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Contractive modulus and common fixed point for three asymptotically regular and weakly compatible self-maps

T. Phaneendra,[a,*] and Swatmaram[b]

[a] Applied Analysis Division, School of Advanced Sciences, VIT-University, Vellore-632014, Tamil Nadu, India.

[a] Department of Mathematics, Chathanya Bharathi Institute of Technology, Hyderabad-500 075, Andhra Pradesh India.

Abstract

A common fixed point theorem for three self-maps on a metric space has been proved through the notions of orbital completeness, asymptotic regularity and weak compatibility. Our result generalizes those of Singh and Mishra, and the first author.

Keywords: Orbit, asymptotic regularity, weakly compatible self-maps, common fixed point.

2010 MSC: 52H25.

1 Introduction

Throughout this paper, (X, d) denotes a metric space, Sx the image of $x \in X$ under a self-map S on X and SA, the composition of self-maps S and A on X.

Definition 1.1. Self-maps S and A on X are compatible [1] if

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0 \tag{1.1}$$

whenever $\langle x_n\rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = p \quad \text{for some} \quad p \in X. \tag{1.2}$$

If $x_n = x$ for all n, compatibility of (S, A) implies that $SAx = ASx$ whenever $Ax = Sx$. Self-maps which commute at their coincidence points are weakly compatible [2].

Definition 1.2. Let $\psi \equiv \psi : [0, \infty) \to [0, \infty)$ be a contractive modulus [3] with the choice $\psi(0) = 0$ and $\psi(t) < t$ for $t > 0$. A contractive modulus ψ is upper semicontinuous (abbriviated as usc) if and only if $\limsup_{n\to\infty} \psi(t_n) \le \psi(t_0)$ for all $t = t_0$ and all $\langle t_n\rangle_{n=1}^{\infty} \subset [0, \infty)$ with $\lim_{n\to\infty} t_n = t_0$.

Using these ideas, Singh and Mishra [5] proved the following result:

Theorem 1.1. *Let S, T and A be self-maps on X satisfying the inclusions*

$$S(X) \subset A(X) \quad \text{and} \quad T(X) \subset A(X) \tag{1.3}$$

and the contractive-type condition

$$d(Sx, Ty) \le \psi\left(\max\left\{d(Ax, Ay), d(Sx, Ax), d(Ty, Ay), \frac{d(Ty, Ax) + d(Sx, Ay)}{2}\right\}\right) \text{ for all } x, y \in X, \tag{1.4}$$

where ψ is an usc contractive modulus. Suppose that

*Corresponding author.

E-mail address: drtp.indra@gmail.com (T. Phaneendra) and ramuswatma@yahoo.com (Swatmaram)

(a) one of $S(X), T(X)$ and $A(X)$ is a complete subspace of X,

(b) (A, S) and (A, T) are weakly compatible.

Then the three maps S, T and A will have a unique common fixed point.

In this paper, we generalize Theorem 1.1 by using the notion of asymptotic regularity (cf. Section 2) and by weakening the condition (b) under a weaker form of the inequality (1.4), when the contractive modulus ψ is nondecreasing. Our result also generalizes a result of the first author under an alternate condition.

2 Main Result

We need the following definitions from [4]:

Definition 2.1. Given $x_0 \in X$ and f, g and h self-maps on X, if we can find points $x_1, x_2, ..., x_n, ...$, then the associated sequence $\langle y_n \rangle_{n=1}^{\infty}$ with the choice

$$y_{2n-1} = Sx_{2n-2} = Ax_{2n-1}, y_{2n} = Tx_{2n-1} = Ax_{2n}, \text{ for } n = 1, 2, 3, ... \tag{2.1}$$

is called an (S, T, A)-orbit or simply an orbit $O(x_0)$ at x_0.

Definition 2.2. The space X is (S, T, A)-orbitally complete or orbitally complete at x_0 if every Cauchy sequence in some orbit $O(x_0)$ converges in X.

