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Application of Rothe's method to fractional differential equations

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Abstract

In this paper we consider an initial value problem for a fractional differential equation formulated in a Banach space X where the fractional derivative is Riemann-Liouville type of order $0 < \alpha < 1$. We establish the existence and uniqueness of a strong solution of the problem by applying the method of semi-discretization in time, also known as the method of lines or more popularly as Rothe's method. The dual space X^* of X is assumed to be uniformly convex. In the final section, we illustrate the applicability of the theoretical results with the help of an example.

Keywords

Riemann-Liouville fractional derivative, Rothe's method, Basset problem, accretive operator, strong solution.

AMS Subject Classification

65M20, 34G10, 34A08, 35D35, 34A12.

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1. Introduction

In recent years, many researchers have developed their interest in fractional differential equations (FDEs). These type of equations have various applications in different areas of science and engineering such as viscoelasticity, fluid flows, control theory, food science, electromagnetic, mathematical modeling of real life problems and diffusion process (see- [22],[23],[24],[25],[8]). Different theories has been proposed and developed by researchers to investigate the existence of solutions of FDEs (see- [21],[19],[11],[12]). The nonlocal character and memory effect of fractional derivatives is quite useful to model real life problems and some experimental setups in a better way for example fractional model for viscoelasticity of soft-tissue materials improves the diagnosis in MRE and stress-strain relationship for a viscoelastic material can be well understood by fractional model.

In [1] , Ashyralyev demonstrated the well-posedness of the following initial value problem for an FDE,

$$\frac{dy}{dt} + D_{0+}^{\frac{1}{2}} y(t) + Ay(t) = f(t), \quad 0 < t < 1, \tag{1.1}$$

$$y(0) = 0, \tag{1.2}$$

in a Banach Space, where the linear operator A is a strongly positive. This problem corresponds to the Basset problem studied in [6], which is a well-known problem in fluid dynamics describing the motion of a accelerated particle in a viscous fluid in the influence of gravity. Govindaraj and Balachandran [10] discussed some stablizability criteria of Basset equation in different range of arbitrary constants by using duality results considered in the case of controllability and observability of fractional systems and feedback control. They discussed some numerical examples and graphical illustration of stability results. In [7], Lona considered the following Basset initial value problem

$$\begin{cases} \frac{dx}{dt} + D_{a+}^{\alpha} x(t) = f(t, x(t)), 0 < \alpha < 1, 0 < t < T \\ x(0) = \phi. \end{cases}$$

In the present work, we prove the existence of a unique strong solution of following initial value problem for an FDE,

$$\frac{dy}{dt} + D_{0+}^{\alpha} y(t) + Ay(t) = f(t), \quad 0 < t < T, \, 0 < \alpha < 1 \tag{1.3}$$

$$y(0) = 0, \tag{1.4}$$

in a Banach space X whose dual X^* is uniformly convex. Here the operator $-A$ is the infinitesimal generator of an analytic semigroup of contractions in X and D_{0+}^{α} is the Riemann-Liouville fractional derivative. For $\alpha = \frac{1}{2}$, this problem reduces to the Basset problem (1.1).

Rothe's method, firstly introduced by by Rothe in 1930 [31], has a long history in solving various type of problems. Many researchers adopted this method for solving various type of differential equations. The role of Rothe's method in the study of integer order and fractional order differential equations has been seen in various papers (see-[5],[3],[15],[4], [14], [16],[30]). This method has been well-founded as an efficient tool in solving partial differential equations and gives a numerical approach to find approximate solution. In this method, we discretize the time variable using some discretization scheme to approximate the problem at some equally spaced discrete points and approximate the solution over entire interval using linear approximation. The sequence of approximate solutions are called Rothe's functions. Here, we discretize the problem in time and show that the limit function of Rothe's functions gives the solution of the problem. This method was firstly introduced by Rothe in 1930 to solve parabolic differential equations with one space variables[31]. Later many researchers adopted this method for solving various type of problems. Ladyzenshaja [16] used this method to study quasilinear and linear parabolic problems of second order. Further, Rektorys found the solution of parabolic boundary value problem and smooth solutions of certain differential equations [29], [30].

Although, many analytical methods for example the method of Laplace transform, Mellin transform, Fourier transform and the Green function etc. have been developed to find the analytical solution of FDEs, while there are only few cases in which these methods are effective to give analytical solutions. Solving FDEs accurately and efficiently is more difficult than integer order DEs. Hence researchers focused on developing different numerical methods to discretize fractional derivatives so that approximate solutions could be find for FDEs with less order of errors. But presence of memory term in fractional deriavtives produces difficulties in developing efficient numerical methods. Currently, there are various numerical methods to solve FDEs such as the finite element, the finite difference, fractional multi-step methods, spectral collocation method and the spectral methods are available in literature. There are various numerical techniques and methods to approximate Riemann-Liouville fractional derivative such as Grünwald-Letnikov approximation, the sifted Grünwald-Letnikov formulae, matrix method, $L1, L2$ and $L2C$ schemes (for details, see Ref.[20]). The $L2, L2C$ and $L1$ schemes for discretization can be extended to approximate the Caputo derivative. In 2011, Changpin Li et. al [17], proposed some new piecewise interpolation based numerical methods for fractional calculus and Simpson method based some new improved methods for FDEs. In 2014, Gao et. al introduced a modification of $L1$ formula i.e. $L1-2$ formula to give an approximation of the Caputo derivative of order α $(0<\alpha<1)$. In 2013, Ongun et. al, [26] discussed nonstandard finite difference schemes for fractional order problem. To study discretization methods for Caputo derivative we refer the readers to [18],[13]. The numerical methods for FDEs use mainly two approaches, first by discretizing directly fractional derivatives and second by discretizing the corresponding fractional integral equation.

