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A frictionless contact problem for elastic-visco-plastic materials with adhesion and thermal effects

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Abstract

We consider a mathematical problem for frictionless contact between a thermo-elastic-viscoplastic body with adhesion and an obstacle. We employ the thermo-elastic-viscoplastic with damage constitutive law for the material. The evolution of the damage is described by an inclusion of parabolic type. The evolution of the adhesion field is governed by the differential equation $\dot{\beta} = H_{ad}(\beta, \xi_\beta, R_\nu(u_\nu), R_\tau(u_\tau))$. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

Keywords

Damage, adhesion, normal compliance, temperature, elastic-visco-plastic materials.

AMS Subject Classification

49J40, 74C10, 74H25.

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Contents

1. Introduction

The constitutive laws with internal variables has been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the absolute temperature and the damage field. See for examples [4, 19, 22, 27, 28] for the case of hardening, temperature and other internal state variables and the references [14, 15] for the case of damage field. The importance of this paper is to make the coupling of the elastic-visco-plastic problem contact with adhesion. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [23, 25, 26] and recently in the monographs [21]. In these papers, the bonding field, denoted by β, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [11, 12], the bonding field satisfies the restriction $0 \leq \beta \leq 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. The novelty of this work lies in the analysis of a system that contains strong couplings in the multivalued boundary conditions: both the normal compliance contact condition and tangential contact condition depend on the adhesion (see (2.11) and (2.12)), and the adhesion be written by the differential equation of the general form

$$\dot{\beta} = H_{ad}(\beta, \xi_\beta, R_\nu(u_\nu), R_\tau(u_\tau)).$$

Here, H_{ad} is the *adhesion evolution rate function*. Then, the adhesion rate function was assumed to depend, in addition to

β, $R_\nu(u_\nu)$, $R_\tau(u_\tau)$ and ξ_β , where

$$\xi_\beta(x,t) = \int_0^t \beta(x,s)ds \quad on \quad \Gamma_3 \times (0,T).$$

We use it in H_{ad}, since usually when the glue is stretched beyond the limit L it does not contribute more to the bond strength. An example of such a function, used in [6], the following form of the evolution of the bonding field was employed:

$$H_{ad}(\beta,\xi_\beta,R_1,R_2) = -\beta_+\gamma_n R_1^2,$$

where γ_n is the normal rate coefficient and $\gamma_n L$ is the maximal tensile normal traction that the adhesive can provide and $\beta_+ = \max(0,\beta)$. We note that in this case, only debonding is allowed. A different rate equation for the the evolution of the bonding field is

$$H_{ad}(\beta,\xi_\beta,R_1,R_2) = -\Big(\beta\left(\gamma_n R_1^2 + \gamma_t |R_2|^2\right) - \varepsilon_a\Big)_+,$$

see, e.g., [7, 16, 17]. Here, γ_t is the tangential rate coefficient, which may also be interpreted as the tangential stiffness coefficient of the interface when the adhesion is complete ($\beta = 1$). Another example, in which H_{ad} depends on all variables is

$$H_{ad}(\beta,\xi_\beta,R_1,R_2) = -\gamma_n\beta_+ R_1^2 - \gamma_t\beta_+|R_2|^2 + \gamma_r\frac{\beta_+(1-\beta)_+}{1+\xi_\beta^2},$$

where γ_r is the rebonding rate coefficient. However, the bonding cannot exceed $\beta = 1$ and, moreover, the rebonding becomes weaker as the process goes on, which is represented by the factor $1+\xi_\beta^2$ in the denominator.

The aim of this paper is to study the dynamic evolution of damage in thermo-electroelastic materials. For this, we use an thermo-elastic-viscop-lastic materials. For this, we consider a rate-type constitutive equation with two internal variables of the form

$$\begin{aligned}\sigma(t) = \mathscr{A}\big(\varepsilon(\dot u(t))\big) + \mathscr{E}\big(\varepsilon(u(t))\big) + \\ \int_0^t \mathscr{G}\big(\sigma(s) - \mathscr{A}\big(\varepsilon(\dot u(s))\big), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big)ds,\end{aligned} \tag{1.1}$$

in which u, σ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, θ represents the absolute temperature, ς is the damage field, $\mathscr{A}$ and $\mathscr{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathscr{G}$ is a nonlinear constitutive function which describes the visco-plastic behavior of the material. It follows from (1.1) that at each time moment, the stress tensor $\sigma(t)$ is split into two parts: $\sigma(t) = \sigma^V(t) + \sigma^R(t)$, where $\sigma^V(t) = \mathscr{A}(\varepsilon(\dot u(t)))$ represents the purely viscous part of the stress, whereas $\sigma^R(t)$ satisfies a rate-type elastic-viscoplastic relation with absolute temperature and damage

$$\begin{aligned}\sigma^R(t) = \mathscr{E}\big(\varepsilon(u(t))\big) + \\ \int_0^t \mathscr{G}\big(\sigma^R(s), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big)ds.\end{aligned} \tag{1.2}$$

When $\mathscr{G} = 0$ in (1.1) reduces to the Kelvin-Voigt viscoelastic constitutive law given by

$$\sigma(t) = \mathscr{A}\big(\varepsilon(\dot u(t))\big) + \mathscr{E}\big(\varepsilon(u(t))\big). \tag{1.3}$$

The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. The models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [20]. The new idea of [13, 14] was the introduction of the *damage function* $\alpha = \alpha(x,t)$, which is the ratio between the elastic moduli of the damage and damage-free materials. In an isotropic and homogeneous elastic material, let E_Y be the Young modulus of the original material and E_{eff} be the current modulus, then the damage function is defined by

$$\alpha = \alpha(x,t) = \frac{E_{eff}}{E_Y}.$$

Clearly, it follows from this definition that the damage function α is restricted to have values between zero and one. When $\alpha = 1$, there is no damage in the material, when $\alpha = 0$, the material is completely damaged, when $0 < \alpha < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [17, 29]. The differential inclusion used for the evolution of the damage field is

$$\begin{aligned}\dot\alpha - \kappa\Delta\alpha + \partial\psi_K(\alpha) \ni \phi\big(\sigma - \mathscr{A}\varepsilon(\dot u), \varepsilon(u), \theta, \alpha\big) \\ \text{in } \Omega\times(0,T),\end{aligned} \tag{1.4}$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in H^1(\Omega);\ 0 \le \xi \le 1,\ \text{a.e. in } \Omega\}, \tag{1.5}$$

κ is a positive coefficient, $\partial\psi_K$ represents the subdifferential of the indicator function of the set K and ϕ is a given constitutive function which describes the sources of the damage in the system.

Examples and mechanical interpretation of elastic-visco-plastic can be found in [9, 18]. Dynamic and quasi-static contact problems are the topic of numerous papers, e.g. [1–3, 8, 10, 24]. More recently in [17], we study an electro-elastic-visco-plastic frictionless contact problem with damage and adhesion. The mathematical problem modelled the quasi-static evolution of damage in thermo-visco-plastic materials has been studied in [20]. In this paper we study a dynamic frictionless contact problem with damage and temperature between an elastic-visco-plastic body and a conductive foundation. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled

with a surface variable, the bonding field. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

The paper is organized as follows. In section 2 we describe the mathematical models for the dynamic evolution of damage and adhesion in elastic-viscop-lastic materials. The contact is modelled with normal compliance and adhesion. We introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. We prove in section 3 the existence and uniqueness of the solution, where it is carried out in several steps and is based on a classical existence and uniqueness result on parabolic inequalities, evolutionary variational equalities, differential equations and fixed point arguments.

2. Statement of the Problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2,3)$ be a bounded domain with a Lipschitz boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$u \cdot v = u_i v_i \quad \forall u, v \in \mathbb{R}^n, \quad \sigma \cdot \tau = \sigma_{ij}\tau_{ij} \quad \forall \sigma, \tau \in \mathbb{S}_n,$$
$$|u| = (u \cdot u)^{1/2} \quad \forall u \in \mathbb{R}^n, \quad |\sigma| = (\sigma \cdot \sigma)^{1/2} \quad \forall \sigma \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We shall use the notation

$$H = L^2(\Omega)^n = \{u = \{u_i\} : u_i \in L^2(\Omega)\},$$
$$\mathscr{H} = \{\sigma = \{\sigma_{ij}\} : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{u \in H : \varepsilon(u) \in \mathscr{H}\},$$
$$\mathscr{H}_1 = \{\sigma \in \mathscr{H} : \text{Div}(\sigma) \in H\},$$
$$V = H^1(\Omega).$$

Here $\varepsilon : H_1 \to \mathscr{H}$ and $\text{Div} : \mathscr{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div}(\sigma) = (\sigma_{ij,j}).$$

The sets H, $\mathscr{H}$, H_1, $\mathscr{H}_1$ and V are real Hilbert spaces endowed with the canonical inner products:

