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Estimation of certain integrals with extended multi-index Bessel function

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Abstract

This article is refers to the study of the generalized multiindex Bessel functions, which play a ubiquitous role in wide range of diverse fields such as acoustic field, electromagnetism, heat, hydrodynamics, wave motion, elasticity and optical science. Here we aim at presenting an extension of generalized multi-index Bessel function and established some interesting integral transforms involving the extended generalized multi-index Bessel function, which are expressed in terms of generalized Wright hypergeometric function ${}_r\Psi_s[\cdot]$ and Fox H-function. We also point out their relevance with known results.

Keywords

Generalized multiindex Bessel function, Fox-Wright function, Fox-H function and Integrals.

AMS Subject Classification

33C20, 33B15.

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Contents

1. Introduction

The Bessel function has gained importance and popularity due to its applications in the problem of wave propagation, cylindrical coordinate system, heat conduction in cylindrical object and static potential etc. In the recent years, some generalizations (unification) and number of integral transforms of Bessel functions have been given by many mathematicians and physicist as well as engineers for example: Choi et al. [3], Kiryakova [7], Kamarujjama and Khan [8], Khan et al. [9], Kamarujjama et al. [10], Khan et al. [12] and Khan and Ghayasuddin [13]. Recently, Choi and Agarwal [4] introduced and studied various properties of generalized multiindex Bessel function and then discussed an interesting unified integrals formulas involving the generalized multiindex Bessel function. Motivated by the above-mentioned work, in the present paper, we establish extension of generalized multiindex Bessel function and then evaluate new class of integrals involving extended generalized multiindex Bessel function, which are expressed in terms of Fox-Wright and Fox H-functions.

For the present study, we consider the following definitions:

Definition 1.1 The generalized Wright hypergeometric function ${}_r\Psi_s[x]$, also called Fox-Wright function (see [20], [21]) is defined as

$$r\Psi_s[x] = {}_r\Psi_s\left[\begin{matrix}(\gamma_1,\acute{\gamma}_1),....,(\gamma_r,\acute{\gamma}_s);\\ (l_1,\acute{l}_1),....,(l_r,\acute{l}_s);\end{matrix}\; x\right] \tag{1.1}$$

$$= \sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+\acute{\gamma}_1 k),.....,\Gamma(\gamma_r+\acute{\gamma}_s k)}{\Gamma(l_1+\acute{l}_1 k),.....,\Gamma(l_r+\acute{l}_s k)}\frac{x^k}{k!} \tag{1.2}$$

$$= H^{1,r}_{r,s+1}\left[-x\left|\begin{matrix}(1-\gamma_1,\acute{\gamma}_1),...,(1-\gamma_r,\acute{\gamma}_r)\\ (0,1),(1-l_1,\acute{l}_1),...,(1-l_s,\acute{l}_s)\end{matrix}\right.\right], \tag{1.3}$$

where $H^{1,r}_{r,s+1}[x]$ denotes the Fox H-function [6], coefficients $\gamma_1',\cdots,\gamma_r', l_1',\cdots,l_s' \in \mathbb{R}^+$ and the series absolutely converges for all $x\in\mathbb{C}$ when $1+\sum_{j=1}^{s} l_j' - \sum_{m=1}^{r}\gamma_m' > 0$.

When $\acute{\gamma}_1 =,....,= \acute{\gamma}_r = 1$, $\acute{l}_1 =,....,= \acute{l}_s = 1$ in (1.1), Fox-Wright function reduces to simpler generalized hypergeometric function ${}_rF_s[x]$ (see [21])

$$ {}_r\Psi_s\left[\begin{array}{c}(\gamma_1,\acute{\gamma}_1),....,(\gamma_r,\acute{\gamma}_s);\\(l_1,\acute{l}_1),....,(l_r,\acute{l}_s);\end{array}\; x\right] \tag{1.4}$$

$$ = \frac{\Gamma(\gamma)_1,...,\Gamma(\gamma)_r}{\Gamma(l)_1,...,\Gamma(l)_s}\; {}_rF_s(\gamma_1,....,\gamma_r;l_1,....,l_r;x). \tag{1.5}$$

