Malaya Journal of Matematik, Vol. 7, No. 2, 147-160, 2019

https://doi.org/10.26637/MJM0702/0004

Parameter uniform numerical method for a singularly perturbed boundary value problem for a linear system of parabolic second order delay differential equations

Parthiban Saminathan[1], Valarmathi Sigamani[2]* and Franklin Victor[3]

Abstract

A singularly perturbed boundary value problem for a linear system of two parabolic second order delay differential equations of reaction-diffusion type is considered. As the highest order space derivatives are multiplied by singular perturbation parameters, the solution exhibits boundary layers. Also, the delay term that occurs in the space variable gives rise to interior layers. A numerical method which uses classical finite difference scheme on a Shishkin piecewise uniform mesh is suggested to approximate the solution. The method is proved to be first order convergent uniformly for all the values of the singular perturbation parameters. Numerical illustrations are presented so that the theoretical results are supported.

Keywords

Singular perturbation problems, boundary layers, parabolic delay-differential equations, finite difference scheme, Shishkin mesh, parameter uniform convergence.

AMS Subject Classification

2010 MSC: 34K10,34K20,34K26,34K28.

[1,2,3] *Department of Mathematics, Bishop Heber College, Tiruchirappalli-620 017, Tamil Nadu, India.*

*Corresponding author; [1] parthitsaminathan@gmail.com; [2] valarmathi07@gmail.com; [3]franklinvicto@gmail.com

Article History: Received **18** October **2018**; Accepted **14** March **2019**

©2019 MJM.

Contents

1. Introduction

Singularly perturbed differential equations with delay have a wide range of applications - from population dynamics [1] to human physiology and bio system dynamics [2, 3]. The solutions of these equations exhibit boundary layers due to the presence of the singular perturbation parameter and interior layers due to the presence of the delay term. The derivatives of the solution of these problems exhibit propagating discontinuities depending on the nature of the problem. Hence classical finite difference schemes on uniform meshes are inadequate in providing good approximations. In a series of papers published by Lange and Miura, [4–7] various aspects of solutions of singularly perturbed delay differential equations were studied through asymptotic analysis and numerical experiments. In [9], parameter uniform convergence for a parabolic system of singularly perturbed differential equations is established. In [11], parameter uniform numerical method has been suggested to solve systems of singularly perturbed delay differential equations.

Here in this paper, a numerical method which uses standard finite difference scheme on a Shishkin piecewise uniform mesh is constructed. It is proved that the numerical approximations obtained by this method converge to the exact solution uniformly for all the values of the parameter in the maximum norm. The plan of the paper is as follows. In Section 2, the problem is defined and existence and regularity of the solution of the problem are discussed. In Section 3, the maximum principle for the differential operator is proved and consequently the stability result is established. And also standard estimates of the derivatives of the solution are presented. Further, improved estimates for the derivatives of components of the solution are presented. In Section 4, piecewise-uniform Shishkin meshes are introduced and in Section 5, the discrete problem is defined and the discrete maximum principle and the discrete stability properties are established. In Section 6, numerical analysis is presented and the error bounds are established. In Section 7, numerical illustrations are presented.

2. The Continuous Problem

A singularly perturbed boundary value problem for a system of two linear parabolic second order delay differential equations of reaction - diffusion type is considered as follows

$$\begin{aligned}\vec{L}\vec{u}(x,t) &= \frac{\partial\vec{u}}{\partial t}(x,t)-E\frac{\partial^2\vec{u}}{\partial x^2}(x,t)+A(x,t)\vec{u}(x,t)\\ &\quad +B(x,t)\vec{u}(x-1,t)=\vec{f}(x,t) \text{ on } \Omega, \quad (2.1)\\ &\quad \vec{u} \text{ given on } \Gamma,\\ &\quad \vec{u}(x,t)=\vec{\chi}(x,t), (x,t)\in[-1,0]\times[0,T],\end{aligned}$$

where $\Omega=\{(x,t):0<x<2, 0<t\leq T\}$, $\overline{\Omega}=\Omega\cup\Gamma, \tilde{\Omega}=((0,1-)\times(0,T])\cup((1+,2)\times(0,T]), \overline{\tilde{\Omega}}=([0,1-]\times[0,T])\cup([1+,2]\times[0,T])$, $\Gamma=\Gamma_L\cup\Gamma_B\cup\Gamma_R$ with $\vec{u}(0,t)=\vec{\chi}(0,t)$ on $\Gamma_L=\{(0,t):0\leq t\leq T\}, \vec{u}(x,0)=\vec{\phi}_B(x)$ on $\Gamma_B=\{(x,0):0\leq x\leq 2\}$, and $\vec{u}(2,t)=\vec{\phi}_R(t)$ on $\Gamma_R=\{(2,t):0\leq t\leq T\}$. For all $(x,t)\in\overline{\Omega}, \vec{u}(x,t)=(u_1(x),u_2(x))^T$ and $\vec{f}(x,t)=(f_1(x),f_2(x))^T$. $E, A(x,t)$ and $B(x,t)$ are 2×2 matrices.
$E=diag(\bar{\varepsilon}), \bar{\varepsilon}=(\varepsilon_1,\varepsilon_2)$ with $0<\varepsilon_1<\varepsilon_2<<1$,
$B(x,t)=diag(\vec{b}(x,t)), \vec{b}(x,t)=(b_1(x,t),b_2(x,t))$.
For all $(x,t)\in[0,2]\times[0,T]$, it is also assumed that the entries $a_{ij}(x,t)$ of $A(x,t)$ and the components $b_i(x,t)$ of $\vec{b}(x,t)$ satisfy

$$\begin{aligned} b_i(x,t), a_{ij}(x,t)\leq 0 \text{ for } 1\leq i\neq j\leq 2,\\ a_{ii}(x,t)>\sum_{i\neq j}|a_{ij}(x,t)+b_i(x,t)| \end{aligned}\tag{2.2}$$

$$\text{and } 0<\alpha<\min_{(x,t)\in[0,2]\times[0,T]}\left(\sum_{j=1}^{2}a_{ij}(x)+b_i(x)\right), \text{ for some } \alpha \tag{2.3}$$

The problem (2.1) can be rewritten as,

$$\begin{aligned}\vec{L}_1\vec{u}(x,t) &= \frac{\partial\vec{u}}{\partial t}(x,t)-E\frac{\partial^2\vec{u}}{\partial x^2}(x,t)+A(x,t)\vec{u}(x,t)\\ &= \vec{g}(x,t), \text{ on } \Omega_1=(0,1)\times(0,T] \end{aligned}\tag{2.4}$$

where $\vec{g}(x,t)=\vec{f}(x,t)-B(x,t)\vec{\chi}(x-1,t)$

$$\begin{aligned}\vec{L}_2\vec{u}(x,t) &= \frac{\partial\vec{u}}{\partial t}(x,t)-E\frac{\partial^2\vec{u}}{\partial x^2}(x,t)+A(x,t)\vec{u}(x,t)\\ &\quad +B(x,t)\vec{u}(x-1,t)=\vec{f}(x,t), \quad (2.5)\\ &\quad \text{on } \Omega_2=(1,2)\times(0,T]\end{aligned}$$

$\vec{u}(x,0)=\vec{\phi}_B(x)$ on $\Gamma_{B_1}=\{(x,0):0\leq x\leq 1-\}$,
$\vec{u}(x,0)=\vec{\phi}_B(x)$ on $\Gamma_{B_2}=\{(x,0):1+\leq x\leq 2\}$,
$\vec{u}(1-,t)=\vec{u}(1+,t), \frac{\partial\vec{u}}{\partial x}(1-,t)=\frac{\partial\vec{u}}{\partial x}(1+,t), \vec{u}(0,t)=\vec{\chi}(0,t)$,
$\vec{u}(2,t)=\vec{\phi}_R(t)$ on Γ_R.
The reduced problem corresponding to (2.4) - (2.5) is defined by

$$\begin{aligned}&\frac{\partial\vec{u}_0}{\partial t}(x,t)+A(x,t)\vec{u}_0(x,t)=\vec{g}(x,t), \text{ on } (0,1)\times(0,T]\\ &\vec{u}_0(x,0)=\vec{\phi}_B(x), 0\leq x\leq 1- \quad (2.6)\\ &\frac{\partial\vec{u}_0}{\partial t}(x,t)+A(x,t)\vec{u}_0(x,t)+B(x,t)\vec{u}_0(x-1,t)=\vec{f}(x,t),\\ &\text{on } (1,2)\times(0,T], \vec{u}_0(x,0)=\vec{\phi}_B(x), 1+\leq x\leq 2. \quad (2.7)\end{aligned}$$

In general as $\vec{u}_0(x,t)$ need not satisfy $\vec{u}_0(0,t)=\vec{u}(0,t)$ and $\vec{u}_0(2,t)=\vec{u}(2,t)$, the solution $\vec{u}(x,t)$ exhibits boundary layers at $x=0$ and $x=2$. In addition to that, as $\vec{u}_0(1-,t)$ need not be equal to $\vec{u}_0(1+,t)$, the solution $\vec{u}(x,t)$ exhibits interior layers at $x=1$.
The norms, $||\vec{V}||=\max_{1\leq k\leq n}|V_k|$ for any n-vector $\vec{V}, ||y||_D=\sup\{|y(x,t)|:(x,t)\in D\}$ for any scalar-valued function y and domain D, and $||\vec{y}||=\max_{1\leq k\leq n}||y_k||$ for any vector-valued function $\vec{y}$, are introduced. When $D=\overline{\Omega}$ or Ω the subscript D is usually dropped. In a compact domain D a function is said to be Hölder continuous of degree $\lambda, 0<\lambda\leq 1$, if, for all $(x_1,t_1),(x_2,t_2)\in D$,

$$|u(x_1,t_1)-u(x_2,t_2)|\leq C(|x_1-x_2|^2+|t_1-t_2|)^{\lambda/2}.$$

The set of Hölder continuous functions forms a normed linear space $C^0_\lambda(D)$ with the norm

$$||u||_{\lambda,D}=||u||_D+\sup_{(x_1,t_1),(x_2,t_2)\in D}\frac{|u(x_1,t_1)-u(x_2,t_2)|}{(|x_1-x_2|^2+|t_1-t_2|)^{\lambda/2}},$$

where $||u||_D=\sup_{(x,t)\in D}|u(x,t)|$. For each integer $k\geq 1$, the subspaces $C^k_\lambda(D)$ of $C^0_\lambda(D)$, which contain functions having Hölder continuous derivatives, are defined as follows

$$C^k_\lambda(D)=\{u:\frac{\partial^{l+m}u}{\partial x^l\partial t^m}\in C^0_\lambda(D) \text{ for } l,m\geq 0 \text{ and } 0\leq l+2m\leq k\}.$$

The norm on $C^0_\lambda(D)$ is taken to be
$$||u||_{\lambda,k,D}=\max_{0\leq l+2m\leq k}||\frac{\partial^{l+m}u}{\partial x^l\partial t^m}||_{\lambda,D}.$$
For a vector function $\vec{v}=(v_1,v_2,\ldots,v_n)$, the norm is defined by $||\vec{v}||_{\lambda,k,D}=\max_{1\leq i\leq n}||v_i||_{\lambda,k,D}$.
Sufficient conditions for the existence, uniqueness and regularity of solution of (2.1) are given in the following theorem.