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad (\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx,$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathscr{H}},$$
$$(\sigma,\tau)_{\mathscr{H}_1} = (\sigma,\tau)_{\mathscr{H}} + (\text{Div}(\sigma),\text{Div}(\tau))_H,$$
$$(f,g)_V = (f,g)_{L^2(\Omega)} + (f_{x_i}, g_{x_i})_{L^2(\Omega)}.$$

The associated norms are denoted by $\|.\|_H$, $\|.\|_{\mathscr{H}}$, $\|.\|_{H_1}$, $\|.\|_{\mathscr{H}_1}$ and $\|.\|_V$. Since the boundary Γ is Lipschitz continuous, the unit outward normal vector field ν on the boundary is defined a.e. For every vector field $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

Let $H_\Gamma = (H^{1/2}(\Gamma))^n$ and $\gamma : H_1 \to H_\Gamma$ be the trace map. We denote by $\mathscr{V}$ the closed subspace of H_1 defined by

$$\mathscr{V} = \{v \in H_1 : \gamma v = 0 \text{ on } \Gamma_1\}.$$

We also denote by H_Γ' the dual of H_Γ. Moreover, since $\text{meas}(\Gamma_1) > 0$, Korn's inequality holds and thus, there exists a positive constant C_0 depending only on Ω, Γ_1 such that

$$\|\varepsilon(v)\|_{\mathscr{H}} \geq C_0 \|v\|_{H_1} \quad \forall v \in \mathscr{V}.$$

On the space $\mathscr{V}$ we consider the inner product given by

$$(u,v)_{\mathscr{V}} = (\varepsilon(u),\varepsilon(v))_{\mathscr{H}},$$

and let $\|.\|_{\mathscr{V}}$ be the associated norm, defined by

$$\|v\|_{\mathscr{V}} = \|\varepsilon(v)\|_{\mathscr{H}}. \tag{2.1}$$

It follows from Korn's inequality that $\|.\|_{H_1}$ and $\|.\|_{\mathscr{V}}$ are equivalent norms on $\mathscr{V}$. Therefore $(\mathscr{V}, \|.\|_{\mathscr{V}})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a positive constant C_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^n} \leq C_0 \|v\|_{\mathscr{V}} \ \forall v \in \mathscr{V}. \tag{2.2}$$

Furthermore, if $\sigma \in \mathscr{H}_1$ there exists an element $\sigma\nu \in H_\Gamma'$ such that the following Green formula holds

$$(\sigma, \varepsilon(v))_{\mathscr{H}} + (\text{Div}(\sigma), v)_H = \int_\Gamma \sigma\nu \cdot \gamma v d\Gamma \quad \forall v \in H_1. \tag{2.3}$$

Similarly, for a regular tensor field $\sigma : \Omega \to \mathbb{S}_n$ we define its normal and tangential components on the boundary by

$$\sigma_\nu = \sigma\nu \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Moreover, we denote by $\mathscr{V}'$ and V' the dual of the spaces $\mathscr{V}$ and V, respectively. Identifying H, respectively $L^2(\Omega)$, with its own dual, we have the inclusions

$$\mathscr{V} \subset H \subset \mathscr{V}', \quad V \subset L^2(\Omega) \subset V'.$$

We use the notation $\langle \cdot, \cdot \rangle_{\mathscr{V}' \times \mathscr{V}}$, $\langle \cdot, \cdot \rangle_{V' \times V}$ to represent the duality pairing between $\mathscr{V}', \mathscr{V}$ and V', V, respectively. Let $T > 0$. For every real space X, we use the notation $C(0,T;X)$, and $C^1(0,T;X)$ for the space of continuous an continuously differentiable functions from $[0,T]$ to X respectively, $C(0,T;X)$ is a real Banach space with the norm

$$|f|_{C(0,T;X)} = \max_{t \in [0,T]} |f(t)|_X.$$

While $C^1(0,T;X)$ is a real Banach space with the norm

$$|f|_{C^1(0,T;X)} = \max_{t \in [0,T]} |f(t)|_X + \max_{t \in [0,T]} |\dot{f}(t)|_X.$$

Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue space $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, for a real number r, we use r_+ to represent its positive part that is $r_+ = \max(0,r)$, and if X_1 and X_2 are real Hilbert spaces, than $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

The physical setting is the following. A body occupies the domain Ω, and is clamped on Γ_1 and so the displacement field vanishes there. Surface tractions of density f_0 act on $\Gamma_2 \times (0,T)$ and a volume force of density f is applied in $\Omega \times (0,T)$. We assume that the body is in adhesive frictionless contact with an obstacle, the so-called foundation, over the potential contact surface Γ_3. We admit a possible external heat source applied in $\Omega \times (0,T)$, given by the function q. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. We use an elastic-visco-plastic constitutive law with damage to model the material's behaviour and an ordinary differential equation to describe the evolution of the adhesion field.
The mechanical formulation of the frictionless problem with normal compliance is as follow.

Problem P

Find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^n$, the stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}_n$, the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$, the damage field $\varsigma : \Omega \times [0,T] \to \mathbb{R}$ and the adhesion field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that

$$\sigma(t) = \mathscr{A}\big(\varepsilon(\dot u(t))\big) + \mathscr{E}\big(\varepsilon(u(t))\big) + \int_0^t \mathscr{G}\big(\sigma(s) - \mathscr{A}\big(\varepsilon(\dot u(s))\big), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big) ds \quad \text{in } \Omega\times(0,T), \tag{2.4}$$

$$\rho \ddot u = \mathrm{Div}(\sigma) + f \quad \text{in } \Omega\times(0,T), \tag{2.5}$$

$$\rho \dot\theta - k_0 \Delta\theta = \psi\big(\sigma - \mathscr{A}\big(\varepsilon(\dot u)\big), \varepsilon(u), \theta, \varsigma\big) + q \quad \text{in } \Omega\times(0,T), \tag{2.6}$$

$$\rho \dot\varsigma - k_1\Delta\varsigma + \partial_K \varphi(\varsigma) \ni \phi\big(\sigma - \mathscr{A}\big(\varepsilon(\dot u)\big), \varepsilon(u), \theta, \varsigma\big) \quad \text{in } \Omega\times(0,T), \tag{2.7}$$

$$u = 0 \quad \text{on } \Gamma_1\times(0,T), \tag{2.8}$$

$$\sigma\nu = f_0 \quad \text{on } \Gamma_2\times(0,T), \tag{2.9}$$

$$\dot\beta = H_{ad}(\beta, \xi_\beta, R_\nu(u_\nu), R_\tau(u_\tau \text{ on } \Gamma_3\times(0,T), \tag{2.10}$$

$$\sigma_\nu = -p_\nu(u_\nu) + \gamma_\nu \beta^2 R_\nu(u_\nu) \ \text{ on } \Gamma_3\times(0,T), \tag{2.11}$$

$$\sigma_\tau = -p_\tau(\beta) R_\tau(u_\tau) \ \text{ on } \Gamma_3\times(0,T), \tag{2.12}$$

$$k_0 \frac{\partial\theta}{\partial\nu} + \alpha\theta = 0 \quad \text{on } \Gamma\times(0,T), \tag{2.13}$$

$$\frac{\partial\varsigma}{\partial\nu} = 0 \quad \text{on } \Gamma\times(0,T), \tag{2.14}$$

$$u(0) = u_0,\ \dot u(0) = w_0,\ \theta(0) = \theta_0,\ \varsigma(0) = \varsigma_0 \text{ in } \Omega, \tag{2.15}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.16}$$

Here, equation (2.4) is the thermo-elastic-visco-plastic constitutive law where $\mathscr{A}$ and $\mathscr{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathscr{G}$ is a nonlinear constitutive function which describes the visco-plastic behavior of the material. (2.5) represents the equation of motion in which the dot above denotes the derivative with respect to the time variable and ρ is the density of mass. Equation (2.6) represents the energy conservation where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source. Inclusion (2.7) describes the evolution of damage field. Equalities (2.8) and (2.9) are the displacement-traction boundary conditions, respectively. Condition (2.11) represents the normal compliance condition with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2 R_\nu(u_\nu)$ the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but only as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The contact condition (2.11) was used in various papers, see e.g. [6, 7, 26, 29]. Condition (2.12) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and R_τ is the truncation operator given by

$$R_\tau(v) = \begin{cases} v & \text{if } |v| \le L, \\ L\frac{v}{|v|} & \text{if } |v| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the adhesion field and on the tangential displacement, but only as long as it does not exceed the adhesion length L. The frictional tangential traction is assumed to be much smaller than the adhesive one, and therefore omitted. The introduction of the operator R_ν, together with the operator R_τ defined above, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows.
Next, equation (2.10) describes the evolution of the bonding field and it was already used in [6, 7], see also [29] for more

details. (2.13) and (2.14) represent, respectively a Fourier boundary condition for the temperature and a homogeneous Neumann boundary condition for the damage field on Γ. Finally the functions u_0, w_0, θ_0 and ς_0 in (2.15) and β_0 in (2.16) are the initial data. To obtain the variational formulation of the problem(2.4)–(2.16) we introduce for the adhesive field the set

$$\mathscr{Z}=\{\omega\in L^{\infty}\big(0,T;L^2(\Gamma_3)\big):0\leq\omega(t)\leq 1,t\in[0,T],\text{a.e. on }\Gamma_3\}.$$

