Malaya Journal of Matematik, Vol. 7, No. 1, 7-10, 2019
https://doi.org/10.26637/MJM0701/0002

Quantum finite automata using quantum logic

Jismy Joseph[1]* and K.S. Dersanambika[2]

Abstract
Two types of Quantum Finite Automata are, the Measure once quantum finite automata (MO-QFA) proposed by Moore and Crutchfield [5] and the Many measure one-way quantum finite automata(MM-QFA) proposed by Kondacs and Waltrous [2]. In both cases it is proved that the language accepted is a subset of regular language. In this paper we define a Quantum Finite Automata using quantum logic. The logic underlying Quantum mechanics is not a Boolean algebra. It is an orthomodular lattice. This logic is called quantum logic By using this logic we study about various properties of QFA's.

Keywords
Quantum Logic, Orthomodular lattice, Quantum Finite Automata, Quantum Regular Language.

AMS Subject Classification
68QXX, 68Q05, 03B70, 81P68.

[1] *Department of Mathematics, Carmel College, Mala, Thrissur-680732, Kerala, India.*
[2] *Department of Mathematics, Fatima Mata National College, Kollam-691001, Kerala , India.*
***Corresponding author**: [1]jismykjoseph@gmail.com; [2]dersanapdf@yahoo.com
Article History: Received **24** July **2018**; Accepted **22** November **2018**

Contents

1. Introduction

The quantum logic was first introduced by Birkhoff and von Neumann [1] in connection with Quantum Mechanics. In Von Neumann's Hilbert space formalism of quantum mechanics the behavior of a quantum mechanical system is described by a closed subspace of a Hilbert space. Since the set of closed subspaces of a Hilbert space is an orthomodular lattice, Birkhoff and Von Neumann suggested to use orthomodular lattice as the logic of quantum mechanics.
The quantum computational model of Finite Automata has been introduced by multiple authors with two different definitions. The Measure once one way quantum finite automata (MO-1QFA) proposed by Moore and Crutchfield [5] and the Many measure one-way quantum finite automata(MM-1QFA) proposed by Kondacs and Waltrous [2]. A lot of works were done to study about the power of QFA. In this paper we define a QFA with the help of quantum logic. This logical approach helps to study about the properties of QFA in a different way. The automata theory based on quantum logic was proposed by Ying in [3]. In his work he introduced an orthomodular lattice valued classical Automata and he discussed about its properties. Many works were done on this line after his work, like [4] In our QFA model using quantum logic, we used the concept probability measurement in quantum logic. Detailed study about probability measurement in orthomodular lattice were done in [7] and [6].
The rest of the paper is organized as follows. In section 2 we recall some definitions that we used in this paper. In section 3 we gave the definition of Quantum Finite Automata using Quantum Logic. Then we give an example of a QFA using Quantum logic. In section 5 we studied about the closure properties of Quantum Regular Languages.

2. Preliminaries

In this section, we recall the definitions of two types of Quantum Finite Automata. Then we discussed about the complete orthomodular lattice which is called the quantum logic.

Definition 2.1. *A Measure Once Quantum Finite Automata is defined as a 5-tuple*

$$M = (Q, \Sigma, \delta, q_0, Q_{acc})$$

where,
Q is the finite set of quantum states
Σ is the set of input symbols
q_0 is the initial quantum state
Q_{acc} is the set of accepting states
For each $\sigma \in \Sigma$, δ_σ is the unitary transformation defined on the Hilbert space spanned by the states in Q
For a given input $w = \sigma_1\sigma_2\cdots\sigma_n$ automata starts from the initial state q_0. After reading the input σ_1 unitary transformation δ_{σ_1} is applied to the state q_0. This process continues until it reads the last input symbol and ends in the state $q = \delta_{\sigma_n}\delta_{\sigma_{n-1}}\cdots\delta_{\sigma_1}q_0$. At the end a measurement is performed on q and the accepting probability of the input w is
$P(w) = ||P_a q||^2$ where P_a is the projection on to the subspace spanned by $\{q : q \in Q_{acc}\}$

