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H-V- super magic labeling of H-factorable graphs

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Abstract

An H-magic labeling in an H-decomposable graph G is a bijection $f: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that for every copy H in the decomposition, $\sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e)$ is constant. The function f is said to be H-V-super magic labeling if $f(V(G)) = \{1,2,\ldots,p\}$. In this article, we give a few fundamental properties of H-V-super magic labeling. Obtained the magic constant for H-factorable graphs which are H-V-super magic. Further we gave a necessary and sufficient condition for an even regular graph to be 2-factor-V-super magic.

Keywords

H-decomposable graph, H-factorable graph, H-magic labeling, H-V-super magic labeling, 2-factor-V-super magic.

AMS Subject Classification

05C78.

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Contents

1. Introduction

In this paper, we discuss only finite, simple and undirected graphs. The set of vertices and edges of a graph $G(p,q)$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$.

A *labeling* of a graph G is a mapping that carries a set of graph elements, usually vertices and or edges into a set of numbers, usually integers. Many kinds of labelings have been defined and studied by many authors and an excellent survey of graph labelings can be found in [3].

In 1963, Sedlàček [8] introduced the concept of magic labeling in graphs. A graph G is *magic* if the edges of G can be labeled by the set of numbers $\{1,2,\ldots,q\}$ so that the sum of labels of all the edges incident with any vertex is the same [6].

A graph G is said to be H-decomposable if G has a family of subgraphs $H_1, H_2, \ldots, H_h$ such that all the subgraphs are isomorphic to a graph H, $E(H_i) \bigcap E(H_j) = \phi$ for $j \neq i$ and $\bigcup_{i=1}^{h} E(H_i) = E(G)$. If each H_i is a spanning subgraph of G, then G is said to be H-factorable. When H is a m-regular graph then G is said to be a m-factorable. If G is a m-factorable graph, then necessarily G is r-regular for some integer r that is a multiple of m.

In 2014, P. Subbiah and J. Pandimadevi [9] introduced H-E-super magic. A function $f: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ is called an H-magic labeling of a H-decomposable graph G if there exists a positive integer k (called magic constant) such that for every copy H in the decomposition, $\sum f(H) = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e) = k$. A graph G that admits such a labeling is called an H-magic docomposable graph. An H-magic labeling f is called an H-E-super magic labeling if $f(E(G)) = \{1,2,\ldots,q\}$. A graph that admits an H-E-super magic labeling is called an H-E-super magic decomposable graph.

By using this definition of H-magic labeling, we define a new labeling called H-V-super magic. An H-magic labeling $f: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ in an H-factorable graph G is called H-V-super magic if $f(V(G)) = \{1,2,\ldots,p\}$ and for every factor H of G, $\sum f(H) = M$, an integer. In this paper, we study some basic properties of H-V-super magic labeling. The magic constant for H-factorable graphs which are H-V-super magic, has been obtained. Further, we provide a necessary and sufficient condition for an even regular graph to be 2-factor-V-super magic. Through out this paper, we use the symbol h to denote the number of H-factors of G when G is H-factorable.

2. H-V-Super magic graphs

This section will explore some fundamental properties of H-V-super magic graphs.

Lemma 2.1. *If G is H-V-super magic, then the magic constant is given by $M = \frac{p(p+1)}{2} + \frac{pq}{h} + \frac{q(q+1)}{2h}$, where h is the number of H-factors of G.*

Proof. Let f be an H-V-super magic labeling of a graph G with the magic constant M. Then $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$, $f(V(G)) = \{1, 2, \ldots, p\}$ and $M = \sum\limits_{v \in V(H')} f(v) + \sum\limits_{e \in E(H')} f(e)$ for every factor H' in the factorization of G. Then $hM = h \sum\limits_{v \in V(G)} f(v) + \sum\limits_{e \in E(G)} f(e) = h[1+2+\ldots+p] + [(p+1)+(p+2)+\ldots+(p+q)] = h\frac{p(p+1)}{2} + pq + \frac{q(q+1)}{2}$ and so $M = \frac{p(p+1)}{2} + \frac{pq}{h} + \frac{q(q+1)}{2h}$. □

If G is a H-factorable graph and G possesses a H-V-super magic labeling, then it is easy to find the sum of the vertex labels (denoted by k_v). This provides the following result.