In the study of the Problem **P**, we consider the following assumptions:

The *viscosity operator* $\mathscr{A}:\Omega\times\mathbb{S}_n\to\mathbb{S}_n$satisfies :

$$\begin{cases}\text{(a) There exists a constant } L_{\mathscr{A}}>0 \text{ such that}\\ |\mathscr{A}(x,\varepsilon_1)-\mathscr{A}(x,\varepsilon_2)|\leq L_{\mathscr{A}}|\varepsilon_1-\varepsilon_2| \text{ for all}\\ \varepsilon_1,\varepsilon_2\in\mathbb{S}_n,\ \text{a.e. } x\in\Omega.\\ \text{(b) There exists a constant } m_{\mathscr{A}} \text{ such that}\\ (\mathscr{A}(x,\varepsilon_1)-\mathscr{A}(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\geq m_{\mathscr{A}}|\varepsilon_1-\varepsilon_2|^2\\ \text{for all } \varepsilon_1,\varepsilon_2\in\mathbb{S}_n \text{ a.e. } x\in\Omega.\\ \text{(c) The mapping } x\mapsto\mathscr{A}(x,\varepsilon) \text{ is Lebesgue}\\ \text{measurable on } \Omega \text{ for all } \varepsilon\in\mathbb{S}_n.\\ \text{(d)The mapping } x\mapsto\mathscr{A}(x,0)\in\mathscr{H}.\end{cases}\tag{2.17}$$

The *relaxation function* $\mathscr{E}:\Omega\times\mathbb{S}_n\to\mathbb{S}_n$ satisfies:

$$\begin{cases}\text{(a)There exists a constant } L_{\mathscr{E}}>0 \text{ such that}\\ |\mathscr{E}(x,\varepsilon_1)-\mathscr{E}(x,\varepsilon_2)|\leq L_{\mathscr{E}}|\varepsilon_1-\varepsilon_2|\\ \text{for all } \varepsilon_1,\varepsilon_2\in\mathbb{S}_n,\text{a.e. } x\in\Omega.\\ \text{(b)The mapping } x\mapsto\mathscr{E}(x,\varepsilon)\\ \text{is Lebesgue measurable on } \Omega \text{ for all } \varepsilon\in\mathbb{S}_n.\\ \text{(c)The mapping } x\mapsto\mathscr{E}(x,0)\in\mathscr{H}.\end{cases}\tag{2.18}$$

The *plasticity operator* $\mathscr{G}:\Omega\times\mathbb{S}_n\times\mathbb{S}_n\times\mathbb{R}\times\mathbb{R}\to\mathbb{S}_n$ satisfies:

$$\begin{cases}\text{(a)There exists a constant } L_{\mathscr{G}}>0 \text{ such that}\\ |\mathscr{G}(x,\sigma_1,\varepsilon_1,\theta_1,\varsigma_1)-\mathscr{G}(x,\sigma_2,\varepsilon_2,\theta_2,\varsigma_2)|\leq\\ L_{\mathscr{G}}(|\sigma_1-\sigma_2|+|\varepsilon_1-\varepsilon_2|+|\theta_1-\theta_2|+|\varsigma_1-\varsigma_2|)\\ \text{for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in\mathbb{S}_n \text{ for all } \theta_1,\theta_2\in\mathbb{R},\\ \text{for all } \varsigma_1,\varsigma_2\in\mathbb{R},\ \text{a.e. } x\in\Omega.\\ \text{(b)The mapping } x\mapsto\mathscr{G}(\mathbf{x},\sigma,\varepsilon,\theta,\varsigma)\\ \text{is Lebesgue measurable on } \Omega\\ \text{for all } \sigma,\varepsilon\in\mathbb{S}_n,\text{for all } \theta,\varsigma\in\mathbb{R}.\\ \text{(c)The mapping } x\mapsto\mathscr{G}(x,0,0,0,0)\in\mathscr{H}.\end{cases}\tag{2.19}$$

The function $\psi:\Omega\times\mathbb{S}_n\times\mathbb{S}_n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies:

$$\begin{cases}\text{(a)There exists a constant } L_{\psi}>0 \text{ such that}\\ |\psi(x,\sigma_1,\varepsilon_1,\theta_1,\varsigma_1)-\psi(x,\sigma_2,\varepsilon_2,\theta_2,\varsigma_2)|\leq\\ L_{\psi}(|\sigma_1-\sigma_2|+|\varepsilon_1-\varepsilon_2|+|\theta_1-\theta_2|+|\varsigma_1-\varsigma_2|)\\ \text{for all } \sigma_1,\sigma_2\in\mathbb{S}_n,\ \text{for all } \varepsilon_1,\varepsilon_2\in\mathbb{S}_n,\\ \text{for all } \theta_1,\theta_2\in\mathbb{R},\ \text{for all } \varsigma_1,\varsigma_2\in\mathbb{R} \text{ a.e. } x\in\Omega.\\ \text{(b)The mapping } x\mapsto\psi(x,\sigma,\varepsilon,\theta,\varsigma) \text{ is}\\ \text{Lebesgue measurable on } \Omega,\ \text{for all } \sigma,\varepsilon\in\mathbb{S}_n,\\ \text{for all } \theta,\varsigma\in\mathbb{R}.\\ \text{(c)The mapping } x\mapsto\psi(x,0,0,0,0)\in L^2(\Omega).\end{cases}\tag{2.20}$$

The *adhesion rate function* $H_{ad}:\Gamma_3\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{d-1}\to\mathbb{R}$ satisfies:

$$\begin{cases}\text{(a) There exists } L_{ad}>0 \text{ such that :}\\ |H_{ad}(x,\beta_1,\xi_1,r_1,d_1)-H_{ad}(x,\beta_2,\xi_2,r_2,d_2)|\leq\\ L_{ad}\big(|\beta_1-\beta_2|+|\xi_1-\xi_2|+|r_1-r_2|+|d_1-d_2|\big),\\ \forall\,\beta_1,\beta_2,\xi_1,\xi_2,r_1,r_2\in\mathbb{R},\ d_1,d_2\in\mathbb{R}^{d-1},\ \text{a.e. } x\in\Gamma_3.\\ \text{(b) The mapping } x\mapsto H_{ad}(x,\beta,\xi,r,d) \text{ is}\\ \text{measurable on}\Gamma_3,\quad \text{for any } \beta,\xi,r\in\mathbb{R},\ d\in\mathbb{R}^{d-1},\\ \text{(c) The mapping } (\beta,\xi,r,d)\mapsto H_{ad}(x,\beta,\xi,r,d) \text{ is}\\ \text{continuous on } \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{d-1},\ \text{a.e. } x\in\Gamma_3,\\ \text{(d) } H_{ad}(x,0,\xi,r,d)=0,\forall\,\xi,r\in\mathbb{R},\\ d\in\mathbb{R}^{d-1},\text{a.e. } x\in\Gamma_3,\\ \text{(e) } H_{ad}(x,\beta,\xi,r,d)\geq 0,\quad \forall\,\beta\leq 0,\xi,r\in\mathbb{R},\\ d\in\mathbb{R}^{d-1},\text{a.e. } x\in\Gamma_3,\\ \text{(f) } H_{ad}(x,\beta,\xi,r,d)\leq 0,\quad \forall\,\beta\geq 1,\xi,r\in\mathbb{R},\\ d\in\mathbb{R}^{d-1},\text{a.e. } x\in\Gamma_3,\end{cases}\tag{2.21}$$

The *normal compliance* $p_\nu:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}_+$ satisfies:

$$\begin{cases}\text{(a) There exists a constant } L_\nu>0 \text{ such that}\\ |p_\nu(x,r_1)-p_\nu(x,r_2)|\leq L_\nu|r_1-r_2|,\\ \forall\ r_1,r_2\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3.\\ \text{(b)The mapping } x\mapsto p_\nu(x,r) \text{ is Lebesgue}\\ \text{measurable on , } \Gamma_3,\ \forall r\in\mathbb{R}.\\ \text{(c)The mapping } x\mapsto p_\nu(x,r)=0\\ \text{for any } r\leq 0,\ \text{a.e. } x\in\Gamma_3.\end{cases}\tag{2.22}$$

The *damage source* $\phi:\Omega\times\mathbb{S}_n\times\mathbb{S}_n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies:

$$\begin{cases}\text{(a)There exists a constant } L_\phi>0 \text{ such that}\\ |\phi(x,\sigma_1,\varepsilon_1,\theta_1,\varsigma_1)-\phi(x,\sigma_2,\varepsilon_2,\theta_2,\varsigma_2)|\leq\\ L_\phi(|\sigma_1-\sigma_2|+|\varepsilon_1-\varepsilon_2|+|\theta_1-\theta_2|+|\varsigma_1-\varsigma_2|)\\ \text{for all } \sigma_1,\sigma_2\in\mathbb{S}_n,\text{for all } \varepsilon_1,\varepsilon_2\in\mathbb{S}_n,\\ \text{for all } \theta_1,\theta_2,\varsigma_1,\varsigma_2\in\mathbb{R},\ \text{a.e. } x\in\Omega.\\ \text{(b)The mapping } x\mapsto\phi(x,\sigma,\varepsilon,\theta,\varsigma) \text{ is}\\ \text{Lebesgue measurable on } \Omega\text{for all } \sigma,\varepsilon\in\mathbb{S}_n,\\ \text{for all } \theta,\varsigma\in\mathbb{R}.\\ \text{(c)The mapping } x\mapsto\phi(x,0,0,0,0)\in L^2(\Omega).\end{cases}\tag{2.23}$$