Now we introduce and studies the extension of generalized multiindex Bessel, is called extended multiindex Bessel function as follows:

Definition 1.2 Let α_j, β_j, γ, δ, b, c $\in \mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^{m}\mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0, \mathbb{R}(\beta)_j>0$, $\mathbb{R}(\gamma)>0$ and $\mathbb{R}(\delta)>0$, then

$$ \mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(z) = \sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{kn}}{\prod\limits_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})(\delta)_n}(-z)^n, \quad m\in\mathbb{N}. \tag{1.6}$$

Here $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda\in\mathbb{C}$) by

$$ (\lambda)_n = \begin{cases} 1 \;\; (n=0) \\ \lambda(\lambda+1),....,(\lambda+n-1)\;(n\in\mathbb{N}) \end{cases}$$

$$ = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad (\lambda\in\mathbb{C}\;\mathbb{Z}_0^-) \tag{1.7}$$

and $\mathbb{Z}_0^-$ denotes the set of non positive integers and $\Gamma(\lambda)$ is the familiar Gamma function.

We enumerate the connections and special cases of extended generalized multiindex Bessel function with other families function in the literature of special functions ie., Bessel-maitland function, generalized Wright hypergeometric function, Fox-H function, generalized multiindex Mittag-Leffler function [18].

(i) If we put k=0, b=c=m=1, α_1=δ =1, $\beta_1=\nu$ and replace z by $z^2/4$ in (1.6), we get

$$ \mathbb{J}^{1,\gamma,1}_{\nu,0,1,1}(z) = \left(\frac{2}{z}\right)^{\nu}\mathbb{J}_\nu(z), \quad (z,\nu\in\mathbb{C};\mathbb{R}(\nu)>0), \tag{1.8}$$

where $\mathbb{J}(z)$ is Bessel function of order ν [17].

(ii) If we put k=0, m=1, δ=b=c=1 in (1.6), we get

$$ \mathbb{J}^{\alpha_1,\gamma,1}_{\beta_1,0,1,1}(z) = \mathbb{J}^{\alpha_1}_{\beta_1}(z), \quad (z,\in\mathbb{C};\mu>0), \tag{1.9}$$

where $\mathbb{J}^{\alpha_1}_{\beta_1}$(z) is the Besel-maitland function [19].

(iii) If we put $\delta=1$ and b=c=-1 in (1.6), we obtained

$$ \mathbb{J}^{(\alpha_j)_m,\gamma,1}_{(\beta_j)_m,k,-1,1}(z) = \mathbb{E}^{(\alpha_j)_m,\;\gamma}_{(\beta_j)_m,k}(z) = \mathbb{E}_{\gamma,k}[(\gamma_j,\beta_j)_{1,m}], \tag{1.10}$$

where $\mathbb{E}_{\gamma,k}[(\gamma_j,\beta_j)_{1,m}]$ is the multiindex Mittag-Leffler function [18].

(iv) Connection with Fox-Wright function ${}_r\Psi_s[z]$:

$$ \mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_2\Psi_{m+1}\left[\begin{array}{c}(\gamma,k),(1,1)\\(\beta_j+\frac{1+b}{2},\alpha_j)_1^m,(\delta,1)\end{array}\;\Big|-cz\right]. \tag{1.11}$$

(v) Connection with Fox H-function:

$$ \mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)}H^{1,2}_{2,3}\left[cz\Big|\begin{array}{c}(1-\gamma,k),(0,1)\\(0,1),(\frac{1-b}{2}-\beta_j,\alpha_j)_1^m,(1-\delta,1)\end{array}\right]. \tag{1.12}$$

Also we recall here the following interesting and useful results as follows:

1. Oberhettinger [15] established the result (1.13) for $0<R(\rho)<R(\sigma)$

$$ \int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma}du = 2\sigma\,a^{-\sigma}\left(\frac{a}{2}\right)^{\rho}\frac{\Gamma(2\rho)\Gamma(\sigma-\rho)}{\Gamma(\sigma+\rho+1)}. \tag{1.13}$$