Lemma 2.2. *If G is H-V-super magic, then the sum of the edge labels of each factor is constant and is given by $k_e = \frac{pq}{h} + \frac{q(q+1)}{2h}$, where h is the number of H-factors of G.*

Proof. As each H-factor G' is a spanning subgraph of G, it results that $k_v = \frac{p(p+1)}{2}$ for every H-factor G'. Since M is constant and $M = k_v + k_e$, k_e must be constant and so from Lemma 2.1, it follows that $k_e = \frac{pq}{h} + \frac{q(q+1)}{2h}$. □

The next lemma gives a necessary and sufficient condition for an H-factorable graph to be H-V-super magic. This lemma is useful in deciding whether a particular graph is H-V-super magic or not.

Lemma 2.3. *Let G be an H-factorable graph and let g be a bijection from $E(G)$ onto $\{p+1, p+2, \ldots, p+q\}$. Then g can be extended to an H-V-super magic labeling of G if and only if $\sum\limits_{e \in E(H')} g(e)$ is constant for every H-factor H' in the factorization of G.*

Proof. Suppose g can be extended to an H-V-super magic labeling of G, say 'f'. Since f is an extension of g, $f(e) = g(e)$ for every $e \in E(G)$. Thus by Lemma 2.2, $\sum\limits_{e \in E(H')} f(e)$ is a constant for every H-factor H' in the factorization of G and so $\sum\limits_{e \in E(H')} g(e)$ is also a constant for every H-factor H' in the factorization of G.

Conversely, assume that $\sum\limits_{e \in E(H')} g(e)$ is constant for every H-factor H' in the factorization of G. Define a function f which is an extension of g, by $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(e) = g(e)$ for $e \in E(G)$ and $f(v_i) = i$ for all $i = 1, 2, \ldots, p$. Then $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$ and $f(V(G)) = \{1, 2, \ldots, p\}$ and so $k_v = \frac{p(p+1)}{2}$ for every H-factor H' of G. Therefore $k_v + k_e$ is constant for every H-factor of G. Thus f is an H-V-super magic labeling of G. □

3. 2-factor-V-super magic labeling

In this section, we explore the 2-factor-V-super magic labeling of 2-factorable graphs. Petersen [7] have proved the next theorem which is helpful in obtaining classes of graphs that are not 2-factor-V-super magic.

Theorem 3.1. *[7] Every $2r$-regular graph has a $2k$-factor for every integer k, $0 < k < r$.*

Lemma 3.2. *Let G be an even regular graph of odd order. If h is even, then G is not 2-factor-V-super magic.*

Proof. Let G be an even regular graph of odd order. Then by Theorem 3.1, G is 2-factorable and so $q = ph$. Suppose G admits an 2-factor-V-super magic labeling. By Lemma 2.1, we have $M = \frac{p(p+1)}{2} + \frac{pq}{h} + \frac{q(q+1)}{2h} = \frac{p(p+1)}{2} + \frac{p(ph)}{h} + \frac{ph(ph+1)}{2h} = \frac{p(p+1)}{2} + p^2 + \frac{p(ph+1)}{2}$, which is not an integer since p is odd and h is even, a contradiction. □

Magic squares are one of the most admired mathematical recreations. A classical reference on magic squares is [1], while one of the better recent book is [2]. A magic square of side n is an $n \times n$ array whose entries are an arrangement of a set of integers $\{1, 2, \ldots, n^2\}$ in which all elements in any row, any column or either main diagonal or back-diagonal, add to the same sum. Furthermore, we represent this sum as magic number $MN = \frac{1}{2}n(n^2+1)$.

Theorem 3.3. *An even regular graph G of odd order is 2-factor-V-super magic if and only if h is odd, where h is the number of 2-factors of G.*

Proof. Let G be an even regular graph of odd order p. If G is 2-factor-V-super magic, then by Lemma 3.2, h is odd.

	F_1	F_2	…	F_h
h edges of F_i		($h \times h$ magic square)$+p$		
$p-h$ edges of F_i	h^2+p+1 h^2+p+2h $h^2+p+2h+1$ h^2+p+4h … … $h^2+p+(p-h-2)h+1$ $h^2+p+(p-h)h$	h^2+p+2 $h^2+p+2h-1$ $h^2+p+2h+2$ $h^2+p+4h-1$ … … $h^2+p+(p-h-2)h+2$ $h^2+p+(p-h)h-1$	… … … … … … … …	h^2+p+h $h^2+p+h+1$ h^2+p+3h $h^2+p+3h+1$ … … $h^2+p+(p-h-1)h$ $h^2+p+(p-h-1)h+1$

Table 1.

Conversely suppose h is odd. Then by Theorem 3.1, G is 2-factorable. Let $F_1, F_2, \ldots, F_h$ be the 2-factors of G. We label the edges of G by using the set of numbers $\{p+1, p+2, \ldots, p+ph\}$ as shown in Table 1.