Definition 2.2. *A Measure Many Quantum Finite Automata is defined as a 6-tuple*

$$M = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej})$$

where $Q, \sigma, \delta, q_0, Q_{acc}$ are same as those defined in the previous definition. $Q_{rej} \subset Q$ is the set of rejecting states.
For any input string $w = \sigma_1\sigma_2\cdots\sigma_n$ the procedure similar to that of Measure Once Quantum Finite automata except that after every transformation measurement is performed on the resulting states. Here the projective measurement consists of $\{P_a, P_r, P_n\}$ where P_a,P_r and P_n are the projections on to the sub spaces spanned by Q_{acc}, Q_{rej} and Q_{non} respectively ($Q_{non} = Q - (Q_{acc} \cup Q_{rej})$). The accepting and rejecting probabilities are given by
$p(M\ accepts\ w) = \sum_{k=0}^{l+1} ||P_a\delta_{\sigma_n}\prod_{i=0}^{k-1}(P_n\delta_{\sigma_i})q_0||^2$
$p(M\ rejects\ w) = \sum_{k=0}^{l+1} ||P_r\delta_{\sigma_n}\prod_{i=0}^{k-1}(P_n\delta_{\sigma_i})q_0||^2$

In this paper we will define a quantum finite automata using quantum logic. So now we will give a brief introduction of quantum logic.

2.1 Quantum Logic

The set of all closed subspace of a Hilbert space $L(H)$ is a lattice under $\subset$. It is also an orthomodular lattice.The fundamental assumption in quantum physics is that the experimental propositions form a logic which is isomorphic with $L(H)$ for some Hilbert space H. So the orthomodular lattice is sometimes called quantum logic.

Definition 2.3. *A 7-tuple $(L, \leq, \wedge, \vee, \perp, 0, 1)$ is called a complete orthomodular lattice if it satisfies the following conditions:*

1. *$(L, \leq, \wedge, \vee, \perp, 0, 1)$ is a complete lattice, 0 and 1 are the least and the greatest elements of L. $\leq$ is the partial ordering in L and for any $M \subseteq L$, $\wedge M$ and $\vee M$ stands for the greatest lower bound and least upper bound of M respectively.*
2. *$\perp$ is a unitary operation on L called orthocomplement and it is required to satisfy the following conditions: for any $a, b \in L$*
   (a) *$a \wedge a^\perp = 0$, $a \vee a^\perp = 1$*
   (b) *$a^{\perp\perp} = a$*
   (c) *$a \leq b$ implies $b^\perp \leq a^\perp$*
   (d) *$a \geq b$ implies $a \wedge (a^\perp \vee b) = b$*

Definition 2.4. *A mapping $p : L \to [0, 1]$ is called a probability measure if*

1. *$p(1) = 1$*
2. *$p(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} p(a_i)$ whenever $a_i \leq {a_j}^\perp$ for any distinct indexes $i, j \in \mathcal{N}$*

3. Quantum Finite Automata using quantum logic

Let $(L, \leq, \wedge, \vee, \perp, 0, 1)$ be a complete orthomodular lattice. Then a quntum finite automata is defined using L as follows.

Definition 3.1. *A qantum finite automata using quantum logic is defined as*

$$M = (Q, \Sigma, \delta, q_0, Q_{acc})$$

where,
Q is finite set of states
Σ finite set of input alphabets
q_0 initial state
$Q_{acc} \subset Q$ is the set of accepting states
δ transition function,

$$\delta : Q \times \Sigma \times Q \to l$$

If $w = \sigma_1\sigma_2\cdots\sigma_n$ then the lattice value of the word w is defined as

$$l_M(w) \stackrel{def}{=} \vee\{\delta(q_0, \sigma_1, q_1) \wedge \cdots \delta(q_{n-1}, \sigma_n, q_n) : q_0, q_1, \cdots q_{n-1} \in Q, q_n \in Q_{acc}\}$$

Then we measure this $l_M(w)$ using a probability measure defined on L and denote it as $p(w)$. Quantum finite automata accepted a language L with probability λ if $p(w) \geq \lambda$ for all w in L. A language accepted by a QFA is called Quantum Regular Language(QRL).