Theorem 2.1. *Assume that* $a_{ij}(x,t), b_i(x,t), i,j = 1,2, \vec{f} \in C^2_\lambda(\overline{\Omega})$, $\vec{\chi} \in C^1([-1,0]\times[0,T]), \vec{\phi}_B \in C^2(\Gamma_B)$, $\vec{\phi}_R \in C^1(\Gamma_R)$ *and that the following compatibility conditions are fulfilled at the corners* $(0,0)$ *and* $(2,0)$ *of* Γ,

$$\begin{aligned} \phi_{B_1}(0) = \chi_1(0,t) \text{ and } \phi_{B_1}(2) = \phi_{R_1}(0) \\ \phi_{B_2}(0) = \chi_2(0,t) \text{ and } \phi_{B_2}(2) = \phi_{R_2}(0) \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{\partial \chi_1}{\partial t}(0,0) - \varepsilon_1 \frac{d^2\phi_{B_1}}{dx^2}(0) + a_{11}(0,0)\phi_{B_1}(0) + a_{12}(0,0)\phi_{B_2}(0) \\ +b_1(0,0)\chi_1(0,0) = f_1(0,0) \\ \frac{\partial \chi_2}{\partial t}(0,0) - \varepsilon_2 \frac{d^2\phi_{B_2}}{dx^2}(0) + a_{21}(0,0)\phi_{B_1}(0) + a_{22}(0,0)\phi_{B_2}(0) \\ +b_2(0,0)\chi_2(0,0) = f_2(0,0) \\ \frac{d\phi_{R_1}}{dt}(0) - \varepsilon_1 \frac{d^2\phi_{B_1}}{dx^2}(2) + a_{11}(2,0)\phi_{B_1}(2) + a_{12}(2,0)\phi_{B_2}(2) \\ +b_1(2,0)\phi_{B_1}(2) = f_1(2,0) \\ \frac{d\phi_{R_2}}{dt}(0) - \varepsilon_2 \frac{d^2\phi_{B_2}}{dx^2}(2) + a_{21}(2,0)\phi_{B_1}(2) + a_{22}(2,0)\phi_{B_2}(2) \\ +b_2(2,0)\phi_{B_2}(2) = f_2(2,0) \end{aligned} \tag{2.9}$$

$$\begin{aligned} \frac{\partial^2 \chi_1}{\partial t^2}(0,0) &= \varepsilon_1 \frac{d^4\phi_{B_1}}{dx^4}(0) - \varepsilon_1[\frac{\partial^2 a_{11}}{\partial x^2}(0,0)\phi_{B_1}(0) \\ &+ \frac{\partial^2 a_{12}}{\partial x^2}(0,0)\phi_{B_2}(0) + 2\frac{\partial a_{11}}{\partial x}(0,0)\frac{d\phi_{B_1}}{dx}(0) \\ &+2\frac{\partial a_{12}}{\partial x}(0,0)\frac{d\phi_{B_2}}{dx}(0) + a_{11}(0,0)\frac{d^2\phi_{B_1}}{dx^2}(0) \\ &+a_{12}(0,0)\frac{d^2\phi_{B_2}}{dx^2}(0) + \frac{\partial^2 b_1}{\partial x^2}(0,0)\chi_1(-1,0) \\ &+2\frac{\partial b_1}{\partial x}(0,0)\frac{\partial \chi_1}{\partial x}(-1,0) \\ &+b_1(0,0)\frac{\partial^2 \chi_1}{\partial x^2}(-1,0)] \\ &-[\frac{\partial a_{11}}{\partial t}(0,0)\phi_{B_1}(0) + \frac{\partial a_{12}}{\partial t}(0,0)\phi_{B_2}(0) + \\ &b_1(0,0)\frac{\partial \chi_1}{\partial t}(-1,0) + \frac{\partial b_1}{\partial t}(0,0)\chi_1(-1,0)] \\ &+\varepsilon_1\frac{\partial^2 f_1}{\partial x^2}(0,0) + \frac{\partial f_1}{\partial t}(0,0) \\ \frac{\partial^2 \chi_2}{\partial t^2}(0,0) &= \varepsilon_2 \frac{d^4\phi_{B_2}}{dx^4}(0) - \varepsilon_2[\frac{\partial^2 a_{21}}{\partial x^2}(0,0)\phi_{B_1}(0) \\ &+ \frac{\partial^2 a_{22}}{\partial x^2}(0,0)\phi_{B_2}(0) + 2\frac{\partial a_{21}}{\partial x}(0,0)\frac{d\phi_{B_1}}{dx}(0) \\ &+2\frac{\partial a_{22}}{\partial x}(0,0)\frac{d\phi_{B_2}}{dx}(0) + a_{21}(0,0)\frac{d^2\phi_{B_1}}{dx^2}(0) \\ &+a_{22}(0,0)\frac{d^2\phi_{B_2}}{dx^2}(0) + \frac{\partial^2 b_2}{\partial x^2}(0,0)\chi_2(-1,0) \\ &+2\frac{\partial b_2}{\partial x}(0,0)\frac{\partial \chi_2}{\partial x}(-1,0) \\ &+b_2(0,0)\frac{\partial^2 \chi_2}{\partial x^2}(-1,0)] \\ &-[\frac{\partial a_{21}}{\partial t}(0,0)\phi_{B_1}(0) + \frac{\partial a_{22}}{\partial t}(0,0)\phi_{B_2}(0) + \\ &b_2(0,0)\frac{\partial \chi_2}{\partial t}(-1,0) + \frac{\partial b_2}{\partial t}(0,0)\chi_2(-1,0)] \\ &+\varepsilon_2\frac{\partial^2 f_2}{\partial x^2}(0,0) + \frac{\partial f_2}{\partial t}(0,0) \end{aligned} \tag{2.10}$$

$$\begin{aligned} \frac{d^2\phi_{R_1}}{dt^2}(0) &= \varepsilon_1 \frac{d^4\phi_{B_1}}{dx^4}(2) - \varepsilon_1[\frac{\partial^2 a_{11}}{\partial x^2}(2,0)\phi_{B_1}(2) \\ &+ \frac{\partial^2 a_{12}}{\partial x^2}(2,0)\phi_{B_2}(2) + 2\frac{\partial a_{11}}{\partial x}(2,0)\frac{d\phi_{B_1}}{dx}(2) \\ &+2\frac{\partial a_{12}}{\partial x}(2,0)\frac{d\phi_{B_2}}{dx}(2) + a_{11}(2,0)\frac{d^2\phi_{B_1}}{dx^2}(2) \\ &+a_{12}(2,0)\frac{d^2\phi_{B_2}}{dx^2}(2) + \frac{\partial^2 b_1}{\partial x^2}(2,0)\phi_{B_1}(1) \\ &+2\frac{\partial b_1}{\partial x}(2,0)\frac{d\phi_{B_1}}{dx}(1) + b_1(2,0)\frac{d^2\phi_{B_1}}{dx^2}(1)] \\ &-[\frac{\partial a_{11}}{\partial t}(2,0)\phi_{B_1}(2) + \frac{\partial a_{12}}{\partial t}(2,0)\phi_{B_2}(2) \\ &+b_1(2,0)\frac{d\phi_{B_1}}{dt}(1) + \frac{\partial b_1}{\partial t}(2,0)\phi_{B_1}(1)] \\ &+\varepsilon_1\frac{\partial^2 f_1}{\partial x^2}(2,0) + \frac{\partial f_1}{\partial t}(2,0) \\ \frac{d^2\phi_{R_2}}{dt^2}(0) &= \varepsilon_2 \frac{d^4\phi_{B_2}}{dx^4}(2) - \varepsilon_2[\frac{\partial^2 a_{21}}{\partial x^2}(2,0)\phi_{B_1}(2) \\ &+ \frac{\partial^2 a_{22}}{\partial x^2}(2,0)\phi_{B_2}(2) + 2\frac{\partial a_{21}}{\partial x}(2,0)\frac{d\phi_{B_1}}{dx}(2) \\ &+2\frac{\partial a_{22}}{\partial x}(2,0)\frac{d\phi_{B_2}}{dx}(2) + a_{21}(2,0)\frac{d^2\phi_{B_1}}{dx^2}(2) \\ &+a_{22}(2,0)\frac{d^2\phi_{B_2}}{dx^2}(2) + \frac{\partial^2 b_2}{\partial x^2}(2,0)\phi_{B_2}(1) \\ &+2\frac{\partial b_2}{\partial x}(2,0)\frac{d\phi_{B_2}}{dx}(1) + b_2(2,0)\frac{d^2\phi_{B_2}}{dx^2}(1)] \\ &-[\frac{\partial a_{21}}{\partial t}(2,0)\phi_{B_1}(2) + \frac{\partial a_{22}}{\partial t}(2,0)\phi_{B_2}(2) \\ &+b_2(2,0)\frac{d\phi_{B_2}}{dt}(1) + \frac{\partial b_2}{\partial t}(2,0)\phi_{B_2}(1)] \\ &+\varepsilon_2\frac{\partial^2 f_2}{\partial x^2}(2,0) + \frac{\partial f_2}{\partial t}(2,0) \end{aligned} \tag{2.11}$$

Then there exists a unique solution $\vec{u}(x,t)$ of (2.1) satisfying $\vec{u}(x,t) \in \mathscr{C} = C^0_\lambda([0,2]\times[0,T]) \cap C^1_\lambda((0,2)\times(0,T]) \cap C^4_\lambda(\bar{\tilde{\Omega}})$.

It is assumed throughout the paper that all of the assumptions (2.2), (2.3), (2.8), (2.9), (2.10) and (2.11) of this section hold. Furthermore, C denotes a generic positive constant, which is independent of x,t and of all singular perturbation and discretization parameters.

3. Analytical results

Lemma 3.1. *Let conditions (2.2) and (2.3) hold. Let $\vec{\psi} = (\psi_1, \psi_2)^T$ be any function in $\mathscr{C}$ such that $\vec{\psi}(x,t) \geq \vec{0}$ on Γ. $\vec{L_1}\vec{\psi}(x,t) \geq \vec{0}$ on $(0,1)\times(0,T], \vec{L_2}\vec{\psi}(x,t) \geq \vec{0}$ on $(1,2)\times(0,T]$ and $[\vec{\psi}](1,t) = \vec{0}, [\frac{\partial\vec{\psi}}{\partial x}](1,t) \leq \vec{0}$ then $\vec{\psi}(x,t) \geq \vec{0}$ on $[0,2]\times[0,T]$.*

Proof. Let i^*, x^*, t^* be such that $\psi_{i^*}(x^*,t^*) = \min_{i=1,2;(x,t)\in[0,2]\times[0,T]} \psi_i(x,t)$.
If $\psi_{i^*}(x^*,t^*) \geq 0$, there is nothing to prove. Therefore suppose that $\psi_{i^*}(x^*,t^*) < 0$. Then $(x^*,t^*) \notin \Gamma, \frac{\partial\psi_{i^*}}{\partial t}(x^*,t^*) \leq 0$ and $\frac{\partial^2\psi_{i^*}}{\partial x^2}(x^*,t^*) \geq 0$. If $(x^*,t^*) \in (0,1)\times(0,T]$, then $(\vec{L_1}\vec{\psi})_{i^*}(x^*,t^*) < 0$, which is a contradiction. And if $(x^*,t^*) \in (1,2)\times(0,T]$, then $(\vec{L_2}\vec{\psi})_{i^*}(x^*,t^*) < 0$, which is also a contradiction. Because of the boundary values, the only other possibility is that $(x^*,t^*) = (1,t^*)$. In this case, the argument depends on whether or not $\vec{\psi}_{i^*}$ is differentiable at $(x,t) = (1,t)$. If $\frac{\partial\psi_{i^*}}{\partial x}(1,t^*)$ does not exist then $\left[\frac{\partial\psi_{i^*}}{\partial x}\right](1,t^*) \neq 0$ and since $\frac{\partial\psi_{i^*}}{\partial x}(1-,t^*) \leq 0, \frac{\partial\psi_{i^*}}{\partial x}(1+,t^*) \geq 0$, it is clear that $\left[\frac{\partial\psi_{i^*}}{\partial x}\right](1,t^*) > 0$, which is a contradiction. On the other hand, let ψ_{i^*} be differentiable at $(x,t) = (1,t)$. As $\sum_{j=1}^{2} a_{i^*j}(x,t)\psi_j(x,t) < 0$ and all the entries of $A(x,t)$ and $\psi_j(x,t)$ are in $C([0,2]\times[0,T])$, there exist an interval $[1-h,1)$ on which $\sum_{j=1}^{2} a_{i^*j}(x,t)\psi_j(x,t) < 0$. If $\frac{\partial^2\psi_{i^*}}{\partial x^2}(\hat{x}) \geq 0$ at any point $(\hat{x},t) \in [1-h,1)\times(0,T]$, then $(\vec{L_1}\vec{\psi})_{i^*}(x,t) < 0$, which is a contradiction. Thus we can assume that $\frac{\partial^2\psi_{i^*}}{\partial x^2}(x,t) < 0$ on $[1-h,1)\times(0,T]$. But this implies that $\frac{\partial\psi_{i^*}}{\partial x}(x,t)$ is strictly decreasing on $[1-h,1)\times(0,T]$. Already we know that $\frac{\partial\psi_{i^*}}{\partial x}(1,t) = 0$ and $\frac{\partial\psi_{i^*}}{\partial x}(x,t) \in C((0,2)\times(0,T])$, so $\frac{\partial\psi_{i^*}}{\partial x}(x,t) > 0$ on $[1-h,1)\times(0,T]$. Consequently the continuous function $\psi_{i^*}(x,t)$ cannot have a minimum at $(x,t) = (1,t)$, which contradicts the assumption $(x^*,t^*) = (1,t^*)$. □

As a consequence of the maximum principle, there is established the stability result for the problem (2.1) in the following.

