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Some inequalities for the q,k-Gamma and Beta functions

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Abstract

Using q-integral inequalities we establish some new inequalities for the q-k Gamma, Beta and Psi functions.

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1 Introduction

The q-analogue Γ_q of the well known Gamma function was initially introduced by Thomae [11] and later deeply studied by Jackson [6]. The reader will find in the research literature more about this feature.
In [1], R. Diaz and C. Truel introduced a q,k-generalized Gamma and Beta functions and they proved integral representations for $\Gamma_{q,k}$ and $B_{q,k}$ functions.

This work is devoted to establish some inequalities for the generalized q,k-Gamma and Beta functions and this has been possible thanks to the inequalities that verify the q-Jackson's integral.

The paper is organized as follows: In section 2, we present some preliminaries and notations that will be useful in the sequel. In section 3, we recall the q-Čebyšev's integral inequality for q-synchronous (q-asynchronous) functions and in direct consequence, we deduce some inequalities involving q,k-Beta and q,k-Gamma functions. In section 4, we establish some inequalities for these functions owing to the q-Holder's inequality. Finally section 5 is devoted to some applications of q-Grüss integral inequality.

2 Notations and preliminaries

To make this paper self containing, we provide in this section a summary of the mathematical notations and definitions useful. All of these results can be found in [4], [8] or [9].
Throughout this paper, we will fix $q \in]0,1[$, $\quad k > 0$ a real number.
For $a \in \mathbb{C}$, we write

$$[a]_q = \frac{1-q^a}{1-q}, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1,2...\infty,$$

$$[n]_q! = [1]_q[2]_q...[n]_q, \qquad n \in \mathbb{N}.$$

The q-derivative D_q of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \text{ if } x \neq 0, \tag{2.1}$$

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and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.
The q-Jackson integrals from 0 to b and from 0 to ∞ are defined by (see [7])

$$\int_0^b f(x)d_qx = (1-q)b\sum_{n=0}^{\infty} f(bq^n)q^n \tag{2.2}$$

and

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{2.3}$$

provided the sums converge absolutely.
The q-Jackson integral in a generic interval $[a,b]$ is given by (see [7])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.4}$$

We denote by I one of the following sets:

$$\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}, \tag{2.5}$$

$$[0,b]_q = \{bq^n : n \in \mathbb{N}\}, \quad b > 0, \tag{2.6}$$

$$[a,b]_q = \{bq^r : \ 0 \le r \le n\}, \quad b > 0, \ a = bq^n, n \in \mathbb{N} \tag{2.7}$$

and we note $\int_I f(x)d_qx$ the q-integral of f on the correspondent I.

Definition 2.1. *let $x, y, s, t \in \mathbb{R}$ and $n \in \mathbb{N}$, we note by*

1. $(x+y)^n_{q,k} := \prod_{j=0}^{n-1}(x+q^{jk}y)$
2. $(1+x)^{\infty}_{q,k} := \prod_{j=0}^{\infty}(1+q^{jk}x)$
3. $(1+x)^t_{q,k} := \frac{(1+x)^{\infty}_{q,k}}{(1+q^{kt}x)^{\infty}_{q,k}}$.

We have $(1+x)^{s+t}_{q,k} = (1+x)^s_{q,k}(1+q^{ks}x)^t_{q,k}$.

We recall the two q,k-analogues of the exponential function (see [1]) given by

$$E^x_{q,k} = \sum_{n=0}^{\infty} q^{\frac{kn(n-1)}{2}} \frac{x^n}{[n]_{q^k}!} = (1+(1-q^k)x)^{\infty}_{q,k} \tag{2.8}$$

and

$$e^x_{q,k} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q^k}!} = \frac{1}{(1-(1-q^k)x)^{\infty}_{q,k}}. \tag{2.9}$$

These q,k-exponential functions satisfy the following relations:
$D_{q^k} e^x_{q,k} = e^x_{q,k}, \qquad D_{q^k} E^x_{q,k} = E^{q^k x}_{q,k} \qquad$ *and* $\qquad E^{-x}_{q,k} e^x_{q,k} = e^x_{q,k} E^{-x}_{q,k} = 1.$

The q,k-Gamma function is defined by [1]

$$\Gamma_{q,k}(x) = \frac{(1-q^k)^{\infty}_{q,k}}{(1-q^x)^{\infty}_{q,k}(1-q)^{\frac{x}{k}-1}} \qquad x > 0. \tag{2.10}$$

When $k = 1$ it reduces to the known q-Gamma function Γ_q.