The *tangential contact function* $p_\tau:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}_+$ satisfies:

$$\begin{cases}\text{(a)There exists a constant } L_\tau>0 \text{ such that}\\ \|p_\tau(x,d_1)-p_\tau(x,d_2)|\leq L_\tau\,|d_1-d_2|\\ \forall\,d_1,d_2\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3,\\ \text{(b)There exists a constant} M_\tau>0 \text{ such that}\\ |p_\tau(x,d)|\leq M_\tau\ \forall\,d\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3.\\ \text{(c)The mapping } x\mapsto p_\tau(x,d) \text{ is}\\ \text{Lebesgue measurable on } \Omega,\ \forall\ d\in\mathbb{R}.\\ \text{(d)The mapping } x\mapsto p_\tau(x,0)\in L^2(\Gamma_3).\end{cases}\tag{2.24}$$

The *body forces*, *surface tractions* and the *volume heat source* have the regularity

$$f\in L^2(0,T;H),\ f_0\in L^2(0,T;L^2(\Gamma_2)^n),\ q\in L^2(0,T;L^2(\Omega)),\tag{2.25}$$

$$u_0\in\mathscr{V},\quad w_0\in H,\quad \theta_0\in V,\quad \varsigma_0\in K,\tag{2.26}$$

$$\beta_0\in L^2(\Gamma_3),\ 0\leq\beta_0\leq 1,\ \text{a.e on }\Gamma_3,\tag{2.27}$$

$$k_i > 0, \quad i = 0,1. \tag{2.28}$$

The *mass density* ρ satisfies:

$$\rho \in L^\infty(\Omega), \exists \rho^* > 0 \text{ such that } \rho \geq \rho^* \text{ a.e. } x \in \Omega. \tag{2.29}$$

The *adhesion coefficient* and the *limit bound* satisfy:

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \gamma_\nu, \gamma_\tau \geq 0. \tag{2.30}$$

Using the Riesz representation theorem, we define the linear mapping $F \to \mathscr{V}'$, by

$$\begin{gathered}\langle F(t), v\rangle_{\mathscr{V}'\times\mathscr{V}} = (f(t), v)_H + (f_0(t), \gamma v)_{L^2(\Gamma_2)^n} \\ \forall v \in \mathscr{V}, \ t \in (0,T).\end{gathered} \tag{2.31}$$

The use of (2.25) permits to verify that

$$F \in L^2(0,T;\mathscr{V}'). \tag{2.32}$$

We introduce the following continuous functionals

$$\begin{gathered}\mathfrak{a}_0 : V \times V \to \mathbb{R}, \quad \mathfrak{a}_0(\zeta,\xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi dx + \\ \alpha \int_\Gamma \zeta\xi d\Gamma, \\ \mathfrak{a}_1 : V \times V \to \mathbb{R}, \quad \mathfrak{a}_1(\zeta,\xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi dx.\end{gathered} \tag{2.33}$$

Finally, we consider the adhesion functional $j : L^\infty(\Gamma_3) \times \mathscr{V} \times \mathscr{V} \to \mathbb{R}$ defined by

$$\begin{gathered}j(\beta,u,v) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu da + \\ \int_{\Gamma_3} (-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + p_\tau(\beta) R_\tau(u_\tau).v_\tau) da.\end{gathered} \tag{2.34}$$

Keeping in mind (2.22) and (2.24), we observe that integrals in (2.34) are well defined. Using standard arguments based on Green's formula (2.3), we can derive the following variational formulation of the frictionless problem with normal compliance (2.4)–(2.16) as follows.

Problem PV

Find the displacement field $u : [0,T] \to \mathbb{R}^n$, the stress field $\sigma : [0,T] \to \mathbb{S}_n$, the temperature $\theta : [0,T] \to \mathbb{R}$, the damage field $\varsigma : [0,T] \to \mathbb{R}$ and the adhesion field $\beta : [0,T] \to \mathbb{R}$ such that

$$\begin{gathered}\sigma(t) = \mathscr{A}\big(\varepsilon(\dot{u}(t))\big) + \mathscr{E}\big(\varepsilon(u(t))\big) + \\ \int_0^t \mathscr{G}\big(\sigma(s) - \mathscr{A}\big(\varepsilon(\dot{u}(s))\big), \varepsilon\big(u(s)\big), \theta(s), \varsigma(s)\big) ds \\ \text{a.e. } t \in (0,T),\end{gathered} \tag{2.35}$$

$$\begin{gathered}\langle \rho\ddot{u}(t), v\rangle_{\mathscr{V}'\times\mathscr{V}} + (\sigma(t), \varepsilon(v))_{\mathscr{H}} + j(\beta(t), u(t), v) \\ = \langle F(t), v\rangle_{\mathscr{V}'\times\mathscr{V}} \ \ \forall v \in \mathscr{V}, \text{ a.e. } t \in (0,T),\end{gathered} \tag{2.36}$$

$$\begin{gathered}\langle \rho\dot{\theta}(t), \omega\rangle_{V'\times V} + \mathfrak{a}_0(\theta(t), \omega) = (q(t), \omega)_{L^2(\Omega)} + \\ \big\langle \psi\big(\sigma(t) - \mathscr{A}\big(\varepsilon(\dot{u}(t))\big), \varepsilon\big(u(t)\big), \theta(t), \varsigma(t)\big), \omega\big\rangle_{V'\times V} \\ \forall \omega \in V, \quad \text{a.e. } t \in (0,T),\end{gathered} \tag{2.37}$$

$$\begin{gathered}\langle \rho\dot{\varsigma}(t), \xi - \varsigma(t)\rangle_{V'\times V} + \mathfrak{a}_1(\varsigma(t), \xi - \varsigma(t)) \geq \\ \big\langle \phi\big(\sigma(t) - \mathscr{A}\big(\varepsilon(\dot{u}(t))\big), \varepsilon\big(u(t)\big), \theta(t), \varsigma(t)\big), \xi - \varsigma(t)\big\rangle_{V'\times V} \\ \forall \xi \in K, \quad \text{a.e. } t \in (0,T), \varsigma(t) \in K,\end{gathered} \tag{2.38}$$

$$\dot{\beta} = H_{ad}\big(\beta, \xi_\beta, R_\nu(u_\nu), R_\tau(u_\tau)\big) \ \text{ a.e. } t \in (0,T), \tag{2.39}$$

$$u(0) = u_0, \ \dot{u}(0) = w_0, \ \theta(0) = \theta_0, \ \varsigma(0) = \varsigma_0, \ \beta(0) = \beta_0. \tag{2.40}$$

3. Main Results

The existence of the unique solution to Problem PV is proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 3.1. We note that, in Problem P and in Problem PV, we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, (2.40) guarantees that $\beta(x,t) \leq \beta_0(x)$ and, therefore, assumption (2.27) shows that $\beta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\beta(x,t_0) = 0$ at time t_0, then it follows from (2.40) that $\beta(x,t) = \beta_0(x)$ for all $t \geq t_0$, and therefore $\beta(x,t) = 0$ for all $t \geq t_0, x \in \Gamma_3$.
We conclude that $0 \leq \beta(x,t) \leq 1$ for all $t \geq t_0, x \in \Gamma_3$.

Theorem 3.2 (Existence and uniqueness)**.** *Under assumptions (2.17)–(2.28), there exists a unique solution $\{u, \sigma, \theta, \varsigma, \beta\}$ to problem PV. Moreover, the solution has the regularity*

$$u \in \mathscr{C}^0(0,T;\mathscr{V}) \cap \mathscr{C}^1(0,T;H), \tag{3.1}$$

$$\dot{u} \in L^2(0,T;\mathscr{V}), \tag{3.2}$$

$$\ddot{u} \in L^2(0,T;\mathscr{V}'), \tag{3.3}$$

$$\sigma \in L^2(0,T;\mathscr{H}), \tag{3.4}$$

$$\theta \in L^2(0,T;V) \cap \mathscr{C}^0(0,T;L^2(\Omega)), \tag{3.5}$$

$$\dot{\theta} \in L^2(0,T;V'), \tag{3.6}$$

$$\varsigma \in L^2(0,T;V) \cap \mathscr{C}^0(0,T;L^2(\Omega)), \tag{3.7}$$

$$\dot{\varsigma} \in L^2(0,T;V'), \tag{3.8}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathscr{Z}. \tag{3.9}$$

A quintuple $(u, \sigma, \theta, \varsigma, \beta)$ which satisfies (2.35)–(2.40) is called a weak solution to the compliance contact Problem P. We conclude that under the stated assumptions, problem (2.4)–(2.16) has a unique weak solution satisfying (3.1)–(3.9).
We turn now to the proof of Theorem 3.2, which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in the following that (2.17)–(2.28) hold. Below, C denotes a generic positive constant which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathscr{A}$, $\mathscr{E}$, $\mathscr{G}$, H_{ad}, ψ, ϕ, p_ν, p_τ, γ_ν, γ_τ, L and T but does not depend on t nor on the rest of input data, and whose value may change from place to place. Moreover, for the sake of simplicity we suppress in what follows the explicit dependence of various functions on $x \in \Omega \cup \Gamma$.
Let $\eta \in L^2(0,T;\mathscr{V}')$ be given. In the first step we consider the following variational problem.