Lemma 3.2. *Let conditions (2.2) and (2.3) hold. Let $\vec{\psi}$ be any function in $\mathscr{C}$, such that $[\vec{\psi}](1,t) = \vec{0}$ and $\left[\frac{\partial\vec{\psi}}{\partial x}\right](1,t) = \vec{0}$, then for each $i = 1,2$ and $(x,t) \in [0,2]\times[0,T]$,*

$$|\psi_i(x,t)| \leq \max\left\{\| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{L_1}\vec{\psi} \|, \frac{1}{\alpha} \| \vec{L_2}\vec{\psi} \|\right\}.$$

Proof. Let

$$M = \max\left\{\| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{L_1}\vec{\psi} \|, \frac{1}{\alpha} \| \vec{L_2}\vec{\psi} \|\right\}.$$

Define two functions $\vec{\theta^{\pm}}(x,t) = M\vec{e} \pm \vec{\psi}(x,t)$ where $\vec{e} = (1,1)^T$. Using the properties of $A(x,t)$ and $B(x,t)$, it is not hard to verify that $\vec{\theta^{\pm}}(x,t) \geq \vec{0}$ for $(x,t) \in \Gamma$ and $\vec{L_1}\vec{\theta^{\pm}} \geq \vec{0}$ on $(0,1)\times(0,T]$ and $\vec{L_2}\vec{\theta^{\pm}} \geq \vec{0}$ on $(1,2)\times(0,T]$. Moreover $[\vec{\theta^{\pm}}](1,t) = \pm[\vec{\psi}](1,t) = \vec{0}$ and $\left[\frac{\partial\vec{\theta^{\pm}}}{\partial x}\right](1,t) = \pm\left[\frac{\partial\vec{\psi}}{\partial x}\right](1,t) = \vec{0}$. It follows from Lemma 3.1 that $\vec{\theta^{\pm}}(x,t) \geq \vec{0}$ on $[0,2]\times[0,T]$. □

Standard estimates of the solution of (2.1) and its derivatives are contained in the following lemma.

Lemma 3.3. *Let conditions (2.2) and (2.3) hold. Let $\vec{u}$ be the solution of (2.1). Then for all $(x,t) \in [0,2]\times[0,T]$ and $i = 1,2$,*

$$|\frac{\partial^k u_i}{\partial t^k}(x,t)| \leq C(||u_i||_\Gamma + \textstyle\sum_{q=0}^{k} ||\frac{\partial^q f_i}{\partial t^q}||), k = 0,1,2$$
$$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{\frac{-k}{2}}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}||), k = 1,2$$
$$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{-1}\varepsilon_1^{\frac{-(k-2)}{2}}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}|| + ||\frac{\partial^2 f_i}{\partial t^2}|| + \varepsilon_1^{\frac{k-2}{2}}||\frac{\partial^{k-2} f_i}{\partial x^{k-2}}||), k = 3,4$$
$$|\frac{\partial^k u_i}{\partial x^{k-1}\partial t}(x,t)| \leq C\varepsilon_i^{\frac{1-k}{2}}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}|| + ||\frac{\partial^2 f_i}{\partial t^2}||), k = 2,3.$$

Proof. The proof is by the method of steps. First, the bounds of $\vec{u}$ and its derivatives are estimated in $[0,1-]\times[0,T]$. Next, these bounds of $\vec{u}$ and its derivatives are used to get the estimates in $[1+,2]\times[0,T]$. The bound on $\vec{u}$ is an immediate consequence of Lemma 3.2 in [9]. To bound $\frac{\partial u_i}{\partial t}(x,t)$ and $\frac{\partial^2 u_i}{\partial t^2}(x,t)$, on $[0,1-]\times[0,T]$ and $[1+,2]\times[0,T]$, differentiating (2.1) partially with respect to time once and twice respectively, and applying Lemma 3.2 in [9] in the domain $[0,1-]\times[0,T]$, the bounds on $\frac{\partial u_i}{\partial t}(x,t)$, respectively $\frac{\partial^2 u_i}{\partial t^2}(x,t)$ are obtained. To bound $\frac{\partial u_i}{\partial x}(x,t)$, on the interval $[0,1-]\times[0,T]$, consider an interval $I = [a, a+\sqrt{\varepsilon_i}], i = 1,2, a \geq 0$ such that $x \in I$. Then for some y such that $a < y < a+\sqrt{\varepsilon_i}$ and $t \in (0,T]$

$$\frac{\partial u_i}{\partial x}(y,t) = \frac{u_i(a+\sqrt{\varepsilon_i},t) - u_i(a,t)}{\sqrt{\varepsilon_i}}$$

$$|\frac{\partial u_i}{\partial x}(y,t)| \leq C\varepsilon_i^{-\frac{1}{2}}||u_i||. \tag{3.1}$$

Then for any $x \in I$,

$$\frac{\partial u_i}{\partial x}(x,t) = \frac{\partial u_i}{\partial x}(y,t) + \int_y^x \frac{\partial^2 u_i}{\partial x^2}(s,t)ds$$

$$\frac{\partial u_i}{\partial x}(x,t) = \frac{\partial u_i}{\partial x}(y,t) + \varepsilon_i^{-1}\int_y^x (\frac{\partial u_i}{\partial t}(s,t) - f_i(s,t) + \textstyle\sum_{j=1}^{2} a_{ij}(s,t)u_i(s,t) + b_i(s,t)\chi_i(x-1,t))ds$$

$$|\frac{\partial u_i}{\partial x}(x,t)| \leq |\frac{\partial u_i}{\partial x}(y,t)| + C\varepsilon_i^{-1}\int_y^x (||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}||)ds$$

Using (3.1) in the above equation

$$|\frac{\partial u_i}{\partial x}(x,t)| \leq C\varepsilon_i^{\frac{-1}{2}}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}||).$$

Rearranging the terms in (2.1), it is easy to get

$$|\frac{\partial^2 u_i}{\partial x^2}(x,t)| \leq C\varepsilon_i{}^{-1}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}||)$$

To bound $\frac{\partial u_i}{\partial x}(x,t)$, $\frac{\partial^2 u_i}{\partial x^2}(x,t)$ on $[1+,2]\times[0,T]$, following the same steps and using the bounds established for $[0,1-]\times[0,T]$, it is not hard to get the bounds in the domain $[1+,2]\times[0,T]$.

Analogous steps are used to get the rest of the estimates. Rearranging the differential equation (2.1) and differentiating once and twice give $\frac{\partial^3 u_i}{\partial x^3}(x,t)$, $\frac{\partial^4 u_i}{\partial x^4}(x,t)$ and the bounds on $\frac{\partial^3 u_i}{\partial x^3}(x,t)$ and $\frac{\partial^4 u_i}{\partial x^4}(x,t)$ follow from those on $\frac{\partial u_i}{\partial x}(x,t)$ and $\frac{\partial^2 u_i}{\partial x^2}(x,t)$. □

The Shishkin decomposition of the exact solution $\vec{u}$ of (2.1) is $\vec{u} = \vec{v} + \vec{w}$ where the smooth component $\vec{v}$ is the solution of

$$\begin{aligned}&\vec{L_1}\vec{v} = \vec{g} \text{ on } (0,1-)\times(0,T],\\ &\vec{v}(0,t) = \vec{u_0}(0,t), \vec{v}(x,0) = \vec{u_0}(x,0), \vec{v}(1-,t) = \vec{u_0}(1-,t)\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\vec{L_2}\vec{v} = \vec{f} \text{ on } (1+,2)\times(0,T],\\ &\vec{v}(2,t) = \vec{u_0}(2,t), \vec{v}(x,0) = \vec{u_0}(x,0), \vec{v}(1+,t) = \vec{u_0}(1+,t)\end{aligned} \tag{3.3}$$

and the singular component $\vec{w}$ is the solution of

$$\begin{aligned}&\vec{L_1}\vec{w} = \vec{0} \text{ on } (0,1)\times(0,T], \vec{L_2}\vec{w} = \vec{0} \text{ on } (1,2)\times(0,T]\\ &\text{with } \vec{w}(0,t) = \vec{u}(0,t) - \vec{v}(0,t), [\vec{w}](1,t) = -[\vec{v}](1,t),\\ &[\frac{\partial \vec{w}}{\partial x}](1,t) = -[\frac{\partial \vec{v}}{\partial x}](1,t), \vec{w}(2,t) = \vec{u}(2,t) - \vec{v}(2,t).\end{aligned} \tag{3.4}$$

The singular component is given a further decomposition

$$\vec{w}(x,t) = \vec{\tilde{w}}(x,t) + \vec{\hat{w}}(x,t) \tag{3.5}$$

where $\vec{\tilde{w}}$ is the solution of

$$\frac{\partial \vec{\tilde{w}}}{\partial t}(x,t) - E\frac{\partial^2 \vec{\tilde{w}}}{\partial x^2}(x,t) + A(x,t)\vec{\tilde{w}}(x,t) = \vec{0} \text{ on } (0,1)\times(0,T],$$
$$\vec{\tilde{w}}(0,t) = \vec{w}(0,t), \vec{\tilde{w}}(1,t) = K_1, \vec{\tilde{w}} = \vec{0} \text{ on } (1,2)\times(0,T]$$

and $\vec{\hat{w}}$ is the solution of

$$\frac{\partial \vec{\hat{w}}}{\partial t}(x,t) - E\frac{\partial^2 \vec{\hat{w}}}{\partial x^2}(x,t) + A(x,t)\vec{\hat{w}}(x,t) + B(x,t)\vec{\hat{w}}(x-1,t) = \vec{0}$$
$$\text{on } (1,2)\times(0,T],$$
$$\vec{\hat{w}}(1,t) = K_2, \vec{\hat{w}}(2,t) = \vec{w}(2,t), \vec{\hat{w}} = \vec{0} \text{on } (0,1)\times(0,T]$$

Here, K_1 and K_2 are constants to be chosen in such a way that the jump conditions at $x = 1$ are satisfied. Bounds on the smooth component and its derivatives are contained in the following lemma.

Lemma 3.4. *Let conditions (2.2) and (2.3) hold. The smooth component $\vec{v}$ and its derivatives satisfy, for each $(x,t) \in [0,2]\times[0,T]$ and $i = 1,2$,*

$$|\frac{\partial^k v_i}{\partial t^k}(x,t)| \leq C, \text{ for } k = 0,1,2$$
$$|\frac{\partial^k v_i}{\partial x^k}(x,t)| \leq C(1 + \varepsilon_i^{1-\frac{k}{2}}), \text{ for } k = 0,1,2,3,4$$
$$|\frac{\partial^k v_i}{\partial x^{k-1}\partial t}(x,t)| \leq C, \text{ for } k = 2,3.$$

Proof. The proof is by the method of steps. Applying lemma 3.2 in [9], the estimates of derivatives of $\vec{v}$ on $[0,1-]\times[0,T]$ follow. The arguments used to bound $\vec{v}$ and its derivatives in the interval $[1+,2]\times[0,T]$ are given below. The bound on $\vec{v}$ is an immediate consequence of the defining equations for $\vec{v}$ and Lemma 3.2 in [9] in the domain $[1+,2]\times[0,T]$. The bounds on the partial derivatives of $\vec{v}$ with respect to x and t are found as follows. Differentiating the equation (3.2) twice partially with respect to x and applying Lemma 3.2 in [9] in the domain $[1+,2]\times[0,T]$ we get

$$|\frac{\partial^2 v_i}{\partial x^2}(x,t)| \leq C(1 + ||\frac{\partial \vec{v}}{\partial x}||). \tag{3.6}$$

Let

$$\frac{\partial v_{i^*}}{\partial x}(x^*,t^*) = ||\frac{\partial \vec{v}}{\partial x}||_{[1,2]\times[0,T]} \text{ for some } i = i^*, x = x^*, t = t^*. \tag{3.7}$$

Using Taylor expansion, it follows that, for some $y \in [1-x^*, 2-x^*]$ and some η, such that $x^* < \eta < x^* + y$,

$$v_{i^*}(x^*+y,t^*) = v_{i^*}(x^*,t^*) + y\frac{\partial v_{i^*}}{\partial x}(x^*,t^*) + \frac{y^2}{2}\frac{\partial^2 v_{i^*}}{\partial x^2}(\eta,t^*). \tag{3.8}$$