The paper is well organised in the following way. It contains 5 sections. In the second section, we give the definitions of some fractional derivatives and fractional integrals and some preliminaries Lemmas. In the third section, we prove some a priori estimates. In the forth section, main result is established for the existence of solution. In last section, an example is given.

2. Assumptions and Preliminaries

In the present section, we recall some notions, definitions and basic facts about fractional calculus. Also, Here we mention certain Lemmas and Hypothesis, which will be used subsequently.

Definition 2.1. *[32] The Riemann-Liouville fractional integral of a function $f(t)$ of order $\alpha>0$ is defined by*

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\ t>0.$$

Definition 2.2. *[32] The Riemann-Liouville fractional derivative of order $0<\alpha<1$ of a function $f(t)$ is defined by*

$$D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(s)}{(t-s)^{\alpha}}ds,\ t>0.$$

Definition 2.3. *[32] The Caputo fractional derivative of order $0<\alpha<1$ of a function $f(t)$ is defined by*

$$D_{\alpha}^{c}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f'(s)}{(t-s)^{\alpha}}ds,\ t>0.$$

Lemma 2.4. *[4] Suppose that the function $\eta(t)\geq 0$ is continuous or piecewise continuous on $0<t\leq T$ and $\xi(t)$ be positive continuous function on $0\leq t\leq T$. If M and $0<\alpha<1$ are such that*

$$\eta(t)\leq M\int_0^t\frac{\eta(p)}{(t-p)^{\alpha}}dp+M\int_0^t\frac{\xi(p)}{(t-p)^{\alpha}}dp,\quad 0\leq t\leq T, \tag{2.1}$$

then

$$\eta(t)\leq C\max_{0\leq t\leq T}f(t),$$

for some positive constant C.

Let $N\in\mathbb{N}$ and $\tau=\frac{T}{N}$.

Theorem 2.5 ([2]). *Suppose that $-A$ with its domain $D(A)$ dense, generates an analytic semigroup. Then it is necessary and sufficient that*

1. $\|R^k(\tau A)\|\leq M$,
2. $\|k\tau AR^k(pA)\|\leq M$,

for all $\tau > 0$ *and* $k = 1, \dots, N$. $R(\tau A) = (I + \tau A)^{-1}$ *and* M *is independent of* k *and* τ.

We define fractional spaces X_β and X'_β for $\beta \in (0,1)$ as follows,
$X_\beta = X_\beta(X, A)$, which consist of all $x \in X$ for which the norm

$$\|x\|_{X_\beta} = \sup_{\lambda>0} \lambda^{1-\beta} \|A\, exp(-\lambda A)x\| \quad \text{is finite.}$$

and $X'_\beta = X'_\beta(X, A)$, which consist of all $x \in X$ for which the norm

$$\|x\|_{X'_\beta} = \sup_{\lambda>0} \lambda^{\beta} \|A(\lambda + A)^{-1}x\| \quad \text{is finite.}$$

We define the the space $C(X_\beta)$ as the space of all continuous functions from $[0,T]$ to the space X_β.
Throughout the paper we have assumed the following hypothesis:

H1 $f(t) \in C(X'_\beta)$ for some $\beta \in (0,1)$.

H2 $\|exp(-tA)\| \le Me^{-\delta t}$, and $\|tA exp(-tA)\| \le M$ for $M, \delta > 0$ and $t > 0$.

H3 $(I + A)^{-1} : X \to X$ is compact.

Theorem 2.6 ([2], Theorem 2.4.1). $X_\beta = X'_\beta$ *for all* $0 < \beta < 1$.

Theorem 2.7 ([2]). *Let* $x \in X_\beta$. *Then*

$$\|x\| \le \frac{M}{\beta} \|x\|_{X_\beta}, \tag{2.2}$$

for some $M \ge 0$.

Proof. Since $-A$ generates the analytic semigroup $exp(-tA)$, $exp(-tA)x$ is continuously differentiable. i.e.

$$\frac{d}{dt}[exp(-tA)x] = -A\, exp(-tA)x.$$

Integrating from 0 to 1, we have

$$(I - exp(-A))x = \int_0^1 A\, exp(-sA)x\, ds.$$

Since $\|exp(-A)\| \le Me^{-\delta}$, there exist a $n_0 \in \mathbb{N}$ such that $\|[exp(-A)]^{n_0}\| \le Me^{-\delta n_0} < 1$. Hence the inverse of $I - exp(-A)$ is bounded. This gives

$$x = (I - exp(-A))^{-1} \int_0^1 A\, exp(-sA)x\, ds.$$

Hence,

$$\begin{aligned}
\|x\| &\le \|(I - exp(-A))^{-1}\| \int_0^1 \|A\, exp(-sA)x\|\, ds \\
&\le \|(I - exp(-A))^{-1}\| \sup_{s>0}\{s^{1-\beta}\|A\, exp(-sA)x\|\} \int_0^1 \frac{ds}{s^{1-\beta}} \\
&= \frac{1}{\beta} \|(I - exp(-A))^{-1}\| \|x\|_{X_\beta}.
\end{aligned}$$

Hence $\|x\| \le \frac{M}{\beta}\|x\|_{X_\beta}$. □

We set $a_l = (l+1)^{1-\alpha} - l^{1-\alpha}$ for $l = 1, 2, \dots$

Lemma 2.8 ([13]). *1.* $a_l > a_{l+1}$ *for* $l = 1, 2, \dots$.