4. Example

Let $\otimes^2\mathbb{C}^2$ be the 2 qubit space , where $\mathbb{C}$ denote set of complex numbers. The set of all closed subspaces of the Hilbert space $\otimes^2\mathbb{C}^2$,l form an orthomodular lattice $(l,\leq,\wedge,\vee,\perp,0,1)$. $q_0 = |0>|0>$, $q_1 = |0>|1>$, $q_2 = |1>|0>$ and $q_3 = |1>|1>$ are the basis states in the 2- qubit state space. The automata is defined as $M = (Q,\Sigma,\delta,q_0,Q_{acc})$ where $Q = \{q_0,q_1,q_2\}$, $\Sigma = \{a,b\}$, $Q_{acc} = \{q_2\}$. $a_{ij} = span\{|i>|j>\}$ denote the closed subspace spanned by $|i>|j>, i, j = 0,1$.
$\delta(q_0,a,q_0) = a_{00}$, $\delta(q_0,b,q_2) = a_{11}$, $\delta(q_0,a,q_1) = a_{00}$, $\delta(q_1,b,q_2) = a_{00}$

Now $l_M(w) = \left\{\begin{array}{cc} a_{00}, & \text{if } w = a^n b,\, n > 0 \\ a_{11}, & \text{if } w = b \\ 0, & \text{otherwise} \end{array}\right\}$

In this example we use the probability measure $p_\phi : L \to [0,1]$ where
$p_\phi(S) = ||P^S\phi||^2$ where P^S is the projection operator corresponding to the closed space S and ϕ is the initial state of the QFA.
Now the language accepted by the QFA is
$L(M) = \{a^n b : n > 0\}$ with probability 1.

5. Closure properties of quantum regular language

Theorem 5.1. *If A and B are quantum regular languages with probability λ and μ respectively then $A\cap B$ is also a quantum regular language with probability less than or equal to $min\{\lambda,\mu\}$.*

Proof. Let A be a QRL with accepting probability λ and B be a QRL with accepting probability μ. That is there exist two Quantum finite automata M_A and M_B such that $L(M_A) = A$ and $L(M_B) = B$. Now we construct a QFA, M_C that accepts $A\cup B$. $M_C = (Q_C,\Sigma,\delta_C,r_0,F_C)$ where
$Q_C = Q_A \times Q_B$
$F_C = F_A \times F_B$
$r_0 = (q_0,s_0)$
$\delta_C((p,q),a,(r,s)) = \delta_A(p,a,r) \wedge \delta_B(q,a,s)$
Let $x = a_1a_2\cdots a_n \in A\cap B$. Then there exist a path $q_0q_1\cdots q_n$ in M_A and a path $s_0s_1\cdots s_n$ in M_B labeled by x and whose lattice value is greater than zero. Therefore there exist atleast one path $(q_0,s_0)(q_1,s_1)\cdots(q_n,s_n)$ which is labeled by x in M_C whose lattice value is greater than zero.
Since $x\in A\cap B$ $p_A(x) \geq \lambda$ and $p_B(x) \geq \mu$.

Now

$$\begin{aligned}
p_C(x) &= p(l_C(x)) \\
l_C(x) &= \vee\{\delta_C((q_0,s_0),a_1,(q_1,s_1)) \\
&\quad \wedge\delta_C((q_1,s_1),a_2,(q_2,s_2)) \wedge\cdots \\
&\quad \wedge\delta_C((q_{n-1},s_{n-1}),a_n,(q_n,s_n))\} \\
&= \vee\{\delta_A(q_0,a_1,q_1)\wedge\delta_B(s_0,a_1,s_1)\wedge\cdots \\
&\quad \wedge\delta_A(q_{n-1},a_n,q_n)\wedge\delta_B(s_{n-1},a_n,s_n)\} \\
&\leq l_A(x)\wedge l_B(x)
\end{aligned}$$

Therefore

$$p_C(x) \leq p(l_A(x)\wedge l_B(x))$$

Since

$$\begin{aligned}
l_A(x)\wedge l_B(x) &\leq l_A(x)\text{and} \\
l_A(x)\wedge l_B(x) &\leq l_A(x), \\
p(l_A(x)\wedge l_B(x)) &\leq min\{p_A(x),p_B(x)\}
\end{aligned}$$

$\Rightarrow p_C(x) \leq min\{\lambda,\mu\}$. Therefore $A\cap B$ is accepted by the QFA M_C with a probability less than or equal to $min\{\lambda,\mu\}$ □

Theorem 5.2. *If A and B are Quantum Regular Languages with accepting probability λ and μ respectively then their union, $A\cup B$ is a QRL with probability greater than or equal to $max\{\lambda,\mu\}$.*

Proof. Let $M_A = (Q_A,\Sigma,\delta_A,q_0,F_A)$ and $M_B = (Q_B,\Sigma,\delta_B,s_0,F_B)$ be the QFA's accepting A and B. To prove the theorem we will construct an automata M_C which will accept the language $A\cup B$. $M_C = (Q_C,\Sigma,\delta_C,r_0,F_C)$ where,
$Q_C = Q_A\cup Q_B\cup\{r_0\}$, we takes the assumption that $Q_A\cap Q_B = \emptyset$