Rearranging (3.8) yields

$$\frac{\partial v_{i^*}}{\partial x}(x^*,t^*) = \frac{v_{i^*}(x^*+y,t^*) - v_{i^*}(x^*,t^*)}{y} - \frac{y}{2}\frac{\partial^2 v_{i^*}}{\partial x^2}(\eta,t^*)$$

$$\Rightarrow |\frac{\partial v_{i^*}}{\partial x}(x^*,t^*)|_{[1,2]\times[0,T]} \leq \frac{2}{y}||\vec{v}||_{[1,2]\times[0,T]} + \frac{y}{2}||\frac{\partial^2 \vec{v}}{\partial x^2}||_{[1,2]\times[0,T]}. \tag{3.9}$$

Using (3.7) and (3.9) in (3.6),

$$|\frac{\partial^2 v_i}{\partial x^2}(x,t)|_{[1,2]\times[0,T]} \leq C(1 + \frac{2}{y}||\vec{v}||_{[1,2]\times[0,T]} + \frac{y}{2}||\frac{\partial^2 \vec{v}}{\partial x^2}||_{[1,2]\times[0,T]}).$$

This leads to

$$(1 - \frac{Cy}{2})||\frac{\partial^2 \vec{v}}{\partial x^2}||_{[1,2]\times[0,T]} \leq C(1 + \frac{2}{y}||\vec{v}||_{[1,2]\times[0,T]}). \tag{3.10}$$

Choosing $y = \min(\frac{1}{C}, 2 - x^*)$, (3.10) then gives

$$||\frac{\partial^2 \vec{v}}{\partial x^2}||_{[1,2]\times[0,T]} \leq C$$

and from (3.9)

$$||\frac{\partial \vec{v}}{\partial x}||_{[1,2]\times[0,T]} \leq C$$

as required. The bounds on $\frac{\partial^3 \vec{v}}{\partial x^3}(x,t), \frac{\partial^4 \vec{v}}{\partial x^4}(x,t)$ are derived by similar arguments. Repeating the above steps with $\frac{\partial \vec{v}}{\partial t}(x,t)$, it is easy to get the required bounds on the mixed derivatives. □

The layer functions $B^L_{1,i}, B^R_{1,i}, B^L_{2,i}, B^R_{2,i}, B_{1,i}, B_{2,i}, i = 1,2$, associated with the solution $\vec{u}$, are defined by

$$B^L_{1,i}(x) = e^{-x\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}}, B^R_{1,i}(x) = e^{-(1-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},$$
$$B_{1,i}(x) = B^L_{1,i}(x) + B^R_{1,i}(x), \text{ on } [0,1]\times[0,T],$$

$$B^L_{2,i}(x) = e^{-(x-1)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}}, B^R_{2,i}(x) = e^{-(2-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},$$
$$B_{2,i}(x) = B^L_{2,i}(x) + B^R_{2,i}(x), \text{ on } [1,2]\times[0,T].$$

It has to be noted that for $i = 1,2, B_{1,i}(x-1) = B_{2,i}(x)$ for $x \in [1,2]$

Definition 3.5. *For $B^L_{1,1}, B^L_{1,2}$, let $x^{(s)}, 1 \leq i \neq j \leq 2, s > 0$ be the point defined by $\frac{B^L_{1,1}(x^{(s)})}{\varepsilon_1} = \frac{B^L_{1,2}(x^{(s)})}{\varepsilon_2}$.*
Then $\frac{B^R_{1,1}(1-x^{(s)})}{\varepsilon_1} = \frac{B^R_{1,2}(1-x^{(s)})}{\varepsilon_2}, \frac{B^L_{2,1}(1+x^{(s)})}{\varepsilon_1} = \frac{B^L_{2,2}(1+x^{(s)})}{\varepsilon_2}$ and $\frac{B^R_{2,1}(2-x^{(s)})}{\varepsilon_1} = \frac{B^R_{2,2}(2-x^{(s)})}{\varepsilon_2}$.

The existence, uniqueness and the properties of $x^{(s)}$ can be verified as in [9, 11]. Bounds on the singular component $\vec{w}$ of $\vec{u}(x,t)$ and their derivatives are contained in the following lemma.

Lemma 3.6. *Let conditions (2.2) and (2.3) hold. Then there exists a constant C, such that, for $(x,t) \in [0,1]\times[0,T]$ and $i = 1,2$,*

$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \leq CB_{1,2}(x)$, for $k = 0,1,2$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\sum_{q=i}^{2} \frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}}}$, for $k = 0,1,2$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\sum_{q=1}^{2} \frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}}}$, for $k = 3$

$|\varepsilon_i \frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\sum_{q=1}^{2} \frac{B_{1,q}(x)}{\varepsilon_q}$, for $k = 4$

$|\frac{\partial^k w_i}{\partial x^{k-1}\partial t}(x,t)| \leq C\sum_{q=i}^{2} \frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}}}$, for $k = 2,3$

and for $(x,t) \in [1,2]\times[0,T]$,

$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \leq CB_{2,2}(x)$, for $k = 0,1,2$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\sum_{q=i}^{2} \frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}}}$, for $k = 0,1,2$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\sum_{q=1}^{2} \frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}}}$, for $k = 3$

$|\varepsilon_i \frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\sum_{q=1}^{2} \frac{B_{2,q}(x)}{\varepsilon_q}$, for $k = 4$

$|\frac{\partial^k w_i}{\partial x^{k-1}\partial t}(x,t)| \leq C\sum_{q=i}^{2} \frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}}}$, for $k = 2,3$.

Proof. First we derive the bound on $\vec{w}$ on $(0,1)\times(0,T]$. To obtain the bound of $\vec{w}$, define the functions

$$\psi_i^{\pm}(x,t) = CB_{1,2} \pm w_i(x,t), i = 1,2.$$

It is clear that, for $(x,t) \in (0,1)\times(0,T]$, $\psi_i^{\pm}(0,t), \psi_i^{\pm}(x,0)$, $\psi_i^{\pm}(1,t)$ and $L_1\psi_i^{\pm}(x,t)$ are non-negative. By Lemma 3.1 in [9], $\psi_i^{\pm}(x,t) \geq 0$ for $(x,t) \in [0,1]\times[0,T]$. It follows that

$$|w_i(x,t)| \leq CB_{1,2}(x) \tag{3.11}$$

Now use the temporary notation $\frac{\partial w_i}{\partial x}(x,t) = y_i(x,t)$. Hence we have

$$L_1 y_i(x,t) = -\sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial x}(x,t) w_j(x,t).$$

Now construct the barrier functions

$$\psi_i^{\pm}(x,t) = C\varepsilon_i^{\frac{-1}{2}} B_{1,2}(x) \pm y_i(x,t)$$

$\psi_i^{\pm}(0,t) = C\varepsilon_i^{\frac{-1}{2}} \pm y_i(0,t) \geq 0$,
$\psi_i^{\pm}(1,t) = C\varepsilon_i^{\frac{-1}{2}} \pm y_i(1,t) \geq 0, \psi_i^{\pm}(x,0) = 0$ and

$$\begin{aligned} L_1\psi_i^{\pm}(x,t) = & C\varepsilon_i^{\frac{-1}{2}}(\textstyle\sum_{j=1}^{2} a_{ij}(x,t) - \alpha)B_{1,2}(x) \\ & \mp \textstyle\sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial x}(x,t) w_j(x,t) \\ & \geq 0 \quad (\text{ since } \textstyle\sum_{j=1}^{2} a_{ij}(x,t) > \alpha \\ & \quad \text{and } |w_i(x,t)| \leq CB_{1,2}(x)). \end{aligned}$$

Thus by Lemma 3.1 in [9], $\psi_i^{\pm}(x,t) \geq 0$ for all $(x,t) \in [0,1]\times[0,T]$.

$$\Rightarrow |\frac{\partial w_i}{\partial x}(x,t)| \leq C\varepsilon_i^{\frac{-1}{2}} B_{1,2}(x). \tag{3.12}$$

The bounds on $\frac{\partial^2 w_i}{\partial x^2}(x,t)$ and $\frac{\partial^3 w_i}{\partial x^3}(x,t)$ are derived by similar arguments.

To obtain the bound for $\frac{\partial w_i}{\partial t}(x,t)$, define the two functions

$$\theta_i^{\pm}(x,t) = CB_{1,2}(x) \pm \frac{\partial w_i}{\partial t}(x,t).$$

Differentiating the homogeneous equation satisfied by w_i, partially with respect to t and rearranging yields

$$\begin{aligned} \frac{\partial^2 w_i}{\partial t^2}(x,t) - \varepsilon_i \frac{\partial^3 w_i}{\partial x^2 \partial t}(x,t) + \sum_{j=1}^{2} a_{ij}(x,t)\frac{\partial w_j}{\partial t}(x,t) \\ = -\sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial t}(x,t) w_j(x,t) \end{aligned} \tag{3.13}$$

and we get
$|L_1 \frac{\partial w_i}{\partial t}(x,t)| \leq CB_{1,2}(x)$
$|\frac{\partial w_i}{\partial t}(0,t)| \leq |\frac{\partial u_i}{\partial t}(0,t)| + |\frac{\partial v_i}{\partial t}(0,t)| \leq C, |\frac{\partial w_i}{\partial t}(1,t)| = C,$
$|\frac{\partial w_i}{\partial t}(x,0)| = 0$ as $w_i(0,t) = w(0), w_i(1,t) = K_1$
and $w_i(x,0) = 0$.
By Lemma 3.2 in [9], for a proper choice of C, it follows that

$$|\frac{\partial w_i}{\partial t}(x,t)| \leq CB_{1,2}(x). \tag{3.14}$$

Now the bound for $\frac{\partial^2 w_i}{\partial x \partial t}$ is obtained by using Lemma 3.3 and Lemma 3.4

$$|\frac{\partial^2 w_i}{\partial x \partial t}(x,t)| \leq |\frac{\partial^2 u_i}{\partial x \partial t}(x,t)| + |\frac{\partial^2 v_i}{\partial x \partial t}(x,t)|$$

$$|\frac{\partial^2 w_i}{\partial x \partial t}(x,t)| \leq C\varepsilon_i^{\frac{-1}{2}}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}|| + ||\frac{\partial^2 f_i}{\partial t^2}||).$$

Similarly,

$$|\frac{\partial^3 w_i}{\partial x^2 \partial t}(x,t)| \leq C\varepsilon_i^{-1}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}|| + ||\frac{\partial^2 f_i}{\partial t^2}||). \tag{3.15}$$

Using (3.11), (3.14) and (3.15) in (3.13),

$$|\frac{\partial^2 w_i}{\partial t^2}(x,t)| \leq C.$$

We now derive the bound on w_i on $[1,2] \times [0,T]$. From the defining equation for w_i, we have

$$L_2 w_i(x,t) = \frac{\partial w_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 w_i}{\partial x^2}(x,t) + \sum_{j=1}^{2} a_{ij}(x,t) w_j(x,t) + b_i(x,t) w_i(x-1,t) = 0 \tag{3.16}$$

or

$$L_1 w_i(x) = \frac{\partial w_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 w_i}{\partial x^2}(x,t) + \sum_{j=1}^{2} a_{ij}(x,t) w_j(x,t) = -b_i(x,t) w_i(x-1,t) \tag{3.17}$$

$$\Rightarrow |L_1 w_i(x,t)| \leq CB_{1,2}(x-1) = CB_{2,2}(x) \tag{3.18}$$

Also, $w_i(1,t) = K_1, w_i(2,t) = 0$.
Consider the differential equation

$$L_2 w_i = 0, x \in (1,2) \times (0,T].$$

Then $\frac{\partial^2 w_i}{\partial x^2}(x,t) = \varepsilon_i^{-1}(\frac{\partial w_i}{\partial t}(x,t) + \sum_{j=1}^{2} a_{ij}(x,t) w_j(x,t) + b_i(x,t) w_i(x-1,t))$.