2. $a_0 = 1$.

3. If $0 < \alpha < 1$ *and* l *is non-negative integer, then there exists a positive constant* $C(\alpha)$ *such that*

$$(l+1)^\alpha - l^\alpha \le C(\alpha)(l+1)^{\alpha-1}, \tag{2.3}$$

where $C(\alpha) = \max\{1, \alpha 2^{1-\alpha}\}$.

3. Discretization scheme and A priori estimates

In this section, we use some discretization scheme to approximate the problem and find a priori estimates on the approximate solution of the problem. Let $h_n = \frac{T}{n}, n \in \mathbb{N}$ and $t^n_k = kh_n$ for $k = 1, 2 \dots n$. Thus for each $n \in \mathbb{N}$ $[0,T]$ is partitioned into n subintervals $[t^n_{j-1}, t^n_j]$, $j = 0, 1, \dots, n$.

Discretization scheme for fractional derivative $D^\alpha_{0+}y(t)$ **:**
At $t = t^n_k$, the approximate value of $D^\alpha_{0+}y(t)$ is given by,

$$\begin{aligned}
D^\alpha_{0+}y(t^n_k) &\approx \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^k a_{k-i} \frac{(y^n_i - y^n_{i-1})}{h_n} h_n^{1-\alpha} && (3.1)\\
&= \sum_{i=1}^k (y^n_i - y^n_{i-1}) b^n_{k,i}. && (3.2)
\end{aligned}$$

where $b^n_{k,i} = a_{k-i}\frac{h_n^{-\alpha}}{\Gamma(2-\alpha)}$.
We denote the approximate value of $D^\alpha_{0+}y(t)$ at $t = t^n_k$ by $D_\alpha y^n_k$.
We replace the equations (1.3)and (1.4) by following approximate equations

$$\frac{y^n_j - y^n_{j-1}}{h_n} + D_\alpha y^n_j + Ay^n_j = f(t^n_j) = f^n_j \quad j = 1, \dots, n. \tag{3.3}$$

$$y^n_0 = 0. \tag{3.4}$$

Equation (3.3) can be written as

$$\frac{y^n_j - y^n_{j-1}}{h_n} + Ay^n_j + b^n_{j,j}y^n_j - \sum_{i=1}^{j-1} \left(b^n_{j,i+1} - b^n_{j,i}\right) y^n_i = f^n_j.$$

Hence

$$y^n_j + h_n Ay^n_j + h_n b^n_{j,j} y^n_j = y^n_{j-1} + h_n f^n_j + h_n \sum_{i=1}^{j-1} \left(b^n_{j,i+1} - b^n_{j,i}\right) y^n_i.$$

The above equation implies that

$$\left[(1 + h_n b^n_{j,j})I + h_n A\right] y^n_j = y^n_{j-1} + h_n f^n_j + h_n \sum_{i=1}^{j-1} \left(b^n_{j,i+1} - b^n_{j,i}\right) y^n_i.$$

Let $E_j^n = \left(\frac{1}{1+h_n b_{j,j}^n} y_{j-1}^n + \frac{h_n}{1+h_n b_{j,j}^n} f_j^n + \frac{h_n}{1+h_n b_{j,j}^n} \sum_{i=1}^{j-1} (b_{j,i+1}^n - b_{j,i}^n) y_i^n \right)$.

Hence, $y_j^n = \left[I + \left(\frac{h_n}{1+h_n b_{j,j}^n} \right) A \right]^{-1} E_j^n, \quad$ for $j = 1, 2, \dots n$

Since $1 + h_n b_{j,j}^n = 1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)} > 0$, hence $\left[I + \left(\frac{h_n}{1+h_n b_{j,j}^n} \right) A \right]^{-1}$ exists and this gives unique $y_j^n \in D(A)$.

Now equation (3.3) can be written as,

$$\frac{y_j^n - y_{j-1}^n}{h_n} + A y_j^n = f_j^n - D_\alpha y_j^n = F_j^n.$$

Arranging the above equation we have

$$(I + h_n A) y_j^n = y_{j-1}^n + h_n F_j^n.$$

This gives $y_j^n = (I + h_n A)^{-1} (y_{j-1}^n + h_n F_j^n)$.
Iterating the above equation n times and using $y_0^n = 0$, we get $y_j^n = \sum_{s=1}^{j} R^{j-s+1}(h_n A) F_s^n h_n$, where $R(h_n A) = (I + h_n A)^{-1}$. Hence,

$$y_j^n = -\sum_{s=1}^{j} R^{j-s+1}(h_n A) D_\alpha y_s^n h_n + \sum_{s=1}^{j} R^{j-s+1}(h_n A) f_s h_n. \quad (3.5)$$