Note that the entries of the $h \times h$ magic square are $1, 2, \ldots, h^2$ and so the entries of ($h \times h$-magic square)$+p$ are $p+1, p+2, \ldots, p+h^2$. From Table 1, the sum of the edge labels of each 2-factor F_i when i is odd, is calculated as follows:

$$k_e = \sum f(E(F_i)) = \frac{1}{2}h(h^2+1) + ph + [(h^2+p)+i] + [(h^2+p)+2h-(i-1)]$$
$$+[(h^2+p)+2h+i] + [(h^2+p)+4h-(i-1)]$$
$$+[(h^2+p)+4h+i] + [(h^2+p)+6h-(i-1)]$$
$$+ \quad \cdots \quad + \quad \cdots$$
$$+[(h^2+p)+(p-h-2)h+i] + [(h^2+p)+(p-h)h-(i-1)]$$
$$= \frac{1}{2}h(h^2+1) + ph + [\frac{(p-h)(h^2+p)}{2} + \frac{(p-h)i}{2}]$$
$$+[0+2+4+\ldots+(p-h-2)h] + [\frac{(p-h)(h^2+p)}{2}]$$
$$+[2+4+\ldots+(p-h)h] + [\frac{(p-h)(-i)}{2}] + [\frac{(p-h)}{2}] = \frac{1}{2}h(h^2+1) + ph + [\frac{2(p-h)(h^2+p)}{2}] + [(\frac{(p-h))}{2})(\frac{(p-h+2))}{2})h]$$
$$+[(\frac{(p-h)}{2})(\frac{(p-h-2))}{2})h] + [\frac{(p-h)}{2}] = \frac{p^2(2+h)+p}{2}.$$

In a same way, we can have $k_e = \frac{p^2(2+h)+p}{2}$ for each factor F_i when i is even. Thus by Lemma 2.3, this labeling can be extended to an 2-factor-V-super magic labeling. □

Example 3.4. Note that the complete graph K_7 is 2-factorable and the number of 2-factors is 3 (by Theorem 3.1), let it be F_1, F_2, F_3. Note that $p=7$ and $h=3$. As discussed in Theorem 3.3, the labels of F_1, F_2 and F_3 are given in Table 2.

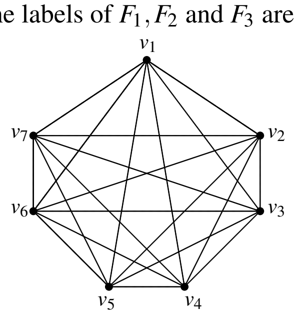


Figure 1 : K_7

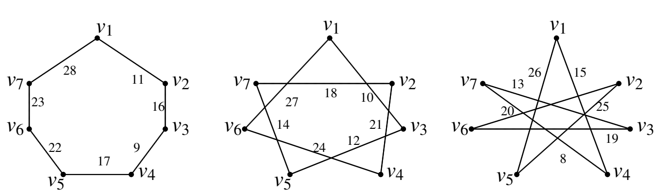


Figure 2: The 2-factors of K_7

	F_1	F_2	F_3
h=3 edges of F_i	11	10	15
	16	12	8
	9	14	13
$p-h=4$ edges of F_i	17	18	19
	22	21	20
	23	24	25
	28	27	26
sum of the labels of edges	126	126	126

Table 2. A 2-factor-V-super magic labeling of K_7

From Table 2, the sum of the edge labels at each factor is $k_e = 126$. Thus K_7 is 2-factor-V-super magic.

Theorem 3.5. *Let G be an even regular graph of even order. Then G is 2-factor-V-super magic.*

Proof. Let G be an even regular graph of even order p. By Theorem 3.1, G is 2-factorable. Let $F_1, F_2, \ldots, F_h$ be the 2-factors of G. We label the edges of G by using the set of numbers $\{p+1, p+2, \ldots, p+ph\}$ as shown in Table 3.

F_1	F_2	$\ldots$	F_{h-1}	F_h
$p+1$	$p+2$	$\ldots$	$p+h-1$	$p+h$
$p+2h$	$p+2h-1$	$\ldots$	$p+2h-(h-2)$	$p+2h-(h-1)$
$p+2h+1$	$p+2h+2$	$\ldots$	$p+2h+(h-1)$	$p+2h+h$
$p+4h$	$p+4h-1$	$\ldots$	$p+4h-(h-2)$	$p+4h-(h-1)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$p+(p-2)h+1$	$p+(p-2)h+2$	$\ldots$	$p+(p-2)h+(h-1)$	$p+(p-2)h+h$
$p+ph$	$p+ph-1$	$\ldots$	$p+ph-(h-2)$	$p+ph-(h-1)$

Table 3.