Hence

$$|\frac{\partial^2 w_i}{\partial x^2}(x,t)| \leq C\varepsilon_i^{-1}(B_{2,2}(x) + B_{2,2}(x) + B_{1,2}(x-1)) \leq C\varepsilon_i^{-1} B_{2,2}(x) \text{ (since } B_{1,2}(x-1) = B_{2,2}(x) \text{ for } x \in [1,2]).$$

Using the mean value theorem and the bound of $w_i(x,t)$, arguments similar to those used to bound $\frac{\partial u_i}{\partial x}$ lead to the bound of $\frac{\partial w_i}{\partial x}(x,t)$. The bounds on $\frac{\partial^3 w_i}{\partial x^3}(x,t)$ and $\frac{\partial^4 w_i}{\partial x^4}(x,t)$ are derived similarly.
To obtain the bound for $\frac{\partial w_i}{\partial t}(x,t)$, define the two functions

$$\theta_i^{\pm}(x,t) = CB_{2,2}(x) \pm \frac{\partial w_i}{\partial t}(x,t).$$

Differentiating the homogeneous equation satisfied by w_i, partially with respect to t and rearranging yields

$$\frac{\partial^2 w_i}{\partial t^2}(x,t) - \varepsilon_i \frac{\partial^3 w_i}{\partial x^2 \partial t}(x,t) + \sum_{j=1}^{2} a_{ij}(x,t) \frac{\partial w_j}{\partial t}(x,t) + b_i(x,t) \frac{\partial w_i}{\partial t}(x-1,t) = -\sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial t}(x,t) w_j(x,t) - \frac{\partial b_i}{\partial t}(x,t) w_i(x-1,t) \tag{3.19}$$

and we get

$$|L_2 \frac{\partial w_i}{\partial t}(x,t)| \leq C(B_{2,2}(x) + B_{1,2}(x-1))$$

$|\frac{\partial w_i}{\partial t}(1,t)| \leq |\frac{\partial u_i}{\partial t}(1,t)| + |\frac{\partial v_i}{\partial t}(1,t)| \leq C, |\frac{\partial w_i}{\partial t}(2,t)| = 0,$
$|\frac{\partial w_i}{\partial t}(x,0)| = 0$ as $w_i(1,t) = K_1, w_i(2,t) = 0$ and $w_i(x,0) = 0$.
By Lemma 3.2 in [9], for a proper choice of C, it follows that

$$|\frac{\partial w_i}{\partial t}(x,t)| \leq CB_{2,2}(x). \tag{3.20}$$

Now the bound for $\frac{\partial^2 w_i}{\partial x \partial t}$ is obtained by using Lemma 3.3 and Lemma 3.4

$$|\frac{\partial^2 w_i}{\partial x \partial t}(x,t)| \leq |\frac{\partial^2 u_i}{\partial x \partial t}(x,t)| + |\frac{\partial^2 v_i}{\partial x \partial t}(x,t)|$$

$$|\frac{\partial^2 w_i}{\partial x \partial t}(x,t)| \leq C\varepsilon_i^{\frac{-1}{2}}(||u_i||_\Gamma + ||f|| + ||\frac{\partial f_i}{\partial t}|| + ||\frac{\partial^2 f_i}{\partial t^2}||).$$

Similarly,

$$|\frac{\partial^3 w_i}{\partial x^2 \partial t}(x,t)| \leq C\varepsilon_i^{-1}(||u_i||_\Gamma + ||f_i|| + ||\frac{\partial f_i}{\partial t}|| + ||\frac{\partial^2 f_i}{\partial t^2}||). \tag{3.21}$$

Using (3.18), (3.20) and (3.21) in (3.19),

$$|\frac{\partial^2 w_i}{\partial t^2}(x,t)| \leq C.$$

□

4. Improved estimates

In the following lemma, sharper estimates of the smooth component are presented.

Lemma 4.1. *Let conditions (2.2) and (2.3) hold. Then the smooth component $\vec{v}$ of the solution $\vec{u}$ of (2.1) satisfies for $i=1,2, k=0,1,2,3$ and for all $(x,t)\in[0,1-]\times[0,T]$,*

$|\frac{\partial v_i^k}{\partial x^k}(x,t)| \le C\left(1+\sum_{q=i}^{2}\frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right)$

and for $(x,t)\in[1+,2]\times[0,T]$,

$|\frac{\partial v_i^k}{\partial x^k}(x,t)| \le C\left(1+\sum_{q=i}^{2}\frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right).$

Proof. Define barrier functions $\psi_i^{\pm}(x,t)=C(1+B_{1,2}(x))\pm\frac{\partial^k v_i}{\partial x^k}(x,t), i=1,2, k=0,1,2$ and $(x,t)\in[0,1-]\times[0,T]$. Using Lemma 3.4, it follows that, for proper choice of C,

$$\psi_i^{\pm}(0,t)=C\pm\frac{\partial^k v_i}{\partial x^k}(0,t)\ge 0$$

$$\psi^{\pm}(1-,t)=C\pm\frac{\partial^k v_i}{\partial x^k}(1-,t)\ge 0,$$

$$\psi^{\pm}(x,0)=C[1+B_{1,2}(x)]\pm\frac{\partial^k v_i}{\partial x^k}\ \ge\ 0$$

and $L\psi_i^{\pm}(x,t)\ge 0$ by lemma 3.2 in [9],

$$|\frac{\partial^k v_i}{\partial x^k}(x,t)|\le C[1+B_{1,2}(x)] \text{ for } k=0,1,2. \tag{4.1}$$

Consider the equation

$$L\frac{\partial^2 v_i}{\partial x^2}(x,t)=\frac{\partial^2 f_i}{\partial x^2}(x,t)-2\sum_{j=1}^{2}\frac{\partial a_{ij}}{\partial x}(x,t)\frac{\partial v_j}{\partial x}(x,t)-\sum_{j=1}^{2}\frac{\partial^2 a_{ij}}{\partial x^2}(x,t)v_j(x,t)-2\frac{\partial b_i}{\partial x}(x,t)\frac{\partial v_i}{\partial x}(x-1,t)-\frac{\partial^2 b_i}{\partial x^2}(x,t)v_i(x-1,t) \tag{4.2}$$

with

$$\frac{\partial^2 v_i}{\partial x^2}(0,t)=0,\frac{\partial^2 v_i}{\partial x^2}(1-,t)=0,\frac{\partial^2 \vec{v}}{\partial x^2}(x,0)=\frac{\partial^2\vec{\phi}_B(x)}{\partial x^2}. \tag{4.3}$$

For convenience, let $\vec{p}$ denote $\dfrac{\partial^2\vec{v}}{\partial x^2}$. Then

$$\vec{L}\vec{p}=\vec{h} \text{ with } \vec{p}(0,t)=0,\vec{p}(1-,t)=0,\vec{p}(x,0)=\vec{s} \tag{4.4}$$

where

$$h_i=\frac{\partial^2 f_i}{\partial x^2}(x,t)-2\sum_{j=1}^{2}\frac{\partial a_{ij}}{\partial x}(x,t)\frac{\partial v_j}{\partial x}(x,t)-\sum_{j=1}^{2}\frac{\partial^2 a_{ij}}{\partial x^2}(x,t)v_j(x,t)-2\frac{\partial b_i}{\partial x}(x,t)\frac{\partial v_i}{\partial x}(x-1,t)-\frac{\partial^2 b_i}{\partial x^2}(x,t)v_i(x-1,t) \text{ and } \vec{s}=\frac{\partial^2\vec{\phi}_B(x)}{\partial x^2}.$$

Let $\vec{q}$ and $\vec{r}$ be the smooth and singular components of $\vec{p}$ given by

$$\vec{L}\vec{q}=\vec{h} \text{ with } \vec{q}(0,t)=\vec{p}_0(0,t),\vec{q}(1-,t)=\vec{p}_0(1,t),\ \vec{q}(x,0)=\vec{p}(x,0) \tag{4.5}$$

where $\vec{p}_0$ is the solution of the reduced problem

$$\frac{\partial\vec{p}_0}{\partial t}+A\vec{p}_0=\vec{g} \text{ with } \vec{p}_0(x,0)=\vec{p}(x,0)=\vec{s}$$

and

$$\vec{L}\vec{r}=\vec{0}, \text{ with } \vec{r}(0,t)=-\vec{q}(0,t),\vec{r}(1-,t)=-\vec{q}(1-,t),\ \vec{r}(x,0)=\vec{0}. \tag{4.6}$$

Using Lemma 3.4 and Lemma 3.6, it follows that, for $i=1,2$ and $(x,t)\in[0,1-]\times[0,T]$,

$$|\frac{\partial q_i}{\partial x}(x,t)|\le C$$

and

$$|\frac{\partial r_i}{\partial x}(x,t)|\le C\left[\frac{B_{1,1}(x)}{\sqrt{\varepsilon_1}}+\frac{B_{1,2}(x)}{\sqrt{\varepsilon_2}}.\right]$$

Hence, for $(x,t)\in[0,1-]\times[0,T]$ and $i=1,2$

$$|\frac{\partial^3 v_i}{\partial x^3}(x,t)|=|\frac{\partial p_i}{\partial x}(x,t)|\le C\left[1+\frac{B_{1,1}(x)}{\sqrt{\varepsilon_1}}+\frac{B_{1,2}(x)}{\sqrt{\varepsilon_2}}\right]. \tag{4.7}$$

Then (4.1) and (4.7), for $k=0,1,2,3$ and $(x,t)\in[0,1-]\times[0,T]$, lead to

$$|\frac{\partial^k v_i}{\partial x^k}(x,t)|\le C\left[1+\varepsilon_1^{1-\frac{k}{2}}B_{1,1}(x)+\varepsilon_2^{1-\frac{k}{2}}B_{1,2}(x)\right]$$

The bounds on $\vec{v}$ and its derivatives are similarly derived when $(x,t)\in[1+,2]\times[0,T]$. □

5. The Shishkin mesh

A piecewise uniform Shishkin mesh with $M\times N$ mesh-intervals is now constructed.
Let $\Omega_t^M=\{t_k\}_{k=1}^M,\overline{\Omega}_t^M=\{t_k\}_{k=0}^M,\Omega_x^N=\{x_j\}_{j=1}^{N-1}$, $\overline{\Omega}_x^N=\{x_j\}_{j=0}^N,\Omega^{M,N}=\Omega_t^M\times\Omega_x^N,\overline{\Omega}^{M,N}=\overline{\Omega}_t^M\times\overline{\Omega}_x^N$, $\Omega_x^{-N}=\{x_j\}_{j=1}^{\frac{N}{2}-1},\overline{\Omega}_x^{-N}=\{x_j\}_{j=0}^{\frac{N}{2}},\Omega_x^{+N}=\{x_j\}_{j=\frac{N}{2}+1}^{N-1}$, $\overline{\Omega}_x^{+N}=\{x_j\}_{j=\frac{N}{2}}^{N},\Omega^{-M,N}=\Omega_t^M\times\Omega_x^{-N},\overline{\Omega}^{-M,N}=\overline{\Omega}_t^M\times\overline{\Omega}_x^{-N}$, $\Omega^{+M,N}=\Omega_t^M\times\Omega_x^{+N},\overline{\Omega}^{+M,N}=\overline{\Omega}_t^M\times\overline{\Omega}_x^{+N}$ and $\Gamma^{M,N}=\Gamma\cap\overline{\Omega}^{M,N}$. The mesh $\overline{\Omega}_t^M$ is chosen to be a uniform mesh with M mesh-intervals on $[0,T]$. The mesh $\overline{\Omega}_x^N$ is chosen to be a piecewise-uniform mesh with N mesh-intervals on $[0,2]$. The interval $[0,1]$ is divided into 5 sub-intervals as follows

$$[0,\tau_1]\cup(\tau_1,\tau_2]\cup(\tau_2,1-\tau_2]\cup(1-\tau_2,1-\tau_1]\cup(1-\tau_1,1].$$

The parameters τ_1, τ_2, which determine the points separating the uniform meshes, are defined by

$$\tau_2 = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_2}}{\sqrt{\alpha}}\ln N\right\} \text{ and } \quad \tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\sqrt{\varepsilon_1}}{\sqrt{\alpha}}\ln N\right\}. \tag{5.1}$$

Then, on the sub-interval $(\tau_2, 1-\tau_2]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $[0,\tau_1], (\tau_1,\tau_2], (1-\tau_2, 1-\tau_1]and(1-\tau_1, 1]$, a uniform mesh of $\frac{N}{16}$ mesh points is placed.
Similarly, the interval $(1,2]$ is also divided into 5 sub-intervals $(1, 1+\tau_1], (1+\tau_1, 1+\tau_2], (1+\tau_2, 2-\tau_2], (2-\tau_2, 2-\tau_1]$ and $(2-\tau_1, 2]$, having a total of $\frac{N}{2}$ mesh points, using the same parameters τ_1 and τ_2. In particular, when both the parameters τ_1 and τ_2 take on their lefthand value, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh throughout from 0 to 2.
In practice, it is convenient to take

$$N = 16k, k \geq 2. \tag{5.2}$$

From the above construction of $\overline{\Omega}^N$, it is clear that the transition points $\{\tau_r, 1-\tau_r, 1+\tau_r, 2-\tau_r\}, r = 1,2$, are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}$ and if $x_j = \tau$, then $h_j^- = x_j - x_{j-1}, h_j^+ = x_{j+1} - x_j, J = \{x_j : h_j^+ \neq h_j^-\}$.