2. MacRobert [14] obtained the result (1.14) for $\mathbb{R}(\rho)>0$, $\mathbb{R}(\sigma)>0$; a and b are non zero constants and $0\le u\le 1$.

$$ \int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[au+b(1-u)]^{-\rho-\sigma}du = \frac{1}{a^\rho b^\sigma}\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}, \tag{1.14}$$

3. Edward [5] established the result (1.15)

$$ \int_0^1\int_0^1 u^\rho(1-v)^{\rho-1}(1-u)^{\sigma-1}(1-uv)^{1-\rho-\sigma}du dv = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}. \tag{1.15}$$

provided $\mathbb{R}(\rho)>0$ and $\mathbb{R}(\sigma)>0$

2. Main results

Here, we investigate some integral formulas of generalized miltiindex Bessel function, which are expressed in terms of Fox-Wright and Fox H-functions.

Theorem 2.1. *Let α_j, β_j, γ, δ, ρ, σ, τ, b, c $\in\mathbb{C}$ (j=1,2,....,m) be such that $\sum_{j=1}^{m}\mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\delta)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$ \int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma}\;\mathbb{J}^{(\alpha_j)_m,\gamma,c}_{(\beta_j)_m,k,b,\delta}\left(\frac{w}{(u+a+\sqrt{u^2+2au})^{\tau}}\right)du$$

$$=\frac{a^{\rho-\sigma}\Gamma(2\rho)\Gamma(\delta)}{2^{\rho-1}\Gamma(\gamma)}{}_4\Psi_{m+3}\left[\begin{array}{c}(\gamma,k),(\sigma-\rho,\tau),\\(\sigma+\rho+1,\tau),(\sigma,\tau),\end{array}\right.$$
$$\left.\begin{array}{c}(\sigma+1,\tau),(1,1)\\(\delta,1),(\beta_j+\frac{1+b}{2},\alpha_j)_1^m\end{array}\middle|-\frac{cw}{a^\tau}\right] \tag{2.1}$$

Proof. First we indicate the L.H.S of (2.1) by I_1 and then using (1.6) and interchanging the order of integration and summation, we get

$$I_1=\sum_{n=0}^{\infty}\frac{(-c)^n\,(\gamma)_{kn}\,(w)^n}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})\,(\delta)_n}$$
$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-(\sigma+\tau n)}du. \tag{2.2}$$

In view of the conditions given in Theorem 1, we can apply the integral formula (1.13) to the integral of (2.2) and obtain the following expression:

$$=\frac{a^{\rho-\sigma}}{2^{\rho-1}}\sum_{n=0}^{\infty}\frac{(-c)^n\,(\gamma)_{kn}\,(w)^n}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})\,(\delta)_n}\frac{\Gamma(2\rho)\Gamma(\sigma+\tau n-\rho)}{\Gamma(\sigma+\tau n+\rho+1)} \tag{2.3}$$

now using (1.7) and interpreting the above result with the help of (1.1), we get the desired result (2.1). This is the completes proof of Theorem 1. □

Remark 2.2. *For m=1, k=0, $\delta=\tau$=1, $\beta_1=\nu$, $\alpha_1=1$, and w is replaced by $-\frac{w^2}{4}$, (2.1) coincides with the known result of Choi et al. [3].*

Theorem 2.3. *Let α_j, β_j, γ, δ, ρ, σ, τ b, c $\in\mathbb{C}$ (j=1,2,....,m) be such that $\sum_{j=1}^m\mathbb{R}(\alpha)_j>max\{0;\mathbb{R}(k)-1\};k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$, $\mathbb{R}(\delta)>0$ and u> 0, then following relation holds true*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma}$$
$$\mathbb{J}_{(\beta_j)_m,k,b,\delta}^{(\alpha_j)_m,\gamma,c}\left(\frac{uw}{(u+a+\sqrt{u^2+2au})^\tau}\right)du$$

