha$ are in the lower and the upper solutions, the solution problem (3.1)-(3.2) is obtained as

$$\begin{aligned}\underline{u}_\alpha(t) &= \left(\underline{\varphi}_\alpha - \underline{u}_{p\alpha}(0)\right)\cos(t)\\ &+\frac{\overline{\psi}_\alpha - (\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0))\cos(1) - \overline{u}_{p\alpha}(1)}{\sin(1)}\sin(t)\\ &+\underline{u}_{p\alpha}(t),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\overline{u}_\alpha(t) &= \left(\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0)\right)\cos(t)\\ &+\frac{\overline{\psi}_\alpha - (\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0))\cos(1) - \overline{u}_{p\alpha}(1)}{\sin(1)}\sin(t)\\ &+\overline{u}_{p\alpha}(t),\end{aligned} \tag{3.7}$$

$$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)] \tag{3.8}$$

The Solution According to the Generalized Differentiability

Let find the solution of the problem (3.1)-(3.2) according to the generalized differentiability, where (i,j) solution means that y is (i)-differentiable, y$'$ is (j) differentiable. (i,j=1,2)

For the (1,1) and (2,2) solutions
Using the generalized differentiability for (1,1) and (2,2) solutions, we have

$$\begin{cases} \underline{u}''_\alpha(t) + \underline{u}_\alpha(t) = \underline{g}_\alpha(t) \\ \underline{u}_\alpha(0) = \underline{\varphi}_\alpha, \underline{u}_\alpha(1) = \underline{\psi}_\alpha \end{cases}, \tag{3.9}$$

$$\begin{cases} \overline{u}''_\alpha(t) + \overline{u}_\alpha(t) = \overline{g}_\alpha(t) \\ \overline{u}_\alpha(0) = \overline{\varphi}_\alpha, \overline{u}_\alpha(1) = \overline{\psi}_\alpha \end{cases}. \tag{3.10}$$

From (3.9) and (3.10), the lower and the upper solution is found as

$$\underline{u}_\alpha(t) = b_1(\alpha)\cos(t) + b_2(\alpha)\sin(t) + \underline{u}_{p\alpha}(t),$$

$$\overline{u}_\alpha(t) = b_3(\alpha)\cos(t) + b_4(\alpha)\sin(t) + \overline{u}_{p\alpha}(t),$$

$$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)],$$

where $\underline{u}_{p\alpha}(t)$ and $\overline{u}_{p\alpha}(t)$ are the nonhomogeneos solutions of the differential equations in (3.9) and (3.10). Using the boundary conditions, the coefficients are obtained as

$$b_1(\alpha) = \underline{\varphi}_\alpha - \underline{u}_{p\alpha}(0),$$

$$b_2(\alpha) = \frac{\underline{\psi}_\alpha - \left(\underline{\varphi}_\alpha - \underline{u}_{p\alpha}(0)\right)\cos(1) - \underline{u}_{p\alpha}(1)}{\sin(1)},$$

$$b_3(\alpha) = \overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0),$$

$$b_4(\alpha) = \frac{\overline{\psi}_\alpha - (\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0))\cos(1) - \overline{u}_{p\alpha}(1)}{\sin(1)}.$$

Then for (1,1) and (2,2) solutions, the solution of the problem (3.1)-(3.2) is

$$\begin{aligned}\underline{u}_\alpha(t) &= \left(\underline{\varphi}_\alpha - \underline{u}_{p\alpha}(0)\right)\cos(t)\\ &+\frac{\overline{\psi}_\alpha - (\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0))\cos(1) - \overline{u}_{p\alpha}(1)}{\sin(1)}\sin(t)\\ &+\underline{u}_{p\alpha}(t),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\overline{u}_\alpha(t) &= \left(\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0)\right)\cos(t)\\ &+\frac{\overline{\psi}_\alpha - (\overline{\varphi}_\alpha - \overline{u}_{p\alpha}(0))\cos(1) - \overline{u}_{p\alpha}(1)}{\sin(1)}\sin(t)\\ &+\overline{u}_{p\alpha}(t),\end{aligned} \tag{3.12}$$

$$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)]. \tag{3.13}$$

For the (1,2) and (2,1) solutions
Using the generalized differentiability for (1,2) and (2,1) solutions, we have

$$\overline{u}''_\alpha(t) + \underline{u}_\alpha(t) = \underline{g}_\alpha(t),$$

$$\underline{u}''_\alpha(t) + \overline{u}_\alpha(t) = \overline{g}_\alpha(t).$$

From here,

$$\underline{u}_{\alpha}^{(4)}(t)-\underline{u}_{\alpha}(t)=\overline{g}_{\alpha}''(t)-\underline{g}_{\alpha}(t),$$

$$\overline{u}_{\alpha}^{(4)}(t)-\overline{u}_{\alpha}(t)=\underline{g}_{\alpha}''(t)-\overline{g}_{\alpha}(t)$$