6. The discrete problem

In this section, a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (2.1), which is shown later to be first order parameter-uniform convergent in time and essentially first order parameter-uniform convergent in the space variable.
The discrete two-point boundary value problem is now defined on any mesh by the finite difference method

$$\begin{aligned}\vec{L}^{M,N}\vec{U}(x_j,t_k) &= D_t^-\vec{U}(x_j,t_k) - E\delta_x^2\vec{U}(x_j,t_k) + \\ &\quad A(x_j,t_k)\vec{U}(x_j,t_k) + B(x_j,t_k)\vec{U}(x_j-1,t_k) \\ &= \vec{f}(x_j,t_k) \text{ on } \Omega^{M,N}\end{aligned} \tag{6.1}$$

$$\vec{U} = \vec{u} \text{ on } \Gamma^{M,N}$$

The problem (6.1) can be rewritten as

$$\begin{aligned}\vec{L_1}^{M,N}\vec{U}(x_j,t_k) &= D_t^-\vec{U}(x_j,t_k) - E\delta_x^2\vec{U}(x_j,t_k) + \\ &\quad A(x_j,t_k)\vec{U}(x_j,t_k) \\ &= \vec{g}(x_j,t_k) \text{ on } \Omega^{-M,N}\end{aligned} \tag{6.2}$$

where $\vec{g}(x_j,t_k) = \vec{f}(x_j,t_k) - B(x_j,t_k)\vec{\chi}(x_j-1,t_k)$

$$\begin{aligned}\vec{L_2}^{M,N}\vec{U}(x_j,t_k) &= D_t^-\vec{U}(x_j,t_k) - E\delta_x^2\vec{U}(x_j,t_k) + \\ &\quad A(x_j,t_k)\vec{U}(x_j,t_k) + B(x_j,t_k)\vec{U}(x_j-1,t_k) \\ &= \vec{f}(x_j,t_k) \text{ on } \Omega^{+M,N}\end{aligned} \tag{6.3}$$

$$\vec{U} = \vec{u} \text{ on } \Gamma^{M,N},\ D_x^-\vec{U}(x_{\frac{N}{2}},t_k) = D_x^+\vec{U}(x_{\frac{N}{2}},t_k)$$

where $D_t^-\vec{U}(x_j,t_k) = \frac{\vec{U}(x_j,t_k)-\vec{U}(x_j,t_{k-1})}{t_k-t_{k-1}}$,
$\delta_x^2\vec{U}(x_j,t_k) = \frac{D_x^+\vec{U}(x_j,t_k)-D_x^-\vec{U}(x_j,t_k)}{\frac{x_{j+1}-x_{j-1}}{2}}$,
$D_x^-\vec{U}(x_j,t_k) = \frac{\vec{U}(x_{j+1},t_k)-\vec{U}(x_j,t_k)}{x_{j+1}-x_j}$ and
$D_x^+\vec{U}(x_j,t_k) = \frac{\vec{U}(x_j,t_k)-\vec{U}(x_{j-1},t_k)}{x_j-x_{j-1}}$
This is used to compute numerical approximations to the exact solution of (2.1). The following discrete results are analogous to those for the continuous case.

Lemma 6.1. *Let conditions (2.2) and (2.3) hold. Then, for any mesh function $\vec{\Psi}(x_j,t_k), 0 \leq j \leq N, 0 \leq k \leq M$, the inequalities $\vec{\Psi} \geq \vec{0}$ on $\Gamma^{M,N}, \vec{L_1}^{M,N}\vec{\Psi}(x_j,t_k) \geq \vec{0}$, on $\Omega^{-M,N}$, $\vec{L_2}^{M,N}\vec{\Psi}(x_j,t_k) \geq \vec{0}$ on $\Omega^{+M,N}$ and $D_x^+\vec{\Psi}(x_{N/2},t_k) - D_x^-\vec{\Psi}(x_{N/2},t_k) \leq \vec{0}$ imply that $\vec{\Psi}(x_j,t_k) \geq \vec{0}$ on $\overline{\Omega}^{M,N}$.*

Proof. Let i^*, j^*, k^* be such that $\Psi_{i^*}(x_{j^*},t_{k^*}) = \min_{i,j,k} \Psi_i(x_j,t_k)$ and assume that the lemma is false. Then $\Psi_{i^*}(x_{j^*},t_{k^*}) < 0$. From the hypothesis it is clear that $(x_{j^*},t_{k^*}) \notin \Gamma^{M,N}$,
$\Psi_{i^*}(x_{j^*},t_{k^*}) - \Psi_{i^*}(x_{j^*},t_{k^*-1}) \leq 0$,
$\Psi_{i^*}(x_{j^*},t_{k^*}) - \Psi_{i^*}(x_{j^*-1},t_{k^*}) \leq 0$,
$\Psi_{i^*}(x_{j^*+1},t_{k^*}) - \Psi_{i^*}(x_{j^*},t_{k^*}) \geq 0$ so $\delta_x^2\Psi_{i^*}(x_{j^*},t_{k^*}) \geq 0$.
It follows that

$$(\vec{L_1}^{M,N}\vec{\Psi})_{i^*}(x_{j^*},t_{k^*}) < 0$$

which is a contradiction. If $(x_{j^*},t_{k^*}) \in \Omega^{+M,N}$, a similar argument shows that

$$(\vec{L_2}^{M,N}\vec{\Psi})_{i^*}(x_{j^*},t_{k^*}) < 0$$

which is a contradiction. Because of the boundary values, the only other possibility is that $x_{j^*} = x_{N/2}$. Then

$$D_x^-\Psi_{i^*}(x_{N/2},t_{k^*}) \leq 0 \leq D_x^+\Psi_{i^*}(x_{N/2},t_{k^*}) \leq D_x^-\Psi_{i^*}(x_{N/2},t_{k^*}),$$

by the hypothesis and so

$$\Psi_{i^*}(x_{\frac{N}{2}-1},t_{k^*}) = \Psi_{i^*}(x_{N/2},t_{k^*}) = \Psi_{i^*}(x_{\frac{N}{2}+1},t_{k^*}) < 0.$$

Then $(\vec{L_1}^{M,N}\vec{\Psi})_{i^*}(x_{\frac{N}{2}-1},t_{k^*}) < 0$, a contradiction. This concludes the proof of the lemma. □

An immediate consequence of this is the following discrete stability result.

Lemma 6.2. *Let conditions (2.2) and (2.3) hold. Then, for any mesh function $\vec{\Psi}$ satisfying $D_x^+\vec{\Psi}(x_{N/2},t_k) = D_x^-\vec{\Psi}(x_{N/2},t_k)$,*
$|\Psi_i(x_j,t_k)| \leq \max\{||\Psi_i||_{\Gamma^{M,N}}, \frac{1}{\alpha}||\vec{L_1}^{M,N}\Psi_i||_{\Omega^{-M,N}}, \frac{1}{\alpha}||\vec{L_2}^{M,N}\Psi_i||_{\Omega^{+M,N}}\}$,
for each $i = 1,2$ and $0 \leq j \leq N, 0 \leq k \leq M$.

Proof. Let $M = \max\{||\Psi_i||_{\Gamma^{M,N}}, \frac{1}{\alpha}||\vec{L}_1^{M,N}\Psi_i||_{\Omega^{-M,N}}, \frac{1}{\alpha}||\vec{L}_2^{M,N}\Psi_i||_{\Omega^{+M,N}}\}$.
Define two functions

$$\vec{\Theta}^{\pm}(x_j,t_k) = M\vec{e} \pm \vec{\Psi}(x_j,t_k) \text{ where } \vec{e} = (1,1)^T.$$

Using the properties of $A(x_j,t_k)$ and $B(x_j,t_k)$, it is not hard to find that $\vec{\Theta}^{\pm}(x_j,t_k) \geq \vec{0}$ for $\Gamma^{M,N}$,
$\vec{L}_1^{M,N}\vec{\Theta}^{\pm}(x_j,t_k) \geq \vec{0}$ for $(x_j,t_k) \in \Omega^{-M,N}$ and
$\vec{L}_2^{M,N}\vec{\Theta}^{\pm}(x_j,t_k) \geq 0$ for $(x_j,t_k) \in \Omega^{+M,N}$. At $j = \frac{N}{2}$,

$$D_x^+\vec{\Theta}^{\pm}(x_{N/2},t_k) - D_x^-\vec{\Theta}^{\pm}(x_{N/2},t_k) = \pm D_x^+\vec{\Psi}(x_{N/2},t_k) - D_x^-\vec{\Psi}(x_{N/2},t_k) = \vec{0}.$$

Hence by Lemma 6.1, $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}^{M,N}$. □

7. Error estimate

Analogous to the continuous case, the discrete solution $\vec{U}$ can be decomposed into $\vec{V}$ and $\vec{W}$ which are defined to be the solutions of the following discrete problems

$$\vec{L}_1^{M,N}\vec{V}(x_j,t_k) = \vec{g}(x_j,t_k), (x_j,t_k) \in \Omega^{-M,N}, \quad 0 \leq j \leq \frac{N}{2} - 1, 0 \leq k \leq M \tag{7.1}$$

$$\vec{V}(0,t_k) = \vec{v}(0,t_k), \vec{V}(x_{N/2-1},t_k) = \vec{v}(1-,t_k), \vec{V}(x_j,0) = \vec{\phi}_B(x_j),$$

$$\vec{L}_2^{M,N}\vec{V}(x_j,t_k) = \vec{f}(x_j,t_k), (x_j,t_k) \in \Omega^{+M,N}, \quad \frac{N}{2} + 1 \leq j \leq N, 0 \leq k \leq M \tag{7.2}$$

$\vec{V}(x_{N/2+1},t_k) = \vec{v}(1+,t_k), \vec{V}(2,t_k) = \vec{v}(2,t_k), \vec{V}(x_j,0) = \vec{\phi}_B(x_j)$
and

$$\begin{aligned} &\vec{L}_1^{M,N}\vec{W}(x_j,t_k) = \vec{0}, (x_j,t_k) \in \Omega^{-M,N}, \vec{W}(0,t_k) = \vec{w}(0,t_k), \\ &\qquad 0 \leq j \leq \tfrac{N}{2} - 1, 0 \leq k \leq M \\ &\vec{L}_2^{M,N}\vec{W}(x_j,t_k) = \vec{0}, (x_j,t_k) \in \Omega^{+M,N}, \vec{W}(2,t_k) = \vec{w}(2,t_k), \\ &\qquad \tfrac{N}{2} + 1 \leq j \leq N, 0 \leq k \leq M \end{aligned} \tag{7.3}$$

$\vec{V}(x_{N/2+1},t_k) + \vec{W}(x_{N/2+1},t_k) = \vec{V}(x_{N/2-1},t_k) + \vec{W}(x_{N/2-1},t_k)$,

$$D_x^-\vec{W}(x_{N/2},t_k) + D_x^-\vec{V}(x_{N/2},t_k) = D_x^+\vec{W}(x_{N/2},t_k) + D_x^+\vec{V}(x_{N/2},t_k).$$

$\vec{W}(x_j,0) = \vec{0}$.
The error at each point $(x_j,t_k) \in \overline{\Omega}^{M,N}$ is denoted by $\vec{e}(x_j,t_k) = \vec{U}(x_j,t_k) - \vec{u}(x_j,t_k)$.
Then the local truncation error $\vec{L}^{M,N}\vec{e}(x_j,t_k)$, for $j \neq N/2$, has the decomposition

$$\vec{L}^{M,N}\vec{e}(x_j,t_k) = \vec{L}^{M,N}(\vec{V} - \vec{v})(x_j,t_k) + \vec{L}^{M,N}(\vec{W} - \vec{w})(x_j,t_k).$$

The error in the smooth and singular components are bounded in the following theorem.