Problem PV$_\eta$
Find the displacement field $u_\eta : [0,T] \to \mathbb{R}^n$, such that

$$\begin{aligned}\langle \rho \ddot{u}_\eta(t), v\rangle_{\mathscr{V}'\times\mathscr{V}} + \big(\mathscr{A}\big(\varepsilon(\dot{u}_\eta(t))\big), \varepsilon(v)\big)_{\mathscr{H}} + \\ \langle \eta(t), v\rangle_{\mathscr{V}'\times\mathscr{V}} = \langle F(t), v\rangle_{\mathscr{V}'\times\mathscr{V}} \\ \forall v \in \mathscr{V},\ \text{a.e. } t \in (0,T),\end{aligned} \tag{3.10}$$

$$u_\eta(0) = u_0, \quad \dot{u}_\eta(0) = w_0 \quad \text{in } \Omega. \tag{3.11}$$

Lemma 3.3. *For all $\eta \in L^2(0,T;\mathscr{V}')$, there exists a unique solution u_η to the auxiliary problem PV$_\eta$ satisfying (3.1)–(3.3).*

Proof. Let us introduce the operator $A : \mathscr{V} \to \mathscr{V}'$,

$$\langle Au, v\rangle_{\mathscr{V}'\times\mathscr{V}} = \big(\mathscr{A}\big(\varepsilon(u)\big), \varepsilon(v)\big)_{\mathscr{H}}. \tag{3.12}$$

Therefore, (3.10) can be rewritten as follows

$$\rho \ddot{u}_\eta(t) + A\big(\dot{u}_\eta(t)\big) = F_\eta(t) \quad \text{on } \mathscr{V}' \text{ a.e. } t \in (0,T), \tag{3.13}$$

where

$$F_\eta(t) = F(t) - \eta(t) \in \mathscr{V}'.$$

It follows from (2.1), (3.12) and hypothesis (2.17) that A is bounded, semi-continuous and coercive on $\mathscr{V}$. We recall that by (2.32) we have $F_\eta \in L^2(0,T;\mathscr{V}')$. Then by using classical arguments of functional analysis concerning parabolic equations [5] we can easily prove the existence and uniqueness of w_η satisfying

$$w_\eta \in L^2(0,T;\mathscr{V}) \cap \mathscr{C}^0(0,T;H), \tag{3.14}$$

$$\dot{w}_\eta \in L^2(0,T;\mathscr{V}'), \tag{3.15}$$

$$\begin{aligned}\rho \dot{w}_\eta(t) + A(w_\eta(t)) = F_\eta(t) \\ \text{on } \mathscr{V}' \text{ a.e. } t \in (0,T),\end{aligned} \tag{3.16}$$

$$w_\eta(0) = w_0. \tag{3.17}$$

Consider now the function $u_\eta : (0,T) \to \mathscr{V}$ defined by

$$u_\eta(t) = \int_0^t w_\eta(s)ds + u_0 \quad \forall t \in (0,T). \tag{3.18}$$

It follows from (3.16) and (3.17) that u_η is a solution of the equation (3.13) and it satisfies (3.1)–(3.3). □

In the second step we use the displacement field u_η obtained in Lemma 3.3 and we consider the following initial value problem.

Problem PV$_\beta$
Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_\eta(t) = H_{ad}\big(\beta_\eta(t), \xi_{\beta_\eta}, R_\nu(u_{\eta\nu}(t)), R_\tau(u_{\eta\tau}(t))\big), \tag{3.19}$$

$$\beta_\eta(0) = \beta_0 \ \text{ in } \ \Omega. \tag{3.20}$$

Lemma 3.4. *Problem PV$_\beta$ has a unique solution β_η such that*

$$\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathscr{Z}.$$

Proof. We consider the mapping $H_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$H_\eta(t,\beta) = H_{ad}\big(\beta_\eta(t), \xi_{\beta_\eta}, R_\nu(u_{\eta\nu}(t)), R_\tau(u_{\eta\tau}(t))\big),$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and R_τ that H_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to H_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy–Lipschitz theorem given in [30, p. 60], we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution of the equation (3.19). Also, the arguments used in Remark 3.1 show that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. This completes the proof. □

Problem PV$_\lambda$
Find the temperature $\theta_\lambda : [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}\langle \rho \dot{\theta}_\lambda(t), \omega\rangle_{V'\times V} + \mathfrak{a}_0(\theta_\lambda(t), \omega) = \\ \langle \lambda(t) + q(t), \omega\rangle_{V'\times V}, \ \forall \omega \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{3.21}$$

$$\theta_\lambda(0) = \theta_0 \text{ in } \Omega. \tag{3.22}$$

Lemma 3.5. *For all $\lambda \in L^2(0,T;V')$, there exists a unique solution θ_λ to the auxiliary problem PV$_\lambda$ satisfying (3.5) and (3.6).*

Proof. By an application of the Poincaré–Friedrichs inequality, we can find a constant $\alpha' > 0$ such that

$$\int_\Omega |\nabla\zeta|^2 dx + \frac{\alpha}{k_0}\int_\Gamma |\zeta|^2 d\gamma \ge \alpha' \int_\Omega |\zeta|^2 dx \quad \forall \zeta \in V.$$

Thus, we obtain

$$\mathfrak{a}_0(\zeta,\zeta) \ge C_1 \|\zeta\|_V^2 \quad \forall \zeta \in V, \tag{3.23}$$

where $C_1 = k_0 \min(1,\alpha')/2$, which implies that $\mathfrak{a}_0$ is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (3.21) has a unique solution θ_λ satisfies (3.5) and (3.6). □

Problem PV$_\mu$
Find the damage field $\varsigma_\mu : [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}\langle \rho \dot{\varsigma}_\mu(t), \xi - \varsigma_\mu(t)\rangle_{V'\times V} + \mathfrak{a}_1(\varsigma_\mu(t), \xi - \varsigma_\mu(t)) \\ \ge \langle \mu, \xi - \varsigma_\mu(t)\rangle_{V'\times V} \\ \forall \xi \in K, \text{ a.e. } t \in (0,T),\ \varsigma_\mu(t) \in K,\end{aligned} \tag{3.24}$$

$$\varsigma_\mu(0) = \varsigma_0 \quad \text{in } \Omega. \tag{3.25}$$

Lemma 3.6. *For all $\mu \in L^2(0,T;V')$, there exists a unique solution ς_μ to the auxiliary problem PV$_\mu$ satisfying (3.7)–(3.8).*

Proof. We know that the form $\mathfrak{a}_1$ is not V-elliptic. To solve this problem we introduce the functions

$$\tilde{\varsigma}_\mu(t) = e^{-k_1 t}\varsigma_\mu(t), \quad \tilde{\xi}(t) = e^{-k_1 t}\xi(t).$$

We remark that if $\varsigma_\mu,\ \xi \in K$ then $\tilde{\varsigma}_\mu,\ \tilde{\xi} \in K$. Consequently, (3.24) is equivalent to the inequality

$$\begin{aligned}\langle \rho\dot{\tilde{\varsigma}}_\mu(t), \tilde{\xi}-\tilde{\varsigma}_\mu(t)\rangle_{V'\times V} + \mathfrak{a}_1(\tilde{\varsigma}_\mu(t), \tilde{\xi}-\tilde{\varsigma}_\mu(t))\\ +k_1(\rho\tilde{\varsigma}_\mu, \tilde{\xi}-\tilde{\varsigma}_\mu(t))_{L^2(\Omega)} \geq \langle e^{-k_1 t}\mu, \tilde{\xi}-\tilde{\varsigma}_\mu(t)\rangle_{V'\times V} \quad (3.26)\\ \forall \tilde{\xi}\in K,\ \text{a.e. } t\in(0,T),\ \tilde{\varsigma}_\mu \in K.\end{aligned}$$

The fact that

$$\mathfrak{a}_1(\tilde{\xi},\tilde{\xi})+k_1(\rho\tilde{\xi},\tilde{\xi})_{L^2(\Omega)} \geq k_1 \min(\rho^*,1)\|\tilde{\xi}\|_V^2 \quad \forall \tilde{\xi}\in V, \tag{3.27}$$

and using classical arguments of functional analysis concerning parabolic inequalities [5], implies that (3.24) has a unique solution $\tilde{\varsigma}_\mu$ having the regularity (3.7) and (3.8). □

Let us consider now the auxiliary problem.