are obtained. That is, we have

$$\begin{cases} \underline{u}_{\alpha}^{(4)}(t)-\underline{u}_{\alpha}(t)=g''(t)-g(t)+1-\alpha \\ \underline{u}_{\alpha}(0)=\underline{\varphi}_{\alpha},\underline{u}_{\alpha}(1)=\underline{\psi}_{\alpha} \end{cases}, \tag{3.14}$$

$$\begin{cases} \overline{u}_{\alpha}^{(4)}(t)-\overline{u}_{\alpha}(t)=g''(t)-g(t)-1+\alpha \\ \overline{u}_{\alpha}(0)=\overline{\varphi}_{\alpha},\overline{u}_{\alpha}(1)=\overline{\psi}_{\alpha} \end{cases}. \tag{3.15}$$

From (3.14) and (3.15), the lower and the upper solution is found as

$$\begin{aligned}\underline{u}_{\alpha}(t) &= c_1(\alpha)e^{t}+c_2(\alpha)e^{-t}-c_3(\alpha)\sin(t) \\ &\quad -c_4(\alpha)\cos(t)+\underline{u}_{p\alpha}(t),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\overline{u}_{\alpha}(t) &= c_1(\alpha)e^{t}+c_2(\alpha)e^{-t}+c_3(\alpha)\sin(t) \\ &\quad +c_4(\alpha)\cos(t)+\overline{u}_{p\alpha}(t),\end{aligned} \tag{3.17}$$

$$[u(t)]^{\alpha}=[\underline{u}_{\alpha}(t),\overline{u}_{\alpha}(t)], \tag{3.18}$$

where $\underline{u}_{p\alpha}(t)$ and $\overline{u}_{p\alpha}(t)$ are the nonhomogeneos solutions of the differential equations in (3.14) and (3.15). Using the boundary conditions, the coefficients are obtained as

$$c_1(\alpha)=\frac{-e^{-1}\left(\underline{\varphi}_{\alpha}+\overline{\varphi}_{\alpha}-\underline{u}_{p\alpha}(0)-\overline{u}_{p\alpha}(0)\right)}{2\left(e-e^{-1}\right)},$$

$$c_2(\alpha)=\frac{-e\left(\underline{\varphi}_{\alpha}+\overline{\varphi}_{\alpha}-\underline{u}_{p\alpha}(0)-\overline{u}_{p\alpha}(0)\right)}{2\left(e^{-1}-e\right)},$$

$$c_3(\alpha)=\frac{\overline{\varphi}_{\alpha}-\underline{\varphi}_{\alpha}+\underline{u}_{p\alpha}(0)-\overline{u}_{p\alpha}(0)}{2},$$

$$\begin{aligned}c_4(\alpha) &= \frac{\left(\overline{\psi}_{\alpha}-\underline{\psi}_{\alpha}+\underline{u}_{p\alpha}(1)-\overline{u}_{p\alpha}(1)\right)}{2\sin(1)} \\ &\quad -\frac{\left(\overline{\varphi}_{\alpha}-\underline{\varphi}_{\alpha}+\underline{u}_{p\alpha}(0)-\overline{u}_{p\alpha}(0)\right)\cos(1)}{2\sin(1)}.\end{aligned}$$

Conclusion 3.1. *The solution (3.6)-(3.8) according to the noniterative method is the same as the solution (3.11)-(3.13) according to the generalized differentiability for the solutions (1,1) and (2,2).*

Example 3.2. *Consider the following problem*

$$u''(t)+u(t)=\left[t^2\right]^{\alpha}, \tag{3.19}$$

$$u(0)=[-1+\alpha,1-\alpha],\ u(1)=[\alpha,2-\alpha]. \tag{3.20}$$

Firstly, let find the solution according to the noniterative method. Then, since

$$u(t)=u_1(t)+\lambda u_2(t),$$

we have

$$\begin{cases} u_1''(t)+u_1(t)=\left[t^2\right]^{\alpha} \\ u_1(0)=[-1+\alpha,1-\alpha],\ u_1'(0)=0 \end{cases}, \tag{3.21}$$

$$\begin{cases} u_2''(t)+u_2(t)=0 \\ u_2(0)=0,u_2'(0)=1 \end{cases}, \tag{3.22}$$