Lemma 7.1. *Let $\vec{v}(x_j,t_k)$ denote the smooth component of the exact solution from (2.1) and $\vec{V}(x_j,t_k)$ the smooth component of the solution from (6.1), then for $j \neq \frac{N}{2}$*

$$||(\vec{L}_1^{M,N}(\vec{V} - \vec{v}))_i(x_j,t_k)|| \leq C(M^{-1} + (N^{-1}\ln N)^2), \quad 0 \leq j \leq \frac{N}{2} - 1, 0 \leq k \leq M, \tag{7.4}$$

$$||(\vec{L}_2^{M,N}(\vec{V} - \vec{v}))_i(x_j,t_k)|| \leq C(M^{-1} + (N^{-1}\ln N)^2), \quad \frac{N}{2} + 1 \leq j \leq N, 0 \leq k \leq M. \tag{7.5}$$

Let $\vec{w}(x_j,t_k)$ denote the singular component of the exact solution from (2.1) and $\vec{W}(x_j,t_k)$ the singular component of the solution from (6.1), then for $j \neq \frac{N}{2}$

$$||(\vec{L}_1^{M,N}(\vec{W} - \vec{w}))_i(x_j,t_k)|| \leq C(M^{-1} + (N^{-1}\ln N)^2), \quad 0 \leq j \leq \frac{N}{2} - 1, 0 \leq k \leq M, \tag{7.6}$$

$$||(\vec{L}_2^{M,N}(\vec{W} - \vec{w}))_i(x_j,t_k)|| \leq C(M^{-1} + (N^{-1}\ln N)^2), \quad \frac{N}{2} + 1 \leq j \leq N, 0 \leq k \leq M. \tag{7.7}$$

Proof. For $j \neq \frac{N}{2}$, as the expression derived for the local truncation error in $\vec{V}$ and $\vec{W}$ and estimates for the derivatives of the smooth and singular components are exactly in the form found in [9], the required bounds hold good.
At the point $x_j = x_{N/2}$,

$$(D_x^+ - D_x^-)\vec{e}(x_{N/2},t_k) = (D_x^+ - D_x^-)(\vec{U} - \vec{u})(x_{N/2},t_k), \quad 0 \leq k \leq M$$

Recall that $(D_x^+ - D_x^-)\vec{U}(x_{N/2},t_k) = 0$.
Let $h^* = h_{N/2}^- = h_{N/2}^+$, where $h_{N/2}^- = x_{N/2} - x_{N/2-1}$ and $h_{N/2}^+ = x_{N/2+1} - x_{N/2}$.
Then

$$\begin{aligned} |(D_x^+ - D_x^-)\vec{e}(x_{N/2},t_k)| &= |(D_x^+ - D_x^-)\vec{u}(x_{N/2},t_k)| \\ &\leq |(D_x^+ - \frac{\partial}{\partial x})\vec{u}(x_{N/2},t_k)| + |(D_x^- - \frac{\partial}{\partial x})\vec{u}(x_{N/2},t_k)| \\ &\leq \frac{1}{2}h_{N/2}^+ \max_{\eta_1 \in (1,2)} |\frac{\partial^2\vec{u}}{\partial x^2}(\eta_1,t_k)| + \frac{1}{2}h_{N/2}^- \max_{\eta_2 \in (0,1)} |\frac{\partial^2\vec{u}}{\partial x^2}(\eta_2,t_k)| \\ &\leq Ch^* \max_{x \in (0,1)\cup(1,2)} \left|\frac{\partial^2\vec{u}}{\partial x^2}(x,t)\right|. \end{aligned}$$

Therefore,

$$|(D_x^+ - D_x^-)\vec{e}(x_{N/2},t_k)| \leq C\frac{h^*}{\varepsilon}. \quad (7.8)$$

Define, for $i = 1,2$ and for each t_k, a set of discrete barrier functions on $\overline{\Omega}^{M,N}$ by

$$\omega_i(x_j,t_k) = \frac{\Pi_{q=1}^{j}(1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_i})}{\Pi_{q=1}^{N/2}(1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_i})}, 0 \leq j \leq N/2 \quad (7.9)$$

$$\frac{\Pi_{q=j}^{N-1}(1+\sqrt{\alpha}h_{q+1}/\sqrt{2\varepsilon_i})}{\Pi_{q=N/2}^{N-1}(1+\sqrt{\alpha}h_{q+1}/\sqrt{2\varepsilon_i})}, N/2 \leq j \leq N. \quad (7.10)$$

Note that

$$\omega_i(0,t_k) = 0,\ \omega_i(1,t_k) = 1,\ \omega_i(2,t_k) = 0 \quad (7.11)$$

and from (7.9), for any i and $0 \leq j \leq N/2$,

$$\omega_i(x_j,t_k) = \frac{1}{\prod_{q=j+1}^{N/2}\left(1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_i}\right)},$$

$$\omega_{i+1}(x_j,t_k) = \frac{1}{\prod_{q=j+1}^{N/2}\left(1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_{i+1}}\right)},$$

$\frac{1}{1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_i}} < \frac{1}{1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_{i+1}}}$ implies that, for any i and $0 \leq j \leq N/2$,

$$\omega_i(x_j,t_k) < \omega_{i+1}(x_j,t_k) \quad (7.12)$$

Similarly, for any i and $N/2 \leq j \leq N$, (7.12) holds.

$$0 \leq \omega_i(x_j,t_k) < \omega_{i+1}(x_j,t_k) \leq 1. \quad (7.13)$$

For $(x_j,t_k) \in \overline{\Omega}^{-M,N}$

$$D_x^+\omega_i(x_j,t_k) = \frac{\omega_i(x_{j+1},t_k) - \omega_i(x_j,t_k)}{h_{j+1}} = \sqrt{\alpha/2\varepsilon_i}\,\omega_i(x_j,t_k).$$

Therefore,

$$D_x^+\omega_i(x_j,t_k) = \sqrt{\alpha/2\varepsilon_i}\,\omega_i(x_j,t_k). \quad (7.14)$$

$$\begin{aligned} D_x^-\omega_i(x_j,t_k) &= \frac{\omega_i(x_j,t_k) - \omega_i(x_{j-1},t_k)}{h_j} \\ &= \frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}(1+\sqrt{\alpha}h_j/\sqrt{2\varepsilon_i})}\omega_i(x_j,t_k). \end{aligned}$$

Therefore,

$$D_x^-\omega_i(x_j,t_k) = \frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}(1+\sqrt{\alpha}h_j/\sqrt{2\varepsilon_i})}\omega_i(x_j,t_k). \quad (7.15)$$

$$\delta_x^2\omega_i(x_j,t_k) = \frac{D_x^+\omega_i(x_j,t_k) - D_x^-\omega_i(x_j,t_k)}{(h_j+h_{j+1})/2} \leq \frac{\alpha}{\varepsilon_i}\omega_i(x_j,t_k).$$

Therefore,

$$\delta_x^2\omega_i(x_j,t_k) \leq \frac{\alpha}{\varepsilon_i}\omega_i(x_j,t_k). \quad (7.16)$$

Similarly, for $(x_j,t_k) \in \overline{\Omega}^{+M,N}$

$$\begin{aligned} D_x^+\omega_i(x_j,t_k) &= -\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}(1+\sqrt{\alpha}h_{j+1}/\sqrt{2\varepsilon_i})}\omega_i(x_j,t_k), \\ D_x^-\omega(x_j,t_k) &= -\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon_i}}\omega_i(x_j,t_k) \text{and} \\ \delta_x^2\omega_i(x_j,t_k) &\leq \frac{\alpha}{\varepsilon_i}\omega_i(x_j,t_k). \end{aligned} \quad (7.17)$$

In particular, at $x_j = x_{N/2}$, using (7.17), (7.15) and (7.11),

$$\begin{aligned} (D_x^+ - D_x^-)\omega_i(x_j,t_k) &= -\sqrt{\alpha/2\varepsilon_i}\frac{1}{(1+\sqrt{\alpha}h_{N/2}^+/\sqrt{2\varepsilon_i})} \\ &\quad -\sqrt{\alpha/2\varepsilon_i}\frac{1}{(1+\sqrt{\alpha}h_{N/2}^-/\sqrt{2\varepsilon_i})} \\ &\leq -\frac{C}{\sqrt{\varepsilon_i}}. \end{aligned} \quad (7.18)$$

From (7.16) and (7.17),

$$-\varepsilon_i\delta_x^2\omega_i(x_j,t_k) \geq -\alpha\omega_i(x_j,t_k).$$

Therefore

$$\begin{aligned} (\vec{L}_1^{M,N}\vec{\omega})_i(x_j,t_k) &= D_t^-\omega_i(x_j,t_k) - \varepsilon_i\delta_x^2\omega_i(x_j,t_k) \\ &\quad + \sum_{l=1}^{n} a_{il}(x_j,t_k)\omega_l(x_j,t_k) \\ &> -\alpha\omega_i(x_j,t_k) + \sum_{l=1}^{i} a_{il}(x_j,t_k)\omega_i(x_j,t_k) \\ &\quad + \sum_{l=i+1}^{2} a_{il}(x_j,t_k). \end{aligned} \quad (7.19)$$

And

$$\begin{aligned} (\vec{L}_2^{M,N}\vec{\omega})_i(x_j,t_k) &= D_t^-\omega_i(x_j,t_k) - \varepsilon_i\delta_x^2\omega_i(x_j,t_k) \\ &\quad + \sum_{l=1}^{2} a_{il}(x_j,t_k)\omega_l(x_j,t_k) \\ &\quad + b_i(x_j,t_k)\omega_i(x_j-1,t_k) \\ &\geq -\alpha\omega_i(x_j,t_k) + \sum_{l=1}^{i} a_{il}(x_j,t_k)\omega_i(x_j,t_k) \\ &\quad + \sum_{l=i+1}^{2} a_{il}(x_j,t_k) + b_i(x_j,t_k). \end{aligned} \quad (7.20)$$

□

We now state and prove the main theoretical result of this paper.

Lemma 7.2. *Let $\vec{u}(x_j,t_k)$ denote the exact solution of (2.1) and $\vec{U}(x_j,t_k)$ the solution of (6.1). Then, for $0 \le j \le N, 0 \le k \le M$,*

$$||\vec{U}(x_j,t_k) - \vec{u}(x_j,t_k)|| \le C(M^{-1} + N^{-1}\ln N). \quad (7.21)$$

Proof: Consider the mesh function $\vec{\Psi}$ given by $\vec{\Psi}(x_j,t_k) = C_1(M^{-1}+N^{-1}\ln N) + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\omega_i(x_j,t_k) \pm e_i(x_j,t_k)$, $i = 1,2, 0 \le j \le N, 0 \le k \le M$, where C_1 and C_2 are constants. Then for $x_j \in \Omega_x^{-N}$,

$$(\vec{L_1}^{M,N}\vec{\Psi})_i(x_j,t_k) = C_1\sum_{j=1}^{2} a_{ij}(x_j,t_k)(M^{-1}+N^{-1}\ln N) + C_2\frac{h^*}{\sqrt{\varepsilon_i}}(\vec{L_1}^{M,N}\omega_i)_i(x_j,t_k) \pm (\vec{L_1}^{M,N}\vec{e})_i(x_j,t_k). \quad (7.22)$$

Using (7.19) in (7.22) and Theorem 7.1,

$$\begin{aligned}(\vec{L_1}^{M,N}\vec{\Psi})_i(x_j,t_k) \ge\ & C_1\sum_{j=1}^{2} a_{ij}(x_j,t_k)(M^{-1}+N^{-1}\ln N) \\ & + C_2\frac{h^*}{\sqrt{\varepsilon_i}}[-\alpha\omega_i(x_j,t_k) \\ & + \sum_{l=1}^{i} a_{il}(x_j,t_k)\omega_i(x_j,t_k) \\ & + \sum_{l=i+1}^{2} a_{il}(x_j,t_k)] \\ & \pm C(M^{-1}+N^{-1}\ln N) \\ =\ & C_1\sum_{j=1}^{2} a_{ij}(x_j,t_k)(M^{-1}+N^{-1}\ln N) + \\ & C_2\frac{h^*}{\sqrt{\varepsilon_i}}\left[\sum_{l=1}^{i} a_{il}(x_j,t_k) - \alpha\right]\omega_i(x_j,t_k) \\ & + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\sum_{l=i+1}^{2} a_{il}(x_j,t_k) \\ & \pm C(M^{-1}+N^{-1}\ln N),\end{aligned}$$