Theorem 3.1. *There exist a constant C independent of j, n and h_n such that*

$$\|D_\alpha y_j^n\| \le C$$

Proof. Using equation (3.5) in (3.3), we get

$$\begin{aligned} \frac{y_j^n - y_{j-1}^n}{h_n} &= -D_\alpha y_j^n - A y_j^n + f_j^n \\ &= -D_\alpha y_j^n + \sum_{s=1}^{j} A R^{j-s+1}(h_n A) D_\alpha y_s^n h_n \\ &\quad - \sum_{s=1}^{j} A R^{j-s+1}(h_n A) f_s^n h_n + f_j^n. \quad (3.6) \end{aligned}$$

Using equation (3.6) we get,

$$\begin{aligned} D_\alpha y_j^n &= \sum_{k=1}^{j} a_{j-k} \frac{y_k^n - y_{k-1}^n}{h_n} \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)} \\ &= \sum_{k=1}^{j} a_{j-k} [-D_\alpha y_k^n + f_k^n] \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)} \\ &+ \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{j} a_{j-k} \left(\sum_{s=1}^{k} A R^{k-s+1}(h_n A) D_\alpha y_s^n h_n \right. \\ &- \left. \sum_{s=1}^{k} A R^{k-s+1}(h_n A) f_s^n h_n \right) h_n^{1-\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{j} a_{j-k} h_n^{1-\alpha} [-D_\alpha y_k^n + f_k^n] \\ &+ \frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^{j} \sum_{k=s}^{j} a_{j-k} A R^{k-s+1}(h_n A) D_\alpha y_s^n h_n^{2-\alpha} \\ &- \frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^{j} \sum_{k=s}^{j} a_{j-k} A R^{k-s+1}(h_n A) f_s h_n^{2-\alpha}. \end{aligned}$$

Let us find the estimate for $\| \frac{1}{\Gamma(2-\alpha)} \sum_{k=s}^{j} a_{j-k} A R^{k-s+1}(h_n A) h_n^{1-\alpha} \|$, for $1 \le s < j \le n$.

$$\begin{aligned} &\sum_{k=s}^{j} a_{j-k} A R^{k-s+1}(h_n A) \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=s}^{\left[\frac{s+j}{2}\right]} a_{j-k} A R^{k-s+1}(h_n A) h_n^{1-\alpha} \\ &+ \frac{1}{\Gamma(2-\alpha)} \sum_{k=\left[\frac{s+j}{2}\right]+1}^{j} a_{j-k} A R^{k-s+1}(h_n A) h_n^{1-\alpha} \\ &= S_1 + S_2. \end{aligned}$$

$S_2 = \frac{1}{\Gamma(2-\alpha)} \sum_{k=\left[\frac{s+j}{2}\right]+1}^{j} a_{j-k} A R^{k-s+1}(h_n A) h_n^{1-\alpha}$.

For $k = \left[\frac{s+j}{2}\right] + 1$ to $k = j$, we have

$$\begin{aligned} k - s + 1 &\ge \left[\frac{s+j}{2}\right] + 1 - s + 1 \ge \frac{s+j}{2} - s + 1 \\ &= \frac{j-s+2}{2} > \frac{j-s+1}{2}. \end{aligned}$$

Using estimates of Theorem 2.5 and Lemma 2.8, we get

$$\begin{aligned} \|S_2\| &\le \frac{1}{\Gamma(2-\alpha)} \sum_{k=\left[\frac{s+j}{2}\right]+1}^{j} a_{j-k} \|A R^{k-s+1}(h_n A)\| h_n^{1-\alpha} \\ &\le \frac{2M}{\Gamma(2-\alpha)} \frac{C(\alpha)}{(j-s+1)h_n} \sum_{k=\left[\frac{s+j}{2}\right]+1}^{j} \frac{h_n}{[(j-k+1)h_n]^\alpha} \end{aligned}$$

Since $\frac{h_n}{[(j-k+1)h_n]^\alpha} \le \int_{t_{k-1}^n}^{t_k^n} \frac{dw}{(t_j^n - w)^\alpha} \le \frac{h_n}{[(j-k)h_n]^\alpha}$

$$\begin{aligned} &\|S_2\| \le \\ &\frac{2M}{\Gamma(2-\alpha)} \frac{C(\alpha)}{(j-s+1)h_n} \sum_{k=\left[\frac{s+j}{2}\right]+1}^{j} \int_{t_{k-1}^n}^{t_k^n} \frac{dw}{(t_j^n - w)^\alpha} \\ &= \frac{2M}{\Gamma(2-\alpha)} \frac{C(\alpha)}{(j-s+1)h_n} \int_{t_{\left[\frac{s+j}{2}\right]}^n}^{t_j^n} \frac{dw}{(t_j^n - w)^\alpha} \end{aligned}$$

$$= \frac{2M}{\Gamma(2-\alpha)} \frac{C(\alpha)}{(j-s+1)h_n} \frac{\left(j-\left[\frac{s+j}{2}\right]\right)^{1-\alpha} h_n^{1-\alpha}}{1-\alpha}$$

$$\leq \frac{2M}{\Gamma(2-\alpha)} \frac{C(\alpha)}{(j-s+1)h_n} \frac{(j-s+1)^{1-\alpha} h_n^{1-\alpha}}{1-\alpha}$$