Definition 2.3. The pair (S, T) is asymptotically regular (abbriviated as a.r.) at x_0 with respect to A if there is an orbit $O(x_0)$ with the choice (2.1) such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Remark 2.1. Every complete metric space is orbitally complete at each of its points. However the converse of this statement is not true as in the following corrected form of the example from [4]:

Example 2.1. Let $X = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}_+, p \le q, q > 0 \right\}$ with $d(x, y) = |x - y|$ for all $x, y \in X$, where $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$. Define S, T and $A : X \to X$ by

$$Sx = \tfrac{x}{3},\ Tx = \tfrac{x}{2} \text{ and } Ax = \begin{cases} \frac{2x}{3} & \text{if } x < 1 \\ \frac{3}{4} & \text{if } x = 1. \end{cases}$$

Then X is incomplete. For instance, the sequence 0.7, 0.705, 0.707, 0.7071, 0.707105, 0.7071065, ... is Cauchy which converges to $\frac{1}{\sqrt{2}} \notin X$. Given $x_0 \in X$, we choose $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that $x_n = \left(\frac{1}{2}\right)\left(\frac{3}{8}\right)^{\frac{n-1}{2}} x_0$ or $\left(\frac{2}{3}\right)\left(\frac{3}{8}\right)^{\frac{n}{2}} x_0$ according as n is odd or even. Then $O(x_0) = \left\{ \left(\frac{1}{3}\right) x_0, \left(\frac{1}{6}\right) x_0, \left(\frac{1}{8}\right) x_0, \left(\frac{1}{16}\right) x_0, ... \right\}$ with $y_n = Ax_n = \left(\frac{1}{3}\right)\left(\frac{3}{8}\right)^{\frac{n-1}{2}} x_0$ or $\left(\frac{1}{6}\right)\left(\frac{3}{8}\right)^{\frac{n}{2}-1} x_0$ according as n is odd or even and $O(x_0)$ converges to $0 \in X$. Thus X is orbitally complete at x_0.

The following is our result, which was presented in the National Conference on Applications of Mathematics in Engineering, Physical and Life Sciences, Tirupaty (7-9 December, 2012):

Theorem 2.1. *Let S, T and A be self-maps on X satisfying the inclusions (1.3) and the inequality*

$$d(Sx, Ty) \le \psi(d(Ax, Ay), d(Ax, Sx), d(Ay, Ty), d(Ax, Ty), d(Ay, Sx)) \quad \text{for all} \quad x, y \in X, \tag{2.2}$$

where ψ is a nondecreasing and usc contractive modulus.
Given $x_0 \in X$, suppose that

(c) the pair (S, T) is a.r. at x_0 with respect to A

(d) any one of $S(X), T(X)$ and $A(X)$ is orbitally complete at x_0.

Then S, T and A will have a common coincidence point. Further, if

(e) either (A, S) or (A, T) is a weakly compatible pair,

then A, S and T will have a unique common fixed point.

Proof. Let $x_0 \in X$. Using the inclusions (1.3), we can inductively find points $x_1, x_2, ..., x_n, ...$ in X to define $O(x_0) = \langle y_n \rangle_{n=1}^{\infty}$ with the choice (2.1).

We show that $\langle y_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence. Suppose that it is not Cauchy. Then for some $\epsilon > 0$, we choose sequences $\langle 2m_k \rangle_{k=1}^{\infty}$ and $\langle 2m_k \rangle_{k=1}^{\infty}$ of even integers such that $d(y_{2m_k}, y_{2n_k}) \geq \epsilon$ for $2m_k > 2n_k > k$ for all k. Let $2m_k$ be the smallest even integer with this property so that $d(y_{2m_k-2}, y_{2n_k}) \leq \epsilon$.