$\delta_C(p,a,q) = \left\{\begin{array}{cc} \delta_A(p,a,q) & \text{if } p,q\in Q_A \\ \delta_B(p,a,q) & \text{if } p,q\in Q_B \\ 0 & otherwise \end{array}\right\}$ $\delta_C(r_0,\varepsilon,q_0) = 1$

and $\delta_C(r_0,\varepsilon,s_0) = 1$
$F_C = F_A\cup F_B$
Let $x\in A\cup B$. Then there exist a path in M_A or M_B labeled by x and whose lattice value is greater than zero. Therefore the accepting probability of x in M_C is greater than zero.
We know that

$$\begin{aligned}
p_C(x) &= p(l_C(x)) \\
l_C(x) &= \vee\{\delta_C(r_0,\varepsilon,r_1)\wedge\delta_C(r_1,a_1,r_2)\cdots\wedge\delta_C(r_{n-1},a_n,r_n)\}
\end{aligned}$$

Since $\delta_C(p,a,q) = 0$ if $p\in Q_A$ and $q\in Q_B$,

$$\begin{aligned}
l_C(x) &= \vee\{\delta_A(q_0,a_1,q_1)\wedge\delta_A(q_1,a_1,q_2)\wedge\cdots \\
&\quad \wedge\delta_A(q_{n-1},a_1,q_n)\} \\
&\quad \bigvee\{\vee\{\delta_B(s_0,a_1,s_1)\wedge\delta_B(s_1,a_1,s_2)\wedge\cdots \\
&\quad \wedge\delta_B(s_{n-1},a_1,s_n)\}\} \\
&= l_A(x)\vee l_B(x)
\end{aligned}$$

$p_C(x) = p(l_A(x)\vee l_B(x)) \geq max\{\lambda,\mu\}$ Therefore $A\cup B$ is a QRL with accepting probability greater than or equal to $max\{\lambda,\mu\}$. □

Theorem 5.3. *If A and B are Quantum Regular languages with accepting probability λ and μ respectively then their concatenation, AB is also a QRL with probability less than or equal to $min\{\lambda,\mu\}$.*

Proof. Let $M_A = (Q_A, \Sigma, \delta_A, q_0, F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, s_0, F_B)$ be the QFA's accepting the languages A and B. Now we will construct a QFA, M_C which accepts the language AB.
$M_C = (Q_C, \Sigma, \delta_C, r_0, F_C)$ where,
$Q_C = Q_A \cup Q_B$
$r_0 = q_0$
$F_C = F_B$
$\delta_C(p,a,q) = \begin{Bmatrix} \delta_A(p,a,q) & if\, p,q \in Q_A \\ \delta_B(p,a,q) & if\, p,q \in Q_B \end{Bmatrix}$
$\delta_C(p,\varepsilon,s_0) = 1$ for every $p \in F_A$
Let $x \in AB$. Then $x = \sigma_1\sigma_2$ where, $\sigma_1 \in A$ and $\sigma_2 \in B$. There is a path $q_0q_1\cdots q_m$ in M_A labeled by σ_1 and a path $s_0s_1\cdots s_n$ in M_B labeled by σ_2 whose lattice values are greater than zero. So $q_0q_1\cdots q_m s_0 s_1\cdots s_n$ is a path in M_C labeled by x whose lattice value is greater than zero since $\delta_C(q_m,\varepsilon,s_0) = 1$ $(q_m \in F_A)$.
Let $\sigma_1 = a_1\cdots a_m$ and $\sigma_2 = b_1\cdots b_n$

$$\begin{aligned} p_C(x) &= p(l_C(x)) \\ l_C(x) &= \vee\{\delta_A(q_0,a_1,q_1)\wedge\cdots\wedge\delta_A(q_{m-1},a_m,q_m) \\ &\quad \wedge\delta_C(q_m,\varepsilon,s_0)\wedge\delta_B(s_0,b_1,s_1)\wedge\cdots \\ &\quad \wedge\delta_B(s_{n-1},b_n,s_n)\} \end{aligned}$$