Problem $\mathrm{PV}_{\eta,\lambda,\mu}$

Find the stress field $\sigma_{\eta,\lambda,\mu} : [0,T] \to \mathbb{S}_n$ which is a solution of the problem

$$\begin{aligned}\sigma_{\eta,\lambda,\mu}(t) = \mathscr{E}\big(\varepsilon(u_\eta(t))\big)\\ +\int_0^t \mathscr{G}\big(\sigma_{\eta,\lambda,\mu}(s), \varepsilon\big(u_\eta(s)\big), \theta_\lambda(s), \varsigma_\mu(s)\big)ds \quad (3.28)\\ \forall t\in[0,T].\end{aligned}$$

Lemma 3.7. *There exists a unique solution of Problem $PV_{\eta,\lambda,\mu}$ and it satisfies (3.4). Moreover, if $u_{\eta_i}, \theta_{\lambda_i}, \varsigma_{\mu_i}$ and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solutions of problems PV_{η_i}, PV_{λ_i}, PV_{μ_i} and $PV_{\eta_i,\lambda_i,\mu_i}$, respectively, for $i=1,2$, then there exists $C>0$ such that*

$$\begin{aligned}\|\sigma_{\eta_1,\lambda_1,\mu_1}(t)-\sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathscr{H}}^2 \leq C\big(\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathscr{V}}^2\\ +\int_0^t(\|u_{\eta_1}(s)-u_{\eta_2}(s)\|_{\mathscr{V}}^2+\|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|_V^2+\\ \|\varsigma_{\mu_1}(s)-\varsigma_{\mu_2}(s)\|_V^2)ds\big).\end{aligned} \tag{3.29}$$

Proof. Let $\Sigma_{\eta,\lambda,\mu} : L^2(0,T;\mathscr{H}) \to L^2(0,T;\mathscr{H})$ be the mapping given by

$$\begin{aligned}\Sigma_{\eta,\lambda,\mu}\sigma(t) = \mathscr{E}\big(\varepsilon(u_\eta(t))\big)+\\ \int_0^t \mathscr{G}\big(\sigma(s), \varepsilon\big(u_\eta(s)\big), \theta_\lambda(s), \varsigma_\mu(s)\big)ds.\end{aligned} \tag{3.30}$$

Let $\sigma_i \in L^2(0,T;\mathscr{H})$, $i=1,2$ and $t_1\in(0,T)$.
Using hypothesis (2.19) and Hölder's inequality, we find

$$\begin{aligned}\|\Sigma_{\eta,\lambda,\mu}\sigma_1(t_1)-\Sigma_{\eta,\lambda,\mu}\sigma_2(t_1)\|_{\mathscr{H}}^2 \leq\\ L_{\mathscr{G}}^2 T\int_0^{t_1}\|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2 ds.\end{aligned} \tag{3.31}$$

By reapplication of mapping $\Sigma_{\eta,\lambda,\mu}$, it yields

$$\begin{aligned}\Big\|\Sigma_{\eta,\lambda,\mu}^2\sigma_1(t_1-\Sigma_{\eta,\lambda,\mu}^2\sigma_2(t_1)\Big\|_{\mathscr{H}}^2 \leq\\ L_{\mathscr{G}}^4 T^2\int_0^{t_1}\int_0^{t_2}\|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2 dsdt_2.\end{aligned}$$

Reiterating this inequality m times leads to

$$\begin{aligned}\Big\|\Sigma_{\eta,\lambda,\mu}^m\sigma_1(t_1)-\Sigma_{\eta,\lambda,\mu}^m\sigma_2(t_1)\Big\|_{\mathscr{H}}^2 \leq\\ L_{\mathscr{G}}^{2m}T^m\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_m}\|\sigma_1(s)-\sigma_2(s)\|_{\mathscr{H}}^2 dsdt_m\ldots dt_2.\end{aligned}$$

Integration on the time interval $(0,T)$, it follows that

$$\Big\|\Sigma_{\eta,\lambda,\mu}^m\sigma_1-\Sigma_{\eta,\lambda,\mu}^m\sigma_2\Big\|_{L^2(0,T;\mathscr{H})}^2 \leq \frac{L_{\mathscr{G}}^{2m}T^{2m}}{m!}\|\sigma_1-\sigma_2\|_{L^2(0,T;\mathscr{H})}^2. \tag{3.32}$$

It follows from this inequality that for m large enough, a power m of the mapping $\Sigma_{\eta,\lambda,\mu}$ is a contraction on the space $L^2(0,T;\mathscr{H})$ and, therefore, from the Banach fixed point theorem, there exists a unique element $\sigma_{\eta,\lambda,\mu}\in L^2(0,T;\mathscr{H})$ such that $\Sigma_{\eta,\lambda,\mu}\sigma_{\eta,\lambda,\mu}=\sigma_{\eta,\lambda,\mu}$, which represents the unique solution of the problem $\mathrm{PV}_{\eta,\lambda,\mu}$. Moreover, if $u_{\eta_i}, \theta_{\lambda_i}, \varsigma_{\mu_i}$ and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solutions of the problems PV_{η_i}, PV_{λ_i}, PV_{μ_i} and $\mathrm{PV}_{\eta_i,\lambda_i,\mu_i}$, respectively, for $i=1,2$, then we use (2.1), (2.17)–(2.19) and Young's inequality to obtain

$$\begin{aligned}\|\sigma_{\eta_1,\lambda_1,\mu_1}(t)-\sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathscr{H}}^2 \leq C\Big(\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathscr{V}}^2\\ +\int_0^t(\|\sigma_{\eta_1,\lambda_1,\mu_1}(s)-\sigma_{\eta_2,\lambda_2,\mu_2}(s)\|_{\mathscr{H}}^2+\|u_{\eta_1}(s)-u_{\eta_2}(s)\|_{\mathscr{V}}^2\\ +\|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|_V^2+\|\varsigma_{\mu_1}(s)-\varsigma_{\mu_2}(s)\|_V^2)ds\Big).\end{aligned}$$

Which permits us to obtain, using Gronwall's lemma, the inequality (3.29).

By taking into account the above results and the properties of the operators $\mathscr{E}$ and $\mathscr{G}$ of the functions ψ and ϕ, we may consider the operator

$$\Lambda : L^2(0,T;\mathscr{V}'\times V'\times V') \to L^2(0,T;\mathscr{V}'\times V'\times V'),$$

$$\begin{aligned}\Lambda\big(\eta(t),\lambda(t),\mu(t)\big) = \big(\Lambda^0(\eta(t),\lambda(t),\mu(t)),\\ \Lambda^1(\eta(t),\lambda(t),\mu(t)),\Lambda^2(\eta(t),\lambda(t),\mu(t))\big),\end{aligned} \tag{3.33}$$

defined by

$$\begin{aligned}\big\langle\Lambda^0\big(\eta(t),\lambda(t),\mu(t)\big),v\big\rangle_{\mathscr{V}'\times\mathscr{V}} = \big(\mathscr{E}\big(\varepsilon(u_\eta(t)),\varepsilon(v)\big)_{\mathscr{H}}\\ +\Big(\int_0^t\mathscr{G}\big(\sigma_{\eta,\lambda,\mu}(s),\varepsilon\big(u_\eta(s)\big),\theta_\lambda(s),\varsigma_\mu(s)\big)ds,\varepsilon(v)\Big)_{\mathscr{H}}\\ +j(\beta_\eta(t),u_\eta(t),v)\ \forall v\in\mathscr{V},\end{aligned} \tag{3.34}$$

$$\Lambda^1(\eta(t),\lambda(t),\mu(t)) = \psi\big(\sigma_{\eta,\lambda,\mu}(t),\varepsilon(u_\eta(t)),\theta_\lambda(t),\varsigma_\mu(t)\big) \tag{3.35}$$

$$\Lambda^2(\eta(t),\lambda(t),\mu(t)) = \phi\big(\sigma_{\eta,\lambda,\mu}(t),\varepsilon(u_\eta(t)),\theta_\lambda(t),\varsigma_\mu(t)\big). \tag{3.36}$$

□

Lemma 3.8. *The mapping Λ has a fixed point*

$$(\eta^*,\lambda^*,\mu^*)\in L^2(0,T;\mathscr{Y}'\times V'\times V').$$

Proof. Let $(\eta_1,\lambda_1,\mu_1),(\eta_2,\lambda_2,\mu_2)\in L^2(0,T;\mathscr{Y}'\times V'\times V')$.