Using the triangle inequality of d and asymptotic regularity (c), above inequalities give

$$\lim_{n\to\infty} d(y_{2m_k}, y_{2n_k}) = \lim_{n\to\infty} d(y_{2m_k}, y_{2n_k+1}) = \lim_{n\to\infty} d(y_{2m_k+1}, y_{2n_k+1}) = \lim_{n\to\infty} d(y_{2m_k+1}, y_{2n_k+2}). \tag{2.3}$$

Since ψ is nondecreasing, from the inequality (2.2) we have

$$d(Sx_{2m_k}, T_{2n_k+1}) \leq \psi(\max\{d(Ax_{2m_k}, A_{2n_k+1}), d(Ax_{2m_k}, Sx_{2m_k}), \\ d(Ax_{2n_k+1}, Tx_{2n_k+1}), d(Ax_{2m_k}, Tx_{2n_k+1}), d(Ax_{2n_k+1}, Sx_{2m_k})\}).$$

Proceeding the limit as $k \to \infty$ in this, then using (c), (2.3) and the upper semicontinuity of ψ, we get $0 < \epsilon \leq \psi(\max\{0+\epsilon, 0, 0, 0+\epsilon, \epsilon\}) = \psi(\epsilon) < \epsilon$. This contradiction establishes that $\langle y_n \rangle_{n=1}^{\infty}$ must be a Cauchy sequence and its subsequences $\langle y_{2n} \rangle_{n=1}^{\infty}$ and $\langle y_{2n+1} \rangle_{n=1}^{\infty}$ are also Cauchy.

Case (i): $A(X)$ is orbitally complete at x_0.
Then

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = z = Au \quad \text{for some} \quad u \in X. \tag{2.4}$$

Thus $\langle y_{2n} \rangle_{n=1}^{\infty}$ is a subsequence of the Cauchy sequence $\langle y_n \rangle_{n=1}^{\infty}$ converging to z. Hence $\langle y_n \rangle_{n=1}^{\infty}$ also converges to $z = Au$.

But then, (2.2) with $x = u$ and $y = x_{2n+1}$ gives

$$d(Su, Tx_{2n+1}) \leq \psi(\max\{d(Au, A_{2n+1}), d(Au, Su), d(Ax_{2n+1}, T_{2n+1}), d(Au, Tx_{2n+1}), d(Ax_{2n+1}, Su)\}).$$

Since ψ is usc, applying the limit as $n \to \infty$, this implies

$$d(Su, z) \leq \psi(\max\{d(Au, z), d(Au, Su), 0, 0, d(Su, z)\}) = \psi(d(Su, z))$$

or $Sz = z$. That is

$$Su = Au = z. \tag{2.5}$$

Writing $x = y = z$ in (2.2) and using (2.5), it follows that

$$d(Su, Tu) \leq \psi(\max\{d(Au, Su), d(Au, Tu)\}) = \psi(d(Su, Tu))$$

or $d(Su, Tu) = 0$ so that $Su = Tu$. Thus

$$Su = Au = Tu = z. \tag{2.6}$$

Thus u is a common coincidence point for A, S and T and z, their common point of coincidence.

Now with $x = y = z$, (2.2) again implies

$$d(Sz, Tz) \leq \psi(\max\{d(Az, Sz), d(Az, Tz)\}). \tag{2.7}$$

If (A, S) is weakly compatible, from (2.6) we get $Az = Sz$ and hence (2.7) yields $d(Sz, Tz) = \psi(d(Sz, Tz))$ or $d(Sz, Tz) = 0$ so that $Sz = Tz$.

Similarly if (A, T) is weakly compatible, from (2.6) we get $Az = Tz$, which together with (2.7) implies that $Sz = Tz$. Thus

$$Sz = Az = Tz, \tag{2.8}$$

whenever (e) holds good.

Finally, writing $x = z$ and $y = x_{2n-1}$ in (2.2), we see that

$$d(Sz, Tx_{2n-1}) \leq \psi(\max\{d(Az, Ax_{2n-1}), d(Az, Sz), d(Ax_{2n-1}, Tx_{2n-1}), d(Az, Tx_{2n-1}), d(Ax_{2n-1}, Sz)\}).$$

In the limit as $n \to \infty$, this along with (2.8) will give

$$d(Sz, z) \leq \psi(\max\{d(Sz, z), 0, 0, d(Sz, z), d(z, Sz)\}) = \psi(d(z, Sz))$$

so that $d(Sz, z) = 0$ or $Sz = z$. This again in view of (2.8) reveals that z is a common fixed point of A, S and T.