$$=\frac{a^{\rho-\sigma}\Gamma(2\rho)\Gamma(\delta)}{2^{\rho-1}\Gamma(\gamma)}H_{4,m+5}^{1,4}\left[\frac{cw}{a}\middle|\begin{array}{c}(1-\gamma,k),\\(0,1),(\sigma+\rho,2\tau),\end{array}\right.$$
$$\left.\begin{array}{c}(1-2\rho,\tau),(\sigma,\tau),(0,1)\\(1-\sigma,\tau),(1-\delta,1),(\frac{1-b}{2}-\beta_j,\alpha_j)_1^m\end{array}\right]. \tag{2.4}$$

Proof. We using (1.6) and (1.13) and apply the same arguments as in the proof of Theorem 1, we achieve the desired result (2.4) of Theorem 2. □

Theorem 2.4. *Let α_j, β_j, γ, δ, ρ, σ, τ, b, c $\in\mathbb{C}$ (j=1,2,......,m) be such that $\sum_{j=1}^m\mathbb{R}(\alpha)_j>max\{0;\mathbb{R}(k)-1\};k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\delta)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[\lambda u+\mu(1-u)]^{-\rho-\sigma}$$
$$\mathbb{J}_{(\beta_j)_m,k,b,\delta}^{(\alpha_j)_m,\gamma,c}\left(\frac{w(1-u)^\tau}{(\lambda u+\mu(1-u))^\tau}\right)du=\frac{\Gamma(\rho)\Gamma(\delta)}{\lambda^\rho\mu^\sigma\Gamma(\gamma)}$$

$$\times{}_3\Psi_{m+2}\left[\begin{array}{c}(\gamma,k),(\sigma,\tau),(1,1)\\(\sigma+\rho,\tau),(\delta,1),(\beta_j+\frac{1+b}{2},\alpha_j)_1^m\end{array}\middle|-\frac{cw}{\mu^\tau}\right]. \tag{2.5}$$

Proof. Denoting the left hand side of (2.5) by I_2 and using (1.6) and interchanging the order of integration and summation, we get

$$I_2=\sum_{n=0}^{\infty}\frac{(-c)^n\,(\gamma)_{kn}\,(w)^n}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})\,(\delta)_n}$$
$$\int_0^\infty u^{\rho-1}(1-u)^{\sigma+\tau n-1}[\lambda u+\mu(1-u)]^{-\rho-(\sigma+n\tau)}du. \tag{2.6}$$

In view of the conditions given in Theorem 3, we can apply the integral formula (1.14) to the integral of (2.6) and obtain the following expression:

$$=\frac{\Gamma(\rho)}{\lambda^\rho\mu^{\sigma+\tau n}}\sum_{n=0}^{\infty}\frac{(c)^n\,(\gamma)_{kn}\,(w)^n}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})\,(\delta)_n}\frac{\Gamma(\rho)\Gamma(\sigma+\tau n)}{\Gamma(\sigma+\tau n+\rho)}. \tag{2.7}$$

Now, using (1.7) in (2.7) and after simplification, we get

$$=\frac{\Gamma(\rho)\Gamma(\delta)}{\lambda^\rho\mu^{\sigma+\tau n}\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+kn)\Gamma(\sigma+\tau n)}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}$$
$$\frac{1}{\Gamma(\sigma+\tau n+\rho)\Gamma(\delta+n)}\left(-\frac{cw}{\mu^\tau}\right)^n. \tag{2.8}$$

Finally, solving the above result with the help of (1.1), we get the desired result (2.5) of Theorem 3. □

Remark 2.5. *If we set m=1, k=0, $\delta=\tau$=b=1, c=-1 and w is replaced by $w^2/4$ in (2.5), we get the known result of Khan et al. [12].*