$$= \frac{M_1}{[(j-s+1)h_n]^\alpha}. \tag{3.7}$$

$$S_1 = \frac{1}{\Gamma(2-\alpha)} \sum_{k=s}^{\left[\frac{s+j}{2}\right]} a_{j-k} h_n A R^{k-s+1}(h_n A) h_n^{-\alpha}$$

$$= \sum_{k=s}^{\left[\frac{s+j}{2}\right]} a_{j-k}[I + h_n - I] R^{k-s+1}(h_n A) \frac{h_n^{-\alpha}}{\Gamma(2-\alpha)}$$

$$= \frac{1}{\Gamma(2-\alpha)} a_{j-s} h_n^{-\alpha} - \frac{1}{\Gamma(2-\alpha)} a_{j-\left[\frac{s+j}{2}\right]}$$

$$\times R^{\left[\frac{s+j}{2}\right]-s+1}(h_n A) h_n^{-\alpha}$$

$$+ \sum_{k=s+1}^{\left[\frac{s+j}{2}\right]} (a_{j-k} - a_{j-k+1}) R^{k-s}(h_n A) \frac{h_n^{-\alpha}}{\Gamma(2-\alpha)}.$$

Now

$$a_{j-\left[\frac{s+j}{2}\right]} \leq C(\alpha)\left(j - \left[\frac{s+j}{2}\right] + 1\right)^{-\alpha}$$

$$\leq C(\alpha)\left(j - \frac{s+j}{2} + 1\right)^{-\alpha}$$

$$= \frac{C(\alpha)2^\alpha}{(j-s+2)^\alpha} \leq \frac{C(\alpha)2^\alpha}{(j-s+1)^\alpha}. \tag{3.8}$$

Hence using Theorem 2.5 and Lemma 2.8, we get

$$\|S_1\| \leq \frac{1}{\Gamma(2-\alpha)} \frac{C(\alpha)}{[(j-s+1)h_n]^\alpha}$$

$$+ \frac{M 2^\alpha C(\alpha)}{\Gamma(2-\alpha)[(j-s+1)h_n]^\alpha}$$

$$+ M \sum_{k=s+1}^{\left[\frac{s+j}{2}\right]} (a_{j-k} - a_{j-k+1}) \frac{h_n^{-\alpha}}{\Gamma(2-\alpha)}$$

$$= \frac{C(\alpha)}{[(j-s+1)h_n]^\alpha \Gamma(2-\alpha)} + \frac{M 2^\alpha C(\alpha)}{\Gamma(2-\alpha)[(j-s+1)h_n]^\alpha}$$

$$+ M\left(a_{j-\left[\frac{s+j}{2}\right]} - a_{j-s}\right) \frac{h_n^{-\alpha}}{\Gamma(2-\alpha)}$$

$$\leq \frac{1}{\Gamma(2-\alpha)} \frac{C_\alpha}{[(j-s+1)h_n]^\alpha} + \frac{M 2^\alpha C_\alpha}{\Gamma(2-\alpha)[(j-s+1)h_n]^\alpha}$$

$$+ \frac{M}{\Gamma(2-\alpha)} a_{j-\left[\frac{s+j}{2}\right]} h_n^{-\alpha}$$

Applying inequality (3.8), we get a constant M_2 such that

$$\|S_1\| \leq \frac{M_2}{[(j-s+1)h_n]^\alpha}. \tag{3.9}$$

Using equations (3.7) and (3.9), we have

$$\left\| \frac{1}{\Gamma(2-\alpha)} \sum_{k=s}^{j} a_{j-k} A R^{k-s+1}(h_n A) h_n^{1-\alpha} \right\| \leq \frac{M_3}{[(j-s+1)h_n]^\alpha}. \tag{3.10}$$

Hence using equation (3.10) and Lemma 2.8, we have

$$\|D_\alpha y_j^n\| \leq \sum_{k=1}^{j} a_{j-i}[\|D_\alpha y_k^n\| + \|f_k^n\|] \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$+ \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=1}^{j} \|\sum_{k=s}^{j} a_{j-k} A R^{k-s+1}(h_n A) h_n^{1-\alpha}\| \|D_\alpha y_s^n\| h_n$$

$$+ \frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^{j} \|\sum_{k=s}^{j} a_{j-k} A R^{k-s+1}(h_n A)\| \|f_s^n\| h_n^{2-\alpha}$$

$$\leq M \sum_{s=1}^{j} \frac{h_n}{[(j-s+1)h_n]^\alpha} (\|D_\alpha y_s^n\| + \|f_s^n\|).$$

Using discrete analog of Lemma 2.4, we get

$$\|D_\alpha y_j^n\| \leq M \max_{0 \leq t \leq T} \|f(t)\| \leq C. \tag{3.11}$$

□

Theorem 3.2. *Let $f(t) \in C(X'_\beta)$. Then there is a positive number C independent of j, n and h_n satisfying $\|D_\alpha y_j^n\|_{X'_\beta} \leq C$, $\left\| \frac{y_j^n - y_{j-1}^n}{h_n} \right\|_{X'_\beta} \leq C$ and $\|A y_j^n\|_{X'_\beta} \leq C$.*