Suppose that the supremum over all paths labeled by x occur along the path $q_{0k}q_{1k}\cdots q_{mk}s_{0k}s_{1k}\cdots s_{nk}$.
Then

$$\begin{aligned} l_c(x) &= \delta_A(q_0,a_1,q_1)\wedge\cdots\delta_A(q_{m-1},a_m,q_m)\wedge\delta_C(q_m,\varepsilon,s_0) \\ &\quad \wedge\delta_B(s_0,b_1,s_1)\wedge\cdots \\ &\quad \wedge\delta_B(s_{n-1},b_n,s_n) \\ &\leq \vee\{\delta_A(q_0,a_1,q_1)\wedge\cdots\delta_A(q_{m-1},a_m,q_m)\}\wedge \\ &\quad \vee\{\delta_B(s_0,b_1,s_1)\wedge\cdots\delta_B(s_{n-1},b_n,s_n)\} \\ &\leq l_A(\sigma_1)\wedge l_B(\sigma_2) \\ p_C(x) &= p(l_C(x)) \\ &\leq p(l_A(\sigma_1)\wedge l_B(\sigma_2)) \\ &\leq min\{\lambda,\mu\} \end{aligned}$$

Therefore the concatenation of the QRL's A and B, AB is a QRL with a probability less than or equal to $min\{\lambda,\mu\}$ □

Now we gives a pumping lemma for the Quantum Regular Languages.

6. pumping lemma for Quantum Regular Language

Theorem 6.1. *Let L be an infinite Quantum Regular Language. Then there exist some positive integer m such that for any $w \in L$ with $|w| \geq m$ can be decomposed as $w = xyz$ with $|xy| \leq m$, $|y| \geq 1$ such that $w_i = xy^iz$ is also in L for all $i = 0,1,\cdots$. Also $p(w_i) = p(w)$.*

Proof. The proof is similar to that in classical automata theory.
If L is a QRL then there exist a QFA, M recognizing L. Let $\{q_0,q_1,\cdots,q_n\}$ be the set of states of M. Now consider the string $w \in L$ such that $|w| \geq n+1$. Now consider the path through which M processes the string w. Let it be $p_0,p_1,\cdots,p_f$, where $p_f \in Q_{acc}$. Since this sequence has exactly $|w|+1$ states, atleast one state must be repeated. Therefore the sequence is of the form $p_0,p_1,\cdots,p_r,\cdots,p_r,\cdots,p_f$.
Let $p_0,p_1,\cdots,p_r$ be labeled by x; $p_r,\cdots,p_r$ be labeled by y and $p_r,\cdots,p_f$ be labeled by z. Then $|xy| \leq n+1$ and $|y| \geq 1$. Let $w = a_1a_2\cdots a_n$. Then $l(w) = \vee\{\delta(p_0,a_1,p_1)\wedge\delta(p_1,a_2,p_2)\wedge\cdots\wedge\delta(p_{n-1},a_n,p_n)\}$
Let $w_i = xy^iz$. Then clearly $l(w_i) = l(w)$ from the above formula. Therefore $p(w) = p(w_i)$ □

7. Conclusion

In this paper we defined Quantum Finite Automata using quantum logic and give an example of a QFA using quantum logic. We also studied about some closure properties of QRL's. The quantum logic approach makes it easier to study about the properties of Quantum Finite Automata. Also we introduced a pumping lemma for the Quantum Regular Languages.

References

[1] G. Birkhoff, J. Von Neumann. The logic of quantum mechanics, Ann. of Math. 37(1936)823-843.

[2] A.Kondacs and J.Watrous . On the power of quantum finite state automata. In Proceedings of the 38th Annual symposium on Foundations of Computer Science. Washington DC, USA:IEEE Computer Society, 1997.

[3] M.S. Ying, Automata theory based on quantum logic, Inernational J. of theoretical physics, 39(2000)981-991.

[4] D. Qiu,Notes on Automata theory based on quantum logic,Science in Cina series F: Information Science 50(2007)154-169.

[5] C. Moore and J.P. Crutchfield. Quantum automata and quantum grammars. Theory of Computer Science. Vol.237, page. 275-306, April 2000.

[6] Pavel PTAK and Vladimir ROGALEWICZ. Measures on Orthomodular Partially Ordered Sets. Journal of Pure and Applied Algebra 28(1983)75-80.

[7] V.S.Varadarajan. Probability in Physics and a Theorem on Simultaneous Observability. Communications on pure and applied mathematics, vol.xv,189-217(1962)

★★★★★★★★★
ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
★★★★★★★★★