Theorem 2.6. *Let α_j, β_j, γ, δ, ρ, σ, τ, b, c $\in\mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^m\mathbb{R}(\alpha)_j>max\{0;\mathbb{R}(k)-1\};k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\delta)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then follow-*

ing relation holds true

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[\lambda u+\mu(1-u)]^{-\rho-\sigma} \mathbb{J}_{(\beta_j)_m,k,b,\delta}^{(\alpha_j)_m,\gamma,c}\left(\frac{wu^\tau(1-u)^\tau}{(\lambda u+\mu(1-u))^{2\tau}}\right)du = \frac{\Gamma(\delta)}{\lambda^\rho\mu^\sigma\Gamma(\gamma)} H_{4,m+4}^{1,4}\left[\frac{cw}{(\lambda\mu)^\tau}\middle| \begin{matrix} (1-\gamma,k), (1-\rho,\tau),(1-\sigma,\tau),\ (0,1) \\ (0,1), (1-\sigma+\rho,2\tau),(1-\delta,1),(\frac{1-b}{2}-\beta_j,\alpha_j)_1^m \end{matrix}\right]. \tag{2.9}$$

Proof. We using (1.3), (1.6) and (1.14), apply the same procedure as in the proof of Theorem 3, we get the desire result (2.9) of Theorem 4. □

Theorem 2.7. *Let α_j, β_j, γ, δ, ρ, σ, τ, b, c $\in \mathbb{C}$ (j=1,2,..,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j > 0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\delta)>0$, $\mathbb{R}(\tau)>0$ and u> 0, v> 0, then following relation holds true*

$$\int_0^1\int_0^1 u^\rho(1-v)^{\rho-1}(1-u)^{\sigma-1}(1-uv)^{1-\rho-\sigma} \mathbb{J}_{(\beta_j)_m,k,b,\delta}^{(\alpha_j)_m,\ \gamma,\ c}\left(\frac{wu^\tau(1-u)^\tau}{(1-uv)^{2\tau}}\right)dudv = \frac{\Gamma(\delta)}{\Gamma(\gamma)} {}_4\Psi_{m+3}\left[\begin{matrix}(\gamma,k),(\rho,\tau),(\sigma,\tau),(1,1)\\ (\sigma+\rho,2\tau),(\delta,1),(\beta_j+\frac{1+b}{2},\alpha_j)_1^m\end{matrix}\middle| -cw\right]. \tag{2.10}$$

Proo. Denoting the left hand side of (2.10) by I_3, then using (1.6) and interchanging the order of integration and summation, we get

$$I_3 = \sum_{n=0}^\infty \frac{(c)^n\ (\gamma)_{kn}\ (w)^n}{\prod_{j=1}^m \Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})\ (\delta)_n}\int_0^1\int_0^1 u^{\rho+\tau n} (1-v)^{\rho+\tau n-1}(1-u)^{\sigma+\tau n-1}(1-uv)^{1-\rho-\sigma+2\tau n}dudv. \tag{2.11}$$

In view of the conditions given in Theorem 5, we can apply the integral formula (1.15) to the integral (2.11), we get the following expression:

$$= \sum_{n=0}^\infty \frac{(c)^n\ (\gamma)_{kn}\ (w)^n}{\prod_{j=1}^m \Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})\ (\delta)_n}\frac{\Gamma(\rho+\tau n)\Gamma(\sigma+\tau n)}{\Gamma(\sigma+2\tau n+\rho)}. \tag{2.12}$$

Now using (1.7) in (2.12) and after simplification we get

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^\infty \frac{\Gamma(\gamma+kn)\Gamma(\rho+\tau n)\Gamma(\sigma+\tau n)}{\prod_{j=1}^m \Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})\ \Gamma(\delta+n)} \frac{1}{\Gamma(\sigma+\tau n+\rho)}\left(\frac{cw}{\mu^\tau}\right)^n. \tag{2.13}$$

Finally, solving the above result with the help of (1.1), we get the desired result (2.10) of Theorem 5. □

Remark 2. If we set m=1, k=0, $\delta=\tau$=b=1, c=-1 and w is replaced by $w^2/4$ in (2.10), we get the known result of Khan and Ghayasuddin [13].

3. Applications

In this section, we give some applications of our main results by taking suitable values of the parameters of extended multiindex Bessel function.