Proof. Using the definition of norm in X'_β and repeating the same steps as in Theorem 3.1, we get

$$\|D_\alpha y_j^n\|_{X'_\beta} \leq M \|f\|_{C(X'_\beta)} \tag{3.12}$$

From equation (3.6), we have

$$\frac{y_j^n - y_{j-1}^n}{h_n} = f_j^n - D_\alpha y_j^n + \sum_{s=1}^{j} A R^{j-s+1}(h_n A) D_\alpha y_s^n h_n$$

$$- \sum_{s=1}^{j} A R^{j-s+1}(h_n A) f_s h_n. \tag{3.13}$$

Taking norm and using triangle inequality, equations (3.6), (3.12) and Theorem (2.4.2) in [2], we have

$$\left\| \frac{y_j^n - y_{j-1}^n}{h_n} \right\|_{X'_\beta} \leq \|f_j^n\|_{X'_\beta} + \|D_\alpha y_j^n\|_{X'_\beta}$$

$$+ \|\sum_{s=1}^{j} A R^{j-s+1}(h_n A) D_\alpha y_s^n h_n\|_{X'_\beta}$$

$$+ \|\sum_{s=1}^{j} A R^{j-s+1}(h_n A) f_s h_n\|_{X'_\beta}$$

$$\leq \|f\|_{C(X'_\beta)} + M\|f\|_{C(X'_\beta)} + \frac{M}{\beta(1-\beta)} \max_{1 \leq j \leq N} \{\|D_\alpha y_j^n\|_{X'_\beta}\}$$

$$+ \frac{M}{\beta(1-\beta)} \|f\|_{C(X'_\beta)} \leq \frac{M'}{\beta(1-\beta)} \|f\|_{C(X'_\beta)} = C$$

Now using equation (3.6) and the triangle inequality, we have

$$\|Ay_j^n\|_{X'_\beta} \leq \frac{M'}{\beta(1-\beta)}\|f\|_{C(X'_\beta)}.$$

Hence proved. □

Corollary 3.3. *Let $f(t) \in C(X'_\beta)$. Then we have a positive number C independent of j, h_n and n such that*

$$\|D_\alpha y_j^n\| \leq C, \left\|\frac{y_j^n - y_{j-1}^n}{h_n}\right\| \leq C \text{ and } \|Ay_j^n\| \leq C. \quad (3.14)$$

Proof. The proof follows directly from Theorems 3.2, 2.7 and 2.6. □

Corollary 3.4. *There is a positive number C independent of j, h_n and n such that $\|y_j^n\| \leq C$.*

Proof. From equation (3.3), we obtain

$$y_j^n = y_{j-1}^n + h_n(f_j^n - D_\alpha y_j^n - Ay_j^n)$$

Hence using the Corollary (3.3), we get

$$\begin{aligned}
\|y_j^n\| &\leq \|y_{j-1}^n\| + h_n(\|f_j^n\| + \|D_\alpha y_j^n\| + \|Ay_j^n\|) \\
&\leq \|y_{j-1}^n\| + C_1 h_n \\
&\leq \|y_{j-2}^n\| + 2C_1 h_n \\
&\leq \|y_0^n\| + jC_1 h_n \leq TC_1 \leq C.
\end{aligned}$$

□

We consider a sequences $\mathscr{X}^n$ and $\mathscr{Y}^n : [0,T] \longrightarrow D(A)$ given by

$$\mathscr{X}^n(t) = \begin{cases} 0, & t = 0, \\ y_j^n, & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

and

$$\mathscr{Y}^n(t) = \begin{cases} 0, & t = 0, \\ y_{j-1}^n + \frac{t - t_{j-1}^n}{h_n}(y_j^n - y_{j-1}^n), & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

Further we introduce a sequence of step functions $\tilde{D}_\alpha \mathscr{Y}^n(t)$ by

$$\tilde{D}_\alpha \mathscr{Y}^n(t) = \begin{cases} 0, & t = 0, \\ \sum_{i=1}^j a_{j-i} \frac{y_i^n - y_{i-1}^n}{h_n} \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}, & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

Remark 3.5. *Sequences $\mathscr{Y}^n(t)$ and $\mathscr{X}^n(t)$ are uniformly bounded in X. Furthermore functions $\mathscr{Y}^n(t)$ are uniformly Lipschitz continuous on $[0,T]$ and $\mathscr{Y}^n(t) - \mathscr{X}^n(t) \to 0$ in X as $n \to \infty$ on $[0,T]$.*

For a given $t \in (0,T]$, there exists a j such that $t \in (t_{j-1}^n, t_j^n]$.