We use the notation $u_{\eta_i}=u_i$, $\dot{u}_{\eta_i}=\dot{u}_i$, $\ddot{u}_{\eta_i}=\ddot{u}_i$, $\beta_{\eta_i}=\beta_i$, $\theta_{\lambda_i}=\theta_i,\varsigma_{\mu_i}=\varsigma_i$ and $\sigma_{\eta_i,\lambda_i,\mu_i}=\sigma_i$, for $i=1,2$. Let us start by using (2.1), hypotheses (2.17)–(2.19), (2.23)–(2.24) and the definition of R_η, R_τ and Remark 3.1 we have

$$\begin{aligned}
&\|\Lambda^0\big(\eta_1(t),\lambda_1(t),\mu_1(t)\big)-\Lambda^0\big(\eta_2(t),\lambda_2(t),\mu_2(t)\big)\|^2_{\mathscr{Y}'}\\
&\leq\|\mathscr{E}\big(\varepsilon(u_1(t))\big)-\mathscr{E}\big(\varepsilon(u_2(t))\big)\|^2_{\mathscr{Y}}\\
&+\int_0^t\|\mathscr{G}\big(\sigma_1(s),\varepsilon\big(u_\eta(s)\big),\theta_1(s),\varsigma_1(s)\big)-\\
&\mathscr{G}\big(\sigma_2(s),\varepsilon\big(u_2(s)\big),\theta_2(s),\varsigma_2(s)\big)\|^2_{\mathscr{H}}ds\\
&+C\big(\|p_\nu(u_{1\eta\nu}(t))-p_\nu(u_{2\eta\nu}(t))\|^2_{L^2(\Gamma_3)}\big)\\
&+C\big(\|\beta_1^2(t)R_\nu(u_{1\eta\nu}(t))-\beta_2^2(t)R_\nu(u_{2\eta\nu}(t))\|^2_{L^2(\Gamma_3)}\big)\\
&+C\big(\|p_\tau(\beta_1(t))R_\tau(u_{1\eta\tau}(t))-p_\tau(\beta_2(t))R_\tau(u_{2\eta\tau}(t))\|^2_{L^2(\Gamma_3)}\big),
\end{aligned}$$

so we obtain

$$\begin{aligned}
&\|\Lambda^0(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^0(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{\mathscr{Y}'}\\
&\leq C\Big(\int_0^t\big(\|\sigma_1(s)-\sigma_2(s)\|^2_{\mathscr{H}}+\|u_1(s)-u_2(s)\|^2_{\mathscr{Y}}+\\
&\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}+\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L^2(\Omega)}\big)ds+\\
&\|u_1(t)-u_2(t)\|^2_{\mathscr{Y}}+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\Big).
\end{aligned}\tag{3.37}$$

We use estimate (3.29) to obtain

$$\begin{aligned}
&\|\Lambda^0\big(\eta_1(t),\lambda_1(t),\mu_1(t)\big)-\Lambda^0\big(\eta_2(t),\lambda_2(t),\mu_2(t)\big)\|^2_{\mathscr{Y}'}\\
&\leq C\Big(\int_0^t\big(\|u_1(s)-u_2(s)\|^2_{\mathscr{Y}}+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\\
&+\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L^2(\Omega)}\big)ds+\|u_1(t)-u_2(t)\|^2_{\mathscr{Y}}\\
&+\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\Big).
\end{aligned}\tag{3.38}$$

Also, from the Cauchy problem (3.19)–(3.20) we can write

$$\beta_i(t)=\beta_0-\int_0^t H_{ad}\big(\beta_i(s),\xi_{\beta_i}(s),R_\nu(u_{i\nu}(s)),R_\tau(u_{i\tau}(s))\big)ds$$

and, employing (2.21) we obtain that

$$\begin{aligned}
\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}&\leq C\int_0^t\big\|\beta_1(s)-\beta_2(s)\big\|_{L^2(\Gamma_3)}ds\\
&+C\int_0^t\big\|R_\nu(u_{1\nu}(s))-R_\nu(u_{2\nu}(s))\big\|_{L^2(\Gamma_3)}ds\\
&+C\int_0^t\big\|R_\tau(u_{1\tau}(s))-R_\tau(u_{2\tau}(s))\big\|_{L^2(\Gamma_3)}ds.
\end{aligned}$$

Using the definition of R_ν and R_τ and writing $\beta_1=\beta_1-\beta_2+\beta_2$, we get

$$\begin{aligned}
\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}&\leq C\Big(\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds\\
&+\int_0^t\big\|u_1(s)-u_2(s)\big\|_{L^2(\Gamma_3)^d}ds\Big).
\end{aligned}\tag{3.39}$$

Next, we apply Gronwall's inequality to deduce

$$\begin{aligned}
&\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\\
&\leq C\int_0^t\|u_1(s)-u_2(s)\|_{L^2(\Gamma_3)^d}ds,
\end{aligned}$$

and from the relation (2.1) we obtain that

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)}\leq C\int_0^t\|u_1(s)-u_2(s)\|^2_{\mathscr{Y}}ds\tag{3.40}$$

holds.

On the other hand, since $u_i(t)=u_0+\int_0^t\dot{u}_i(s)ds$, we know that for a.e. $t\in(0,T)$,

$$\|u_1(t)-u_2(t)\|_{\mathscr{Y}}\leq\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|_{\mathscr{Y}}ds.\tag{3.41}$$

Applying Young's and Hölder's inequalities, (3.38) becomes, via (3.40) and (3.41)

$$\begin{aligned}
&\|\Lambda^0\big(\eta_1(t),\lambda_1(t),\mu_1(t)\big)-\Lambda^0\big(\eta_2(t),\lambda_2(t),\mu_2(t)\big)\|^2_{\mathscr{Y}'}\\
&\leq C\Big(\int_0^t\big(\|\dot{u}_1(s)-\dot{u}_2(s)\|^2_{\mathscr{Y}}+\|u_1(s)-u_2(s)\|^2_{\mathscr{Y}}\\
&+\|\theta_1(s)-\theta_2(s)\|^2_V+\|\varsigma_1(s)-\varsigma_2(s)\|^2_V\big)ds\Big)\\
&\text{a.e. } t\in(0,T).
\end{aligned}\tag{3.42}$$

Furthermore, we find by taking the substitution $\eta=\eta_1$, $\eta=\eta_2$ in (3.10) and choosing $v=\dot{u}_1-\dot{u}_2$ as test function

$$\begin{aligned}
&\langle\rho(\ddot{u}_1(t)-\ddot{u}_2(t))+A\dot{u}_1(t)-A\dot{u}_2(t),\dot{u}_1(t)-\dot{u}_2(t)\rangle_{\mathscr{Y}'\times\mathscr{Y}}\\
&=\langle\eta_2(t)-\eta_1(t),\dot{u}_1(t)-\dot{u}_2(t)\rangle_{\mathscr{Y}'\times\mathscr{Y}}\quad\text{a.e. } t\in(0,T).
\end{aligned}$$

By virtue of (2.17) and (2.29), this equation becomes

$$\begin{aligned}
&\tfrac{(\rho^*)^2}{2}\tfrac{d}{dt}\|\dot{u}_1(t)-\dot{u}_2(t)\|^2_H+m_{\mathscr{A}}\|\dot{u}_1(t)-\dot{u}_2(t)\|^2_{\mathscr{Y}}\leq\\
&\|\eta_2(t)-\eta_1(t)\|_{\mathscr{Y}'}\|\dot{u}_1(t)-\dot{u}_2(t)\|_{\mathscr{Y}}.
\end{aligned}$$

Integrating this inequality over the interval time variable $(0,t)$, Young's inequality leads to

$$\begin{aligned}
&(\rho^*)^2\|\dot{u}_1(t)-\dot{u}_2(t)\|^2_H+m_{\mathscr{A}}\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|^2_{\mathscr{Y}}ds\leq\\
&\tfrac{2}{m_{\mathscr{A}}}\int_0^t\|\eta_1(s)-\eta_2(s)\|^2_{\mathscr{Y}'}ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|^2_{\mathscr{Y}}ds\leq C\int_0^t\|\eta_1(s)-\eta_2(s)\|^2_{\mathscr{Y}'}ds,\\
&\text{a.e. } t\in(0,T),
\end{aligned}\tag{3.43}$$

which also implies, using a variant of (3.41), that

$$\|u_1(t)-u_2(t)\|^2_{\mathscr{V}} \leq C\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_{\mathscr{V}'}ds \quad \text{a.e. } t\in(0,T). \tag{3.44}$$

Moreover, if we take the substitution $\lambda=\lambda_1$, $\lambda=\lambda_2$ in (3.21) and subtracting the two obtained equations, we deduce by choosing $\omega=\theta_{\lambda_1}-\theta_{\lambda_2}$ as test function

$$\frac{(\rho^*)^2}{2}\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+C_1\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_V ds \leq \int_0^t \|\lambda_1(s)-\lambda_2(s)\|_{V'}\|\theta_1(s)-\theta_2(s)\|_V ds \quad \text{a.e. } t\in(0,T). \tag{3.45}$$

Employing Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}&\|\theta_{\lambda_1}(t)-\theta_{\lambda_2}(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|^2_V ds\\ \leq& C\int_0^t\|\lambda_1(s)-\lambda_2(s)\|^2_{V'}ds \ \text{ a.e. } t\in(0,T).\end{aligned} \tag{3.46}$$