1. If we set $\delta=b=1$ and c=-1 in (2.1), we evaluate the following result:

Corollary 3.1. *Let α_j, β_j, γ, ρ, σ, τ, $\in\mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma} \mathbb{J}_{(\beta_j)_m,k}^{(\alpha_j)_m,\gamma}\left(\frac{w}{(u+a+\sqrt{u^2+2au})^\tau}\right)du = \frac{a^{\rho-\sigma}\Gamma(2\rho)}{2^{\rho-1}\Gamma(\gamma)} \times {}_3\Psi_{m+2}\left[\begin{matrix}(\gamma,k),(\sigma-\rho,\tau),(\sigma+1,\tau),\\ (\sigma+\rho+1,\tau),(\sigma,\tau),(\beta_j+1,\alpha_j)_1^m\end{matrix}\middle| -\frac{w}{a^\tau}\right], \tag{3.1}$$

where $\mathbb{J}_{(\beta_j)_m,\ k}^{(\alpha_j)_m,\ \gamma}$[z] is the multiindex Bessel function [4].

Remark 3.2. *For m=1 and $\delta=\tau=1$, (3.1) coincides with the known results of Abouzaid et al. [2].*

2. If we set k=0, m=1, $\delta=b=1$, and c=-1 in (2.1), we get following result:

Corollary 3.3. *Let α_1, β_1, ρ, σ, τ, $\in\mathbb{C}$ be such that $\mathbb{R}(\alpha)_1>0$; $\mathbb{R}(\beta)_1>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma} \mathbb{J}_{\beta_1}^{\alpha_1}\left(\frac{w}{(u+a+\sqrt{u^2+2au})^\tau}\right)du = \frac{a^{\rho-\sigma}\Gamma(2\rho)}{2^{\rho-1}} \times {}_2\Psi_{m+2}\left[\begin{matrix}(\sigma-\rho,\tau),(\sigma+1,\tau),\\ (\sigma+\rho+1,\tau),(\sigma,\tau),(\beta_1+1,\alpha_1)_1^m\end{matrix}\middle| -\frac{w}{a^\tau}\right]. \tag{3.2}$$

3. If we set k=0, $\delta=1$, b=-1 and c=-1 in (2.1), we get following result:

Corollary 3.4. *Let α_j, β_j, ρ, σ, τ, $\in\mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j>0$; $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma} \mathbb{E}^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}\left(\frac{w}{(u+a+\sqrt{u^2+2au})^\tau}\right)du = \frac{a^{\rho-\sigma}\Gamma(2\rho)}{2^{\rho-1}}$$

$$\times {}_2\Psi_{m+2}\left[\begin{array}{c}(\sigma-\rho,\tau),(\sigma+1,\tau),\\ (\sigma+\rho+1,\tau),(\sigma,\tau),(\beta_j,\alpha_j)_1^m\end{array}\Bigg|\frac{w}{a^\tau}\right]. \quad (3.3)$$

Remark 3.5. *If we set k=γ=1, $\tau=\delta=c=1$ b=-1 and α_j, is replaced by 1/ α_j in (3.3), we get the known result of khan et al. [11].*

4. If we set b=$\delta=1$ and c=-1 in (2.4), we get the following result:

Corollary 3.6. *Let α_j, β_j, γ, δ, ρ, σ, τ b, $c\in\mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$, $\mathbb{R}(\delta)>0$ and u> 0, then following relation holds true*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma} \mathbb{J}^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}\left(\frac{uw}{(u+a+\sqrt{u^2+2au})^\tau}\right)du$$
$$= \frac{a^{\rho-\sigma}\Gamma(2\rho)}{2^{\rho-1}\Gamma(\gamma)} H^{1,3}_{3,m+4}\left[\frac{w}{a}\Bigg|\begin{array}{c}(1-\gamma,k),(1-2\rho,\tau),(\sigma,\tau),\\ (0,1),(\sigma+\rho,2\tau),(1-\sigma,\tau),(\frac{1-b}{2}-\beta_j,\alpha_j)_1^m\end{array}\right]. \quad (3.4)$$