$$\begin{aligned}
D_{0+}^\alpha \mathscr{Y}^n(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\mathscr{Y}^n(s)}{(t-s)^\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\sum_{k=1}^{j-1} \int_{t_{k-1}^n}^{t_k^n} \frac{\mathscr{Y}^n(s)}{(t-s)^\alpha} ds + \int_{t_{k-1}^n}^t \frac{\mathscr{Y}^n(s)}{(t-s)^\alpha} ds \right) \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{j-1} \frac{d}{dt} \int_{t_{k-1}^n}^{t_k^n} \frac{\mathscr{Y}^n(s)}{(t-s)^\alpha} ds + \frac{d}{dt} \int_{t_{j-1}^n}^t \frac{\mathscr{Y}^n(s)}{(t-s)^\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \Bigg[\sum_{k=1}^{j-1} \Bigg(-\mathscr{Y}^n(t_k^n)(t-t_k^n)^{-\alpha} + \mathscr{Y}^n(t_{k-1}^n) \times \\
&\quad (t-t_{k-1}^n)^{-\alpha} + \frac{y_k^n - y_{k-1}^n}{h_n} \frac{(t-t_{k-1}^n)^{1-\alpha} - (t-t_k^n)^{1-\alpha}}{1-\alpha} \Bigg) \\
&\quad + \frac{y_j^n - y_{j-1}^n}{h_n} \frac{(t-t_{j-1}^n)^{1-\alpha}}{1-\alpha} + \mathscr{Y}^n(t_{j-1}^n)(t-t_{j-1}^n)^{-\alpha} \Bigg] \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{j-1} \frac{y_k^n - y_{k-1}^n}{h_n} [(t-t_{k-1}^n)^{1-\alpha} - (t-t_k^n)^{1-\alpha}] \\
&\quad + \frac{y_j^n - y_{j-1}^n}{h_n} \frac{(t-t_{j-1}^n)^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

Lemma 3.6. *([4]) $\|\tilde{D}_\alpha \mathscr{Y}^n(t) - D_{0+}^\alpha \mathscr{Y}^n(t)\| \to 0$ as $n \to \infty$ uniformly on $(0,T]$.*

Proof. The proof directly follows from a easy calculation using the above equation and definition of functions $\tilde{D}_\alpha \mathscr{Y}^n(t)$. □

Lemma 3.7. *([4]) There exists a subsequence $\{\mathscr{Y}^{n_k}\}$ of $\{\mathscr{Y}^n\}$ such that $\frac{d^- \mathscr{Y}^{n_k}}{dt} \rightharpoonup \frac{dy}{dt}$ and $D_{0+}^\alpha \mathscr{Y}^{n_k}(t) \rightharpoonup D_{0+}^\alpha y(t)$ in $L^2([0,T],X)$, as $n \to \infty$.*

Proof. See the proof of Lemma 10 in [4]. □

Remark 3.8.

$$\tilde{D}_\alpha \mathscr{Y}^n(t) \rightharpoonup D_{0+}^\alpha y(t) \text{ in } L^2([0,T],X) \quad \text{as} \quad n \to \infty.$$

We consider a sequence of functions $f^n(t)$ as,

$$f^n(t) = \begin{cases} f(0), & t = 0, \\ f(t_j^n), & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

We can write equation (3.3) can as

$$\frac{d^-}{dt}\mathscr{Y}^n(t) + \tilde{D}_\alpha \mathscr{Y}^n(t) + A\mathscr{X}^n(t) = f^n(t), \; t \in (0,T]. \quad (3.15)$$

4. Main Results

Theorem 4.1. *Let $-A$ generate an analytic semigroup of contractions in X such that $(H1)-(H3)$ hold. Then the FDE (1.3)-(1.4) has a unique strong solution.*

Proof. Integrating equation (3.15) from 0 to t and then for each $\phi \in X^*$, we get

$$\int_0^t \langle A\mathscr{X}^n(p), \phi\rangle dp = -\langle \mathscr{Y}^n(t), \phi\rangle + \int_0^t \langle f^n(p), \phi\rangle dp - \int_0^t \langle \tilde{D}_\alpha \mathscr{Y}^n(s), \phi\rangle dp. \tag{4.1}$$

Rewriting above equation for the subsequence n_k of n, we have

$$\int_0^t \langle A\mathscr{X}^{n_k}(p), \phi\rangle dp = -\langle \mathscr{Y}^{n_k}(t), \phi\rangle + \int_0^t \langle f^{n_k}(p), \phi\rangle dp - \int_0^t \langle \tilde{D}_\alpha Y^{n_k}(p), \phi\rangle dp. \tag{4.2}$$

From Lebesgue dominated convergence theorem, Lemmas 3.7 and Remark 3.8 and Lemma 2.3 and Theorem 2.1 of [5], as $k \to \infty$ it follows that

$$\int_0^t \langle Ay(p), \phi\rangle ds = -\langle y(t), \phi\rangle + \int_0^t \langle f(p), \phi\rangle ds - \int_0^t \langle D^\alpha_{0+}y(p), \phi\rangle dp. \tag{4.3}$$

Using $\int_0^t \langle D^\alpha_{0+}y(p), \phi\rangle dp = \langle I^{1-\alpha}_{0+}y(t), \phi\rangle$ in equation (4.3), we obtain

$$\langle y(t) + I^{1-\alpha}_{0+}y(t), \phi\rangle = -\int_0^t \langle Ay(p), \phi\rangle dp + \int_0^t \langle f(p), \phi\rangle dp.$$

Continuity of the integrands on the RHS gives the continuous differentiability of $\langle y(t) + I^{1-\alpha}y(t), \phi\rangle$. Now, since $Ay(t)$ is Bochner integrable, the strong derivative of $y(t) + I^{1-\alpha}y(t)$ exists a.e. on the interval $[0,T]$. Hence

$$\frac{d}{dt}\left(y(t) + I^{1-\alpha}_{0+}y(t)\right) = -Ay(t) + f(t), \text{a.e on } [0,T].$$