Substituting now $\{\mu=\mu_1,\xi=\tilde{\xi}_{\mu_1}\}$, $\{\mu=\mu_2,\xi=\tilde{\xi}_{\mu_2}\}$ in (3.26) and subtracting the two inequalities, we obtain

$$\begin{aligned}&\|\tilde{\xi}_1(t)-\tilde{\xi}_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\tilde{\xi}_1(s)-\tilde{\xi}_2(s)\|^2_V ds\\ \leq& C\int_0^t\|e^{-k_1 t}(\mu_1(s)-\mu_2(s))\|^2_{V'}ds \quad \text{a.e. } t\in(0,T),\end{aligned}$$

from which also follows that

$$\begin{aligned}&\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\varsigma_1(s)-\varsigma_2(s)\|^2_V ds\\ \leq& C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{V'}ds \ \text{ a.e. } t\in(0,T).\end{aligned} \tag{3.47}$$

We can infer, using (3.42)–(3.47), that

$$\begin{aligned}&\|\Lambda^0(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^0(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{\mathscr{V}'}\leq\\ &C\big(\|\eta_1(t)-\eta_2(t)\|^2_{\mathscr{V}'}+\|\lambda_1(t)-\lambda_2(t)\|^2_{V'}+\|\mu_1(t)-\mu_2(t)\|^2_{V'}\big).\end{aligned} \tag{3.48}$$

From hypothesis (2.20), (3.29) and (2.23) it follows

$$\begin{aligned}&\|\Lambda^1(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^1(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{V'}\\ =&\|\psi\big(\sigma_1(t),\varepsilon(u_1(t)),\theta_1(t),\varsigma_1(t)\big)-\\ &\qquad\psi\big(\sigma_2(t),\varepsilon(u_2(t)),\theta_2(t),\varsigma_2(t)\big)\|^2_{V'}\\ \leq& C\big(\|u_1(t)-u_2(t)\|^2_{\mathscr{V}}+\|\theta_1(t)-\theta_2(t)\|^2_V+\|\varsigma_1(t)-\varsigma_2(t)\|^2_V\big)\\ &\text{a.e. } t\in(0,T).\end{aligned}$$

This permits us to deduce, via (3.43), (3.46) and (3.47), that

$$\begin{aligned}&\|\Lambda^1(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^1(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{\mathscr{V}'}\leq\\ &C\big(\|\eta_1(t)-\eta_2(t)\|^2_{\mathscr{V}'}+\|\lambda_1(t)-\lambda_2(t)\|^2_{V'}+\|\mu_1(t)-\mu_2(t)\|^2_{V'}\big).\end{aligned} \tag{3.49}$$

Similarly, using (3.29), (3.44), (3.46) and (3.47), we obtain the following estimate for Λ^2

$$\begin{aligned}&\|\Lambda^2(\eta_1(t),\lambda_1(t),\mu_1(t))-\Lambda^2(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{V'}\\ =&\|\phi\big(\sigma_1(t),\varepsilon(u_1(t)),\theta_1(t),\varsigma_1(t)\big)-\\ &\qquad\phi\big(\sigma_2(t),\varepsilon(u_2(t)),\theta_2(t),\varsigma_2(t)\big)\|^2_{V'}\leq\\ &C\big(\|\eta_1(t)-\eta_2(t)\|^2_{\mathscr{V}'}+\|\lambda_1(t)-\lambda_2(t)\|^2_{V'}+\|\mu_1(t)-\mu_2(t)\|^2_{V'}\big).\end{aligned} \tag{3.50}$$

From (3.48), (3.49) and (3.50), we conclude that there exists a positive constant $C>0$ verifying

$$\begin{aligned}&\|\Lambda(\eta_1,\lambda_1,\mu_1)-\Lambda(\eta_2,\lambda_2,\mu_2)\|^2_{\mathscr{V}'\times V'\times V'}\leq\\ &C\|(\eta_1-\eta_2,\lambda_1-\lambda_2,\mu_1-\mu_2)\|^2_{\mathscr{V}'\times V'\times V'}.\end{aligned} \tag{3.51}$$

We generalize this procedure by recurrence on m. Then we obtain the formula

$$\begin{aligned}&\|\Lambda^m(\eta_1,\lambda_1,\mu_1)-\Lambda^m(\eta_2,\lambda_2,\mu_2)\|^2_{L^2(0,T;\mathscr{V}'\times V'\times V')}\\ \leq&\frac{C^m T^m}{m!}\|(\eta_1-\eta_2,\lambda_1-\lambda_2,\mu_1-\mu_2)\|^2_{L^2(0,T;\mathscr{V}'\times V'\times V')}.\end{aligned} \tag{3.52}$$

Thus, for m sufficiently large, Λ^m is a contraction on $L^2(0,T;\mathscr{V}'\times V'\times V')$. Hence, Banach's fixed point theorem shows that Λ admits a unique fixed point $(\eta^*,\lambda^*,\mu^*)\in L^2(0,T;\mathscr{V}'\times V'\times V')$. □

Now, we have all the ingredients to prove Theorem (3.2).

Proof. Let $(\eta^*,\lambda^*,\mu^*)\in L^2(0,T;\mathscr{V}'\times V'\times V')$ be the fixed point of Λ defined by (3.33)–(3.36) and denote by

$$(a)\ u=u_{\eta^*},\ \ (b)\ \theta=\theta_{\lambda^*},\ \ (c)\ \varsigma=\varsigma_{\mu^*}, \tag{3.53}$$

$$(d)\ \sigma=\mathscr{A}\varepsilon(\dot{u})+\sigma_{\eta^*\lambda^*\mu^*}\ ,\ (e)\ \beta=\beta_{\eta^*}. \tag{3.54}$$

We prove that $(u,\sigma,\theta,\varsigma,\beta)$ satisfies (2.35)–(**??**) and (3.1)–(3.9). Indeed, we write (3.28) for $\eta=\eta^*$, $\lambda=\lambda^*$ and $\mu=\mu^*$ using (3.53) and (3.54)(a) to obtain that (2.35) is satisfied. Now we consider (3.10) for $\eta=\eta^*$ and using (3.53)(a) to find

$$\begin{aligned}&\langle\rho\ddot{u},v\rangle_{V'\times V}+(\mathscr{A}(\varepsilon(\dot{u}(t))),\varepsilon(v))_{\mathscr{H}}+\langle\eta^*(t),v\rangle_{V'\times V}\\ &\quad=\langle F(t),v\rangle_{V'\times V},\quad\forall v\in V,\text{a.e.}t\in(0,T).\end{aligned} \tag{3.55}$$

Equalities $\Lambda^1(\eta^*,\lambda^*,\mu^*)=\eta^*$, $\Lambda^2(\eta^*,\lambda^*,\mu^*)=\lambda^*$ and $\Lambda^2(\eta^*,\lambda^*,\mu^*)=\mu^*$ combined with (3.34)–(3.36),(3.53) and (3.54) show that

$$\begin{aligned}&\langle\eta^*(t),v\rangle_{\mathscr{V}'\times\mathscr{V}}=\big(\mathscr{E}(\varepsilon(u(t))),\varepsilon(v)\big)_{\mathscr{H}}+\\ &\big(\int_0^t\mathscr{G}\big(\sigma(s)-\mathscr{A}(\varepsilon(\dot{u}(s))),\varepsilon(u(s)),\varsigma(s)\big)ds,\varepsilon(v)\big)_{\mathscr{H}}\\ &+j(\beta(t),u(t),v),\ \ \forall v\in\mathscr{V},\end{aligned} \tag{3.56}$$

$$\lambda^*(t)=\psi\big(\sigma(t)-\mathscr{A}(\varepsilon(\dot{u}(t))),\varepsilon(u(t)),\theta(t),\varsigma(t)\big), \tag{3.57}$$

$$\mu^*(t)=\phi\big(\sigma(t)-\mathscr{A}(\varepsilon(\dot{u}(t))),\varepsilon(u(t)),\theta(t),\varsigma(t)\big). \tag{3.58}$$

Now we substitute (3.56) in (3.55) and use (2.35) to see that (2.36) is satisfied. We write (3.21) for $\lambda = \lambda^*$ and use (3.53)(b) and (3.57) to find that (2.37) is satisfied, also we write (3.23) for $\mu = \mu^*$ and using (3.53)(c) and (3.58) to find that (2.38) is satisfied. We consider now (3.19) for $\eta = \eta^*$ and use (3.53)(a) and (3.54)(b) to obtain that (2.40) is satisfied. Next (**??**) and the regularities (3.1)–(3.3), (3.5)–(3.9) follow lemmas (3.3), (3.4), (3.5) and (3.6). The regularity (3.4) follows from lemma (3.7). The uniqueness part of theorem (3.2) is a consequence of the uniqueness of the fixed point of the operator Λ defined (3.34)–(3.36) and the unique solvability of the problems PV_η, PV_β, PV_λ, PV_μ and $PV_{\eta,\lambda,\mu}$ which completes the proof. □

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