5. If we put $\delta=b=1$ and c=-1 in (2.5), we get the following result:

Corollary 3.7. *Let α_j, β_j, γ, ρ, σ, τ, $\in\mathbb{C}$ (j=1,2,......,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[\lambda u+\mu(1-u)]^{-\rho-\sigma} \mathbb{J}^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}\left(\frac{w(1-u)^\tau}{(\lambda u+\mu(1-u))^\tau}\right)du$$
$$= \frac{\Gamma(\rho)\Gamma(\delta)}{\lambda^\rho\mu^\sigma\Gamma(\gamma)}\,{}_2\Psi_{m+1}\left[\begin{array}{c}(\gamma,k),(\sigma,\tau),\\ (\sigma+\rho,\tau),(\beta_j+1,\alpha_j)_1^m\end{array}\Bigg|\frac{w}{\mu^\tau}\right]. \quad (3.5)$$

6. If we set b=c=δ=1 in (2.9), we get the following result:

Corollary 3.8. *Let α_j, β_j, γ, δ, ρ, σ, τ, b, $c\in\mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\delta)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[\lambda u+\mu(1-u)]^{-\rho-\sigma} \mathbb{J}^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}\left(\frac{wu^\tau(1-u)^\tau}{(\lambda u+\mu(1-u))^{2\tau}}\right)du = \frac{\Gamma(\delta)\lambda^{-\rho}}{\mu^\sigma\Gamma(\gamma)}$$
$$H^{1,3}_{3,m+3}\left[\frac{w}{(\lambda\mu)^\tau}\Bigg|\begin{array}{c}(1-\gamma,k),(1-\rho,\tau),(1-\sigma,\tau)\\ (0,1),(1-\sigma+\rho,2\tau),(-\beta_j,\alpha_j)_1^m\end{array}\right]. \quad (3.6)$$

7. If we put $\delta=1$, b=-1, and c=1 in (2.5), we get the following result:

Corollary 3.9. *Let α_j, β_j, γ, ρ, σ, τ, $\in\mathbb{C}$ (j=1,2,..,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[\lambda u+\mu(1-u)]^{-\rho-\sigma} \mathbb{E}^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}\left(\frac{w(1-u)^\tau}{(\lambda u+\mu(1-u))^\tau}\right)du = \frac{\Gamma(\rho)\Gamma(\delta)}{\lambda^\rho\mu^\sigma\Gamma(\gamma)}$$
$$\times\ {}_2\Psi_{m+1}\left[\begin{array}{c}(\gamma,k),(\sigma,\tau),\\ (\sigma+\rho,\tau),(\beta_j,\alpha_j)_1^m\end{array}\Bigg|-\frac{w}{\mu^\tau}\right]. \quad (3.7)$$

8. If we put k=0, m=1, $\delta=b=1$ and c=-1 in (2.5), we get the following result:

Corollary 3.10. *Let α_1, β_1, ρ, σ, τ, $\in\mathbb{C}$ be such that $\mathbb{R}(\alpha)_1>0$ $\mathbb{R}(\beta)_1>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$ and u> 0, then following relation holds true*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[\lambda u+\mu(1-u)]^{-\rho-\sigma} \mathbb{J}^\eta_\nu\left(\frac{w(1-u)^\tau}{(\lambda u+\mu(1-u))^\tau}\right)du = \frac{\Gamma(\rho)\Gamma(\delta)}{\lambda^\rho\mu^\sigma\Gamma(\gamma)}$$

$$\times{}_1\Psi_{m+1}\left[\begin{array}{c}(\sigma,\tau),\\ (\sigma+\rho,\tau),\ (\beta_1,\alpha_1)_1^m\end{array}\Bigg|\frac{w}{\mu^\tau}\right]. \quad (3.8)$$