Case (ii): Let $S(X)$ be orbitally complete at x_0. Then $\langle y_n \rangle_{n=1}^{\infty}$ converges to $z \in S(X) \subset A(X)$. The conclusion follows from Case (i).

Case (iii): Let $T(X)$ be orbitally complete at x_0. Then $\langle y_n \rangle_{n=1}^{\infty}$ converges to $z \in T(X)$ and hence $z \in A(X)$, in view of (1.3). Again the conclusion follows from Case (i).

Uniqueness of the common fixed point follows directly from (2.2). □

Remark 2.2. Let $x_0 \in X$ be arbitrary and $r_n = d(y_{n-1}, y_n)$ for $n \geq 2$.
We now show that (1.4) of Theorem 1.1 implies the condition (c) of Theorem 2.1.

In fact, by a routine procure, it follows that

$$r_n \leq \psi(\max\{r_{n-1}, r_n\}) \quad \text{for} \quad n \geq 2. \tag{2.9}$$

If $r_m > r_{m-1}$ for some $m \geq 2$, then the choice of ψ and (2.9) would give a contradiction that $0 < r_m \leq \psi(r_m) < r_m$. Therefore $r_n \leq r_{n-1}$ for all $n \geq 2$. Using this again in (2.9), we get

$$r_n \leq \psi(r_{n-1}) \quad \text{for} \quad n = 2, 3, 4, ... \tag{2.10}$$

Repeated application of (2.10) and the choice of ψ will imply that $r_1 \geq r_2 \geq r_3 \geq ... \geq r_{n-1} \geq r_n \geq ...$, where $r_n \geq 0$ for all n. Hence $\lim_{n\to\infty} r_n = a$ for some $a \geq 0$. Then employing the limit as $n \to \infty$ in (2.10) and the upper semicontinuity of ψ, we get $a \leq \psi(a)$ so that $a = 0$, which the condition (c).

Further if ψ is nondcreasing, we see that the right hand side of (1.4) is less than or equal to the the right hand side of (2.2) due to the fact that $\frac{a+b}{2} \leq \max\{a, b\}$ for any $a \geq 0$ and $b \geq 0$. That is, (2.2) is weaker than (1.4) if ψ is nondcreasing.

Moreover, (a) of Theorem 1.1 implies (d) of Theorem 2.1, in vew of Remark 2.1. Therefore, a unique common fixed point of S, T and A can be ensured by Theorem 2.1. Thus Theorem 1.1 follows as a particular case of Theorem 2.1, when ψ is nondecreasing.

Our proof requires weak compatibility of only one of the pairs (A, S) and (A, T), where as Theorem 1.1 required weak compatibility of both the pairs.

Corollary 2.1. *Let S, T and A be self-maps on X satisfying inclusions* (1.3) *and the inequality*

$$d(Sx, Ty) \leq \omega(d(Ax, Ay), d(Ax, Sx), d(Ay, Ty), d(Ax, Ty), d(Ay, Sx)) \quad \textit{for all} \quad x, y \in X, \tag{2.11}$$

where $\omega : [0, \infty)^5 \to [0, \infty)$ *is nondecreasing and usc in each coordinate variable with* $\omega(t, t, t, t, t) < t$ *for* $t > 0$. *Given* $x_0 \in X$, *suppose that* (c) *holds good and*

(f) X is orbitally complete at x_0,

(g) any one of S, T and A is onto.

If either (A, S) *or* (A, T) *is compatible, then S, T and A will have a unique common fixed point.*

Proof. We write $\psi(t) = \omega(t,t,t,t,t)$ for $t \geq 0$. Then (2.11) is a particular case of (2.2), and the conditions (f) and (g) imply (d). Also every compatible pair is weakly compatible. Therefore A, S and T will have a unique common fixed point, by Theorem 2.1. □

Remark 2.3. *When (g) is replaced by the condition that A is orbitally continuous at x_0 in the sense that A is continuous at every point of some $O(x_0)$, Corollary 2.1 gives Theorem B of the first author ([4], p.46).*

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