From Table 3, the sum of the edge labels of 2-factor F_i when i is odd, is calculated as follows:

$$k_e = \sum f(E(F_i)) = [(p)+i] + [(p+2h)-(i-1)] + [(p+2h)+i] + [(p+4h)-(i-1)] + [(p+4h)+i] + [(p+6h)-(i-1)]$$
$$+ \quad \cdots \quad + \quad \cdots$$
$$+[(p+(p-2)h)+i] + [(p+ph)-(i-1)]$$
$$= p(\frac{p}{2}) + [2+4+\ldots+(p-2)]h + \frac{p}{2}(i)$$
$$+p(\frac{p}{2}) + [2+4+\ldots+p]h - \frac{p}{2}(i) + \frac{p}{2}$$
$$= p^2 + 2[1+2+\ldots+\frac{(p-2)}{2}]h + 2[1+2+\ldots+\frac{p}{2}]h + \frac{p}{2}$$
$$= \frac{p^2(2+h)+p}{2}.$$

Similary, we can prove that $k_e = \frac{p^2(2+h)+p}{2}$ for each factor F_i when i is even. Thus by Lemma 2.3, this labeling can be extended to an 2-factor-V-super magic labeling. □

Example 3.6. The following graph G can be factorized into three 2-factors say F_1, F_2 and F_3. observe that one of the factors is disconnected.

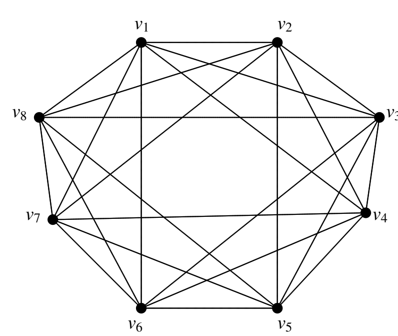


Figure 3 : G

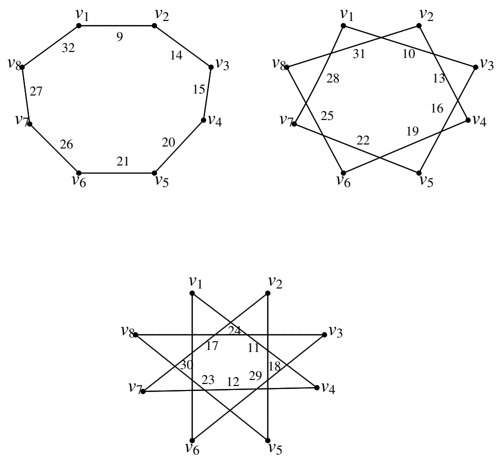


Figure 4: The 2-factor factorization of G:

F_1	F_2	F_3
9	10	11
14	13	12
15	16	17
20	19	18
21	22	23
26	25	24
27	28	29
32	31	30

Table 4. 2-factor-V-super magic labeling of G

The edges of each factor of G are labeled as shown in Table 4. From Table 4, the sum of the edge labels at each factor is $k_e = 164$. Thus the graph G is 2-factor-V-super magic.

4. Conclusion

We have characterized the 2-factor-V-super magic labeling of even regular graphs. Furthermore, we have found a few examples for 1-factor-V-super magic graphs(see Figure 5 and 6). The complete graph K_6 can be factorized into five 1-factors, say F_1, F_2, F_3, F_4 and F_5. From Figure 6, the sum of the edge labels at each factor is $k_e = 42$ and so K_6 is 1-factor-V-super magic.

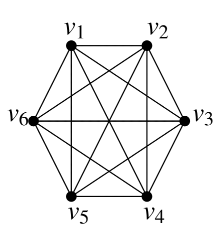
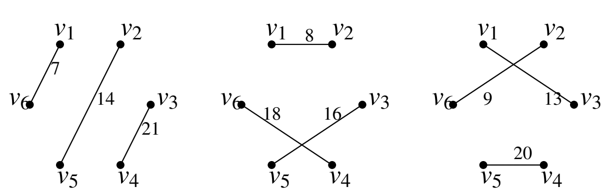


Figure 5: The graph K_6 is 1-factor-V-super magic.



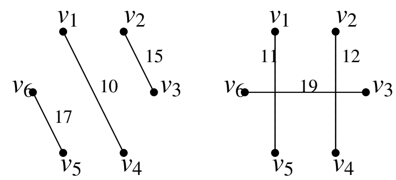


Figure 6: 1-factors of K_6

Thus, we finalize this article followed by an open problem.

Proposition 4.1. *Characterize all the r-factor-V-super magic graphs for* $r \geq 3$.

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