9. If we set $\delta=b=1$ and c=-1 in (2.10), we get the following result:

Corollary 3.11. *Let α_j, β_j, γ, ρ, σ, $\tau\in\mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k>0$, $\mathbb{R}(\beta)_j>0$ with $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\tau)>0$ and u> 0, v> 0, then following relation holds true*

$$\int_0^1\int_0^1 u^\rho(1-v)^{\rho-1}(1-u)^{\sigma-1}(1-uv)^{1-\rho-\sigma} \mathbb{J}^{(\alpha_j)_m,\ \gamma}_{(\beta_j)_m,\ k}\left(\frac{wu^\tau(1-u)^\tau}{(1-uv)^{2\tau}}\right)dudv$$
$$= \frac{1}{\Gamma(\gamma)}\,{}_3\Psi_{m+1}\left[\begin{array}{c}(\gamma,k),(\rho,\tau),(\sigma,\tau),\\ (\sigma+\rho,2\tau),(\beta_j+\frac{1+b}{2},\alpha_j)_1^m\end{array}\Bigg|w\right]. \quad (3.9)$$

10. If we set k=0, m=1, $\delta = b = 1$ and c=-1 in (2.10), yields following result:

Corollary 3.12. *Let α_1, β_1, ρ, σ, $\tau \in \mathbb{C}$ be such that $\mathbb{R}(\alpha)_1 > 0$, $\mathbb{R}(\beta)_1 > 0$ with $\mathbb{R}(\gamma) > 0$, $\mathbb{R}(\tau) > 0$ and u> 0, v> 0, then following relation holds true*

$$\int_0^1 \int_0^1 u^{\rho}(1-v)^{\rho-1}(1-u)^{\sigma-1}(1-uv)^{1-\rho-\sigma} \mathbb{J}_{\nu}^{\eta}\left(\frac{wu^{\tau}(1-u)^{\tau}}{(1-uv)^{2\tau}}\right)dudv = {}_2\Psi_{m+1}\left[\begin{array}{c}(\gamma,k),(\rho,\tau),(\sigma,\tau),\\(\sigma+\rho,2\tau),(\beta_1+1,\alpha_1)_1^m\end{array}\middle|w\right]. \tag{3.10}$$

11. If we set $\delta = c = 1$ and b=-1 in (2.10), we get the following result:

Corollary 3.13. *Let α_j, β_j, γ, ρ, σ, $\tau \in \mathbb{C}$ (j=1,2,...,m) be such that $\sum_{j=1}^m \mathbb{R}(\alpha)_j > max\{0;\mathbb{R}(k)-1\}; k > 0$, $\mathbb{R}(\beta)_j > 0$ with $\mathbb{R}(\gamma) > 0$, $\mathbb{R}(\tau) > 0$ and u> 0, v> 0, then following relation holds true*

$$\int_0^1 \int_0^1 u^{\rho}(1-v)^{\rho-1}(1-u)^{\sigma-1}(1-uv)^{1-\rho-\sigma} \mathbb{E}_{(\beta_j)_m,\,k}^{(\alpha_j)_m,\,\gamma}\left(\frac{wu^{\tau}(1-u)^{\tau}}{(1-uv)^{2\tau}}\right)dudv = \frac{1}{\Gamma(\gamma)}\,{}_3\Psi_{m+1}\left[\begin{array}{c}(\gamma,k),\ (\rho,\tau),(\sigma,\tau),\\(\sigma+\rho,2\tau),(\beta_j,\alpha_j)_1^m\end{array}\middle|-w\right]. \tag{3.11}$$

4. Concluding note

In the present paper, we have investigated a new extension of generalized multiindex-Bessel function and find their connection with other functions scattered in the literature of special function. Certain class of unified integrals involving EGMBF are expressed in terms of Wright hypergeometric function and Fox H-function. Extended generalized multiindex-Bessel function (EGMBF) is the special case of 2m-parametric Mittag-Leffler function [1], generalized multiindex-Bessel function [4], Multi-index Mittag-Leffler function [7] and Mittag-Leffler type function [16] by specializing the suitable values of parameter involved. Also extended generalized multiindex-Bessel function are expressed in terms of Meijer-G function [6] and generalized hypergeometric function ${}_pF_q$ [21]. Therefore the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments.

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