As the function $y(t)$ is Lipschitz continuous, $I^{1-\alpha}_{0+}y(t)$ is differentiable(see [28]), hence $y(t)$ is differentiable. Hence, we have

$$\frac{dy}{dt} + D^\alpha_{0+}y(t) + Ay(t) = f(t), \quad \text{a.e.} \quad t \in [0,T].$$

i.e. $y(t)$ is a strong solution to the problem (1.3)-(1.4).
To prove the uniquness, let y_1 and y_2 be two strong solutions of the problem (1.3)-(1.4), then they will satisfy

$$\frac{dy_i}{dt} + D^\alpha_{0+}y_i(t) + Ay_i(t) = f(t), \quad \text{for} \quad i = 1,2.$$

Let $y = y_1 - y_2$, then $y(t) = y_1(t) - y_2(t)$ satisfies the following fractional differential equation,

$$\frac{dy}{dt} + D^\alpha_{0+}y(t) + Ay(t) = 0. \tag{4.4}$$

$$y(0) = 0. \tag{4.5}$$

Hence there exists a strong solution of the problem (4.4)-(4.5) by Theorem 4.1. Equation (4.4) can also be re-written as

$$\frac{dy}{dt} + Ay(t) = -D^\alpha_{0+}y(t).$$

If the semigroup generated by $-A$ is $exp(-At) = T(t)$, then

$$y(t) = -\int_0^t T(t-p)D^\alpha_{0+}y(p)dp.$$

Differentiating $y(t)$, we obtain

$$y'(t) = -D^\alpha_{0+}y(t) + \int_0^t AT(t-p)D^\alpha_{0+}y(p)dp. \tag{4.6}$$

Using (4.6), in the definition of $D^\alpha_{0+}y(t)$, we get

$$\begin{aligned} D^\alpha_{0+}y(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{-D^\alpha_{0+}y(p)}{(t-p)^\alpha}dp + \\ &\int_0^t \int_p^t \frac{1}{\Gamma(1-\alpha)} AT(s-p)dsD^\alpha_{0+}y(p)dp. \end{aligned}$$

Similarly as in [1], we have

$$\left\| \frac{1}{\Gamma(1-\alpha)} \int_p^t AT(s-p)ds \right\| \le \frac{M}{(t-p)^\alpha} \tag{4.7}$$

for a constant M. This gives

$$\|D^\alpha_{0+}y(t)\| \le M\int_0^t \frac{\|D^\alpha_{0+}y(s)\|}{(t-s)^\alpha}ds. \tag{4.8}$$

Using Lemma (2.4), we have $\|D^\alpha_{0+}y(t)\| = 0$.
Thus $\|y(t)\| = 0$. Hence $y_1(t) = y_2(t)$, i.e. the solution is unique. ☐

Corollary 4.2. *The following initial value problem for the FDE,*

$$\frac{dy}{dt} + D^\alpha_{0+}y(t) + Ay(t) = f(t), \quad 0 < t < T, \tag{4.9}$$

$$y(0) = y_0, \tag{4.10}$$

has a unique strong solution for $y_0 \in D(A^2)$ under the assumptions $(H1)-(H3)$.

5. Example

We consider the following fractional initial boundary value problem

$$\frac{\partial u(t,x)}{\partial t} + D^\alpha_{0+}u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = g(t,x),$$
$$t \in [0,1], x \in [0,1], 0 < \alpha < 1, \delta > 0,$$

$$\begin{aligned} u(t,0) &= 0 = u(t,1), \quad t \in [0,1], \\ u(0,x) &= 0. \end{aligned} \tag{5.1}$$

We set $X = L^2([0,1],\mathbb{R})$ and $g : [0,1]\times[0,1] \longrightarrow \mathbb{R}$ be enough smooth such that $G(t) \in C(X'_\beta)$, where $G(t)(x) = g(t,x)$. We define $U(t) : [0,1] \to \mathbb{R}$ by $U(t)(x) = u(t,x)$ as a function of x and operator $K : D(K) \to X$ by $Kv = -v''$.
$D(K) = \{\omega \in L^2([0,1],\mathbb{R}) : \omega, \omega'$is absolutely continuous $\omega'' \in L^2([0,1],\mathbb{R}), \omega(0) = \omega(1) = 0\}$.
From [27], it is clear that $-K$ is generates a compact analytic semigroup of contractions on X. Consider an operator $A : D(A) \to X$ defined by $Aw = -w'' + \delta w$ with $D(A) = D(K)$. Then $-A = -K - \delta I$ also generates a compact analytic semigroup with contractions satisfying the hypothesis $(H2)-(H3)$.
Then the reformulated problem in in abstract form is

$$\frac{dU}{dt} + D^{\alpha}_{0+}U(t) + AU(t) = G(t), \quad 0 < t < 1,$$
$$U(0) = 0.$$

Thus we may apply the Theorem 4.1 to obtain the existence of a unique strong solution of the above problem.

6. Conclusion

The problem presented in this paper is the generalization of a problem in viscoelasticity, named as Basset problem. Existence and uniqueness of the problem is considered by Rothe's method. Here we defined some fractional spaces and the function takes the values from the fractional Banach space which contains the domain of the infinite dimensional operator A. Some a priori estimates has been established on the approximate solution of the problem, which also guarantees the wellposedness of the discrete problem in fractional spaces. The considered problem (1.3)-(1.4) has zero initial condition, but Corollary 4.2 shows this condition can be dropped by assuming a regularity condition on the initial values.

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