Let $\lambda_i(x_j) = \left(\sum_{l=1}^{i} a_{il}(x_j,t_k) - \alpha\right)\omega_i(x_j,t_k) + \sum_{l=i+1}^{2} a_{il}(x_j,t_k)$, $i = 1,2$. Then choosing $C_1 > C_2||\lambda|| + C$, $(\vec{L_1}^{M,N}\vec{\Psi})_i(x_j,t_k) \ge 0$, for $i = 1,2$.
For $x_j \in \Omega_x^{+N}$,

$$\begin{aligned}(\vec{L_2}^{M,N}\vec{\Psi})_i(x_j,t_k) =\ & C_1\left(\sum_{l=1}^{2} a_{il}(x_j,t_k) + b_i(x_j,t_k)\right) \\ & (M^{-1}+N^{-1}\ln N) \\ & + C_2\frac{h^*}{\sqrt{\varepsilon_i}}(\vec{L_2}^{M,N}\omega_i)_i(x_j,t_k) \\ \pm\ & (\vec{L_2}^{M,N}\vec{e})_i(x_j,t_k). \quad (7.23)\end{aligned}$$

Using (7.20) in (7.23) and Theorem 9,

$$\begin{aligned}(\vec{L_2}^{M,N}\vec{\Psi})_i(x_j,t_k) \ge\ & C_1\left(\sum_{l=1}^{2} a_{il}(x_j,t_k) + b_i(x_j,t_k)\right) \\ & (M^{-1}+N^{-1}\ln N) \\ & + C_2\frac{h^*}{\sqrt{\varepsilon_i}}[-\alpha\omega_i(x_j,t_k) \\ & + \sum_{l=1}^{i} a_{il}(x_j,t_k)\omega_i(x_j,t_k) \\ & + \sum_{l=i+1}^{2} a_{il}(x_j,t_k) + b_i(x_j,t_k)] \\ & \pm C(M^{-1}+N^{-1}\ln N) \\ =\ & C_1\sum_{j=1}^{2} a_{ij}(x_j,t_k)(M^{-1}+N^{-1}\ln N) + \\ & C_2\frac{h^*}{\sqrt{\varepsilon_i}}\left[\sum_{l=1}^{i} a_{il}(x_j,t_k) - \alpha\right]\omega_i(x_j,t_k) \\ & + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\left[\sum_{l=i+1}^{2} a_{il}(x_j,t_k) + b_i(x_j,t_k)\right] \\ \pm\ & C(M^{-1}+N^{-1}\ln N).\end{aligned}$$

Let $\mu_i(x_j) = \left(\sum_{l=1}^{i} a_{il}(x_j,t_k) - \alpha\right)\omega_i(x_j,t_k) + \sum_{l=i+1}^{2} a_{il}(x_j,t_k)$ $+ b_i(x_j,t_k), i = 1,2$.
Then choosing $C_1 > C_2||\mu|| + C$, $(\vec{L_2}^{M,N}\vec{\Psi})_i(x_j,t_k) \ge 0$, for $i = 1,2$.
Further,

$$\begin{aligned}D_x^+\Psi_i(1,t_k) - D_x^-\Psi_i(1,t_k) \le\ & -C_2\frac{Ch^*}{\varepsilon_i} \pm C\frac{h^*}{\varepsilon_i}, \\ & \text{using (7.8) and (7.18)} \\ \le\ & 0, \text{ for proper choice of } C_2. \quad (7.24)\end{aligned}$$

Also, using (7.11), $\Psi_i(0,t_k) = C_1(M^{-1}+N^{-1}\ln N) \ge 0$, $\Psi_i(2,t_k) = C_1(M^{-1}+N^{-1}\ln N) \ge 0$, $\Psi_i(x_j,0) = C_1(M^{-1}+N^{-1}\ln N) \ge 0$.
Therefore, using Lemma 6.1 for $\vec{\Psi}$, it follows that $\Psi_i(x_j,t_k) \ge 0$ for all $i = 1,2, 0 \le j \le N, 0 \le k \le M$. As, from (7.13), $\omega_i(x_j,t_k) \le 1$ for $0 \le j \le N, 0 \le k \le M$

$$|(\vec{U} - \vec{u})(x_j,t_k)| \le C(M^{-1}+N^{-1}\ln N),$$

which completes the proof.

8. Numerical Illustration

The ε–uniform convergence of the numerical method proposed in this paper is illustrated through an example presented in this section.

Example 8.1. *For* $(x,t) \in [0,2] \times [0,T]$,
$\frac{\partial u_1}{\partial t}(x,t) - \varepsilon_1 \frac{\partial^2 u_1}{\partial x^2}(x,t) + 5u_1(x,t) - 2u_2(x,t) - u_1(x-1,t) = 1,$
$\frac{\partial u_2}{\partial t}(x,t) - \varepsilon_2 \frac{\partial^2 u_2}{\partial x^2}(x,t) - u_1(x,t) + 4u_2(x,t) - u_2(x-1,t) = 5$
$u_1(x,t) = 1$ for $x \in [-1,0] \times [0,T], u_1(0,t) = 1,$
$u_1(x,0) = 1, u_1(2,t) = 1,$
$u_2(x,t) = 1$ for $x \in [-1,0] \times [0,T], u_2(0,t) = 1,$
$u_2(x,0) = 1, u_2(2,t) = 1$

Fixing a fine Shishkin mesh with 32 points horizontally, the problem is solved by the method suggested above. The order of convergence and the error constant are calculated for t and the results are presented in Table 1. A fine uniform mesh on t with 32 points is considered and the order of convergence in the variable x is calculated. The results are presented in Table 2. A graph of the numerical solution is presented in the figure 1.

Figure 1

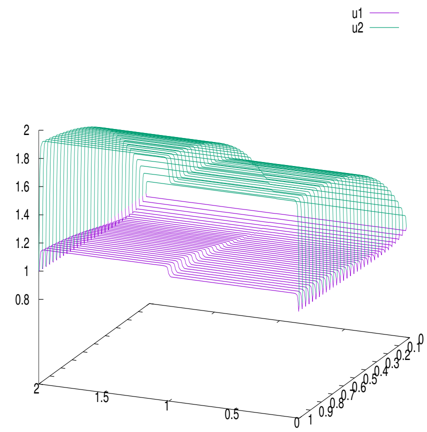

Table 1. Values of D^N, p^N, p^* and $C^N_{p^*}$ for $\varepsilon_1 = \eta/32, \varepsilon_2 = \eta/16$ and $\alpha = 0.9$

η	Number of mesh points N				
	64	128	256	512	1024
2^{-3}	0.411094E-02	0.207861E-02	0.104757E-02	0.526658E-03	0.264056E-03
2^{-6}	0.399695E-02	0.202648E-02	0.102043E-02	0.512040E-03	0.256480E-03
2^{-9}	0.399651E-02	0.202636E-02	0.102040E-02	0.512032E-03	0.256478E-03
2^{-12}	0.399635E-02	0.202632E-02	0.102039E-02	0.512029E-03	0.256477E-03
2^{-15}	0.399630E-02	0.202631E-02	0.102039E-02	0.512028E-03	0.256477E-03
D^N	0.411094E-02	0.207861E-02	0.104757E-02	0.526658E-03	0.264056E-03
p^N	0.983848E+00	0.988567E+00	0.992114E+00	0.996020E+00	
C^N_p	0.497616E+00	0.497616E+00	0.495991E+00	0.493157E+00	0.489014E+00
t-order of convergence= $0.983848E+00$					
The error constant= $0.497616E+00$					

Table 2. Values of D^N, p^N, p^* and $C^N_{p^*}$ for $\varepsilon_1 = \eta/32$, $\varepsilon_2 = \eta/16$ and $\alpha = 0.9$

η	Number of mesh points N				
	64	128	256	512	1024
2^{-3}	0.74375E-02	0.40016E-02	0.20374E-02	0.10228E-02	0.51181E-03
2^{-6}	0.23723E-01	0.95874E-02	0.55212E-02	0.28643E-02	0.14447E-02
2^{-9}	0.93245E-02	0.91796E-02	0.66289E-02	0.40901E-02	0.23577E-02
2^{-12}	0.93245E-02	0.91796E-02	0.66289E-02	0.40901E-02	0.23577E-02
2^{-15}	0.93245E-02	0.91796E-02	0.66289E-02	0.40901E-02	0.23577E-02
D^N	0.23723E-01	0.95874E-02	0.66289E-02	0.40901E-02	0.23577E-02
p^N	0.13071E+01	0.53237E+00	0.69663E+00	0.79475E+00	
C^N_p	0.70366E+00	0.41129E+00	0.41129E+00	0.36703E+00	0.30600E+00
x- order of convergence= $0.53237E+00$					
The error constant= $0.70366E+00$					

9. Conclusion

Thus in this paper, a linear parabolic system of singularly perturbed equations of reaction diffusion type with delay is considered and the suggested numerical method has been proved to be first order convergent, with respect to space and time, theoretically and numerically.

References

[1] A. Martin, S. Raun, Predetor-prey models with delay and prey harvesting, J. Math. Bio., 43, 2001, pp. 247-267.

[2] O. Arino, M.–L. Hbid, E. Ait Dads, Delay Differential Equations and Applications, Springer, The Netherlands, 2006.

[3] A. Asachenkov, G. Marchuk, R. Mohler, S. Zuew, Disease Dynamics, Birkhauser, Boston, 1994.

[4] Charles G. Lange and Robert M. Miura, *Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations* SIAM J. APPL. MATH. Vol. 42, No. 3, June 1982.

[5] Charles G. Lange and Robert M. Miura, *Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations II. Rapid Oscillations and Resonances* SIAM J. APPL. MATH. Vol. 45, No. 5, October 1985.

[6] Charles G. Lange and Robert M. Miura, *Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations III. Turning Point Problems* SIAM J. APPL. MATH. Vol. 45, No. 5, October 1985.

[7] Charles G. Lange and Robert M. Miura, *Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations. VI. Small Shifts with Rapid Oscillations* SIAM J. APPL. MATH. Vol. 54, No. 1, pp. 273-283, February 1994.

[8] J. J. H. Miller, E. O'Riordan, G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific Publishing Co., Singapore, New Jersey, London, Hong Kong (1996).

[9] V. Franklin, M. Paramasivam, J.J.H. Miller and S. Valarmathi, Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear parabolic system, International Journal of Numerical Analysis and Modeling Vol. 10, No. 1, pp. 178-202.

[10] M.Manikandan, N.Shivaranjani, J. J. H. Miller and S. Valarmathi, A parameter uniform first order convergent numerical method for a boundary value problem for a singularly perturbed delay differential equation, Advances in Applied Mathematics, Springer Proceedings in Mathematics and Statistics 87, pp. 71-88.

[11] Manikandan Mariappan, John J. H. Miller and Valarmathi Sigamani, A parameter-uniform first order convergent numerical method for a system of singularly perturbed second order delay differential equations, P.Knobloch(ed.), Boundary and Interior Layers, Computational and Asymptotic Methods - BAIL 2014, Lecture Notes in Computational Science and Engineering 108, Springer International Publishing Switzerland 2015.

[12] Parthiban Saminathan, Valarmathi Sigamani and Franklin Victor, Numerical Method for a Singularly Perturbed Boundary Value Problem for a Linear Parabolic Second Order Delay Differential Equation, Differential Equations and Numerical Analysis, Springer Proceedings in Mathematics and Statistics(2016), Volume172.

[13] A.R. Ansari, S.A. Bakr, and G.I. Shishkin, A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations, Journal of Computational and Applied Mathematics 205 (2007) 552 – 566

[14] Zhongdi Cen, A hybrid finite difference scheme for a class of singularly perturbed delay differential equations, Neural, Parallel and Scientific Computations 16, 303-308 (2008).

[15] J. J. H. Miller, E. O'Riordan, G.I. Shishkin, L.P. Shishkina, Fitted Mesh Methods for Problems with Parabolic Boundary Layers, Mathematical Proceedings of the Royal Irish Academy, 98A(2), 173 - 190 (1998).

[16] E.P. Doolan, J.J.H.Miller and W.H.A. Schilders, Uniform numerical methods for problems with initial and boundary layers, Boole Press, 1980.

[17] P.A. Farrell and A.F. Hegarty and J.J.H. Miller and E.O' Riordan and G.I. Shishkin, Robust computational techniques for boundary layers, Chapman and hall/CRC, Boca Raton, Florida,USA, 2000.

⋆⋆⋆⋆⋆⋆⋆⋆⋆

ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

⋆⋆⋆⋆⋆⋆⋆⋆⋆