$$\lambda=\frac{[\alpha,2-\alpha]-u_1(1)}{u_2(1)}. \tag{3.23}$$

The solution of the equation in (3.21) is

$$\underline{u}_{1\alpha}(t)=c_1(\alpha)\cos t+c_2(\alpha)\sin t+t^2-3+\alpha,$$

$$\overline{u}_{1\alpha}(t)=c_3(\alpha)\cos t+c_4(\alpha)\sin t+t^2-1-\alpha,$$

$$[u_1(t)]^{\alpha}=[\underline{u}_{1\alpha}(t),\overline{u}_{1\alpha}(t)].$$

Using the initial conditions, the coefficients are found as

$$c_1(\alpha)=c_3(\alpha)=2,$$

$$c_2(\alpha)=c_4(\alpha)=0.$$

The solution of the equation in (3.22) is

$$\underline{u}_{2\alpha}(t)=a_1(\alpha)\cos t+a_2(\alpha)\sin t,$$

$$\overline{u}_{2\alpha}(t)=a_3(\alpha)\cos t+a_4(\alpha)\sin t,$$

$$[u_2(t)]^\alpha = [\underline{u}_{2\alpha}(t), \overline{u}_{2\alpha}(t)].$$

Using the initial conditions, the coefficients are found as

$$a_1(\alpha) = a_3(\alpha) = 0,$$

$$a_2(\alpha) = a_4(\alpha) = 1.$$

Then, from (3.23)

$$\lambda = \frac{[\alpha, 2-\alpha] - [2\cos(1) - 2 + \alpha, 2\cos(1) - \alpha]}{[\sin(1), \sin(1)]},$$

$$\underline{\lambda} = \overline{\lambda} = \frac{-2\cos(1) + 2}{\sin(1)} = \frac{2(1-\cos(1))}{\sin(1)}.$$

Consequently, the solution of the problem (3.19)-(3.20) is

$$\underline{u}_\alpha(t) = 2\cos t + \frac{2(1-\cos(1))}{\sin(1)}\sin t + t^2 - 3 + \alpha, \quad (3.24)$$

$$\overline{u}_\alpha(t) = 2\cos t + \frac{2(1-\cos(1))}{\sin(1)}\sin t + t^2 - 1 - \alpha, \quad (3.25)$$

$$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)]. \quad (3.26)$$

Secondly, let find the solution according to the generalized differentiability. For the solutions (1,1) and (2,2),

$$\begin{cases} \underline{u}''_\alpha(t) + \underline{u}_\alpha(t) = t^2 - 1 + \alpha, \\ \underline{u}_\alpha(0) = -1 + \alpha, \underline{u}_\alpha(1) = \alpha \end{cases}$$

$$\begin{cases} \overline{u}''_\alpha(t) + \overline{u}_\alpha(t) = t^2 + 1 - \alpha, \\ \overline{u}_\alpha(0) = 1 - \alpha, \overline{u}_\alpha(1) = 2 - \alpha \end{cases}$$

are solved. Then for the solutions (1,1) and (2,2), the solution of the problem (3.19)-(3.20) is

$$\underline{u}_\alpha(t) = 2\cos t + \frac{2(1-\cos(1))}{\sin(1)}\sin t + t^2 - 3 + \alpha \quad (3.27)$$

$$\overline{u}_\alpha(t) = 2\cos t + \frac{2(1-\cos(1))}{\sin(1)}\sin t + t^2 - 1 - \alpha \quad (3.28)$$

$$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)]. \quad (3.29)$$

For the solutions (1,2) and (2,1),

$$\begin{cases} \overline{u}''_\alpha(t) + \underline{u}_\alpha(t) = t^2 - 1 + \alpha, \\ \underline{u}''_\alpha(t) + \overline{u}_\alpha(t) = t^2 + 1 + \alpha, \\ \underline{u}_\alpha(0) = -1 + \alpha, \underline{u}_\alpha(1) = \alpha, \\ \overline{u}_\alpha(0) = 1 - \alpha, \overline{u}_\alpha(\ell) = 2 - \alpha \end{cases}$$

that is

$$\begin{cases} \underline{u}^{(4)}_\alpha(t) - \underline{u}_\alpha(t) = 3 - t^2 - \alpha, \\ \underline{u}_\alpha(0) = -1 + \alpha, \underline{u}_\alpha(1) = \alpha, \end{cases}$$

$$\begin{cases} \overline{u}^{(4)}_\alpha(t) + \overline{u}_\alpha(t) = 1 - t^2 + \alpha, \\ \overline{u}_\alpha(0) = 1 - \alpha, \overline{u}_\alpha(1) = 2 - \alpha \end{cases}$$

are solved. Then for the solutions (1,2) and (2,1), the solution of the problem (3.19)-(3.20) is

$$\underline{u}_\alpha(t) = \frac{2(e^{-1}-1)}{e^{-1}-e}e^t + \frac{2(1-e)}{e^{-1}-e}e^{-t} + t^2 - 3 + \alpha \quad (3.30)$$

$$\overline{u}_\alpha(t) = \frac{2(e^{-1}-1)}{e^{-1}-e}e^t + \frac{2(1-e)}{e^{-1}-e}e^{-t} + t^2 - 1 - \alpha \quad (3.31)$$

$$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)]. \quad (3.32)$$

The solution (3.24)-(3.26) according to the noniterative method is the same as the solution (3.27)-(3.29) according to the generalized differentiability for the solutions (1,1) and (2,2). Also, all of the solutions are a valid $\alpha-$level set.

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