It satisfies the following functional equation:

$$\Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x), \quad \Gamma_{q,k}(k) = 1 \tag{2.11}$$

and having the following integral representation (see [1])

$$\Gamma_{q,k}(x) = \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} t^{x-1} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} d_q t, \ \ x > 0. \tag{2.12}$$

The previous integral representation, give that $\Gamma_{q,k}$ is an infinitely differentiable function on $]0, \ +\infty[$ and

$$\Gamma^{(i)}_{q,k}(x) = \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} t^{x-1}(lnt)^i E_{q,k}^{-\frac{q^k t^k}{[k]_q}} d_q t, \quad x > 0, \ \ i \in \mathbb{N}. \tag{2.13}$$

The q,k-Beta function is defined by (see [1])

$$B_{q,k}(t,s) = [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1}(1 - q^k \frac{x^k}{[k]_q})_{q,k}^{\frac{s}{k}-1} d_q x, \qquad s > 0, t > 0. \tag{2.14}$$

By using the following change of variable $u = \frac{x}{[k]_q^{\frac{1}{k}}}$, the last equation becomes

$$B_{q,k}(t,s) = \int_0^1 u^{t-1}(1 - q^k u^k)_{q,k}^{\frac{s}{k}-1} d_q u, \qquad s > 0, t > 0. \tag{2.15}$$

It satisfies

$$B_{q,k}(t,s) = \frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)}, \qquad s > 0, t > 0. \tag{2.16}$$

3 q-Čebyšev's integral inequality and applications

We begin this section by recalling the q-Čebyšev's integral inequality for q-synchronous (q-asynchronous) mappings [3] and as applications we give some inequalities for the q,k-Beta and the q,k-Gamma functions.

Definition 3.2. *Let f and g be two functions defined on I. The functions f and g are said q-synchronous (q-asynchronous) on I if*

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0 \quad \forall x, y \in I. \tag{3.17}$$

Note that if f and g are both q-increasing or q-decreasing on I then they are q-synchronous on I.

Proposition 3.1. *Let f, g and h be three functions defined on I such that:*

1. *$h(x) \geq 0, \quad x \in I$,*
2. *f and g are q-synchronous (q-asynchronous) on I.*

Then

$$\int_I h(x) d_q x \int_I h(x) f(x) g(x) d_q x \geq (\leq) \int_I h(x) f(x) d_q x \int_I h(x) g(x) d_q x. \tag{3.18}$$

Proof. We have

$$\int_I h(x) d_q x \int_I h(x) f(x) g(x) d_q x - \int_I h(x) f(x) d_q x \int_I h(x) g(x) d_q x =$$
$$1/2 \int_I \int_I h(x) h(y) \left[f(x) - f(y)\right] \left[g(x) - g(y)\right] d_q x d_q y.$$

So, the result follows from the conditions (1) and (2). □

The following theorem is a direct consequence of the previous proposition.

Theorem 3.1. *Let m, n, p and p' be some positive reals such that*

$$(p-m)(p'-n) \leq (\geq) 0.$$

Then

$$B_{q,k}(p,p')B_{q,k}(m,n) \geq (\leq) B_{q,k}(p,n)B_{q,k}(m,p') \tag{3.19}$$

and

$$\Gamma_{q,k}(p+n)\Gamma_{q,k}(p'+m) \geq (\leq) \Gamma_{q,k}(p+p')\Gamma_{q,k}(m+n). \tag{3.20}$$

Proof. Fix m, n, p and p' in $]0, \ +\infty[$, satisfying the condition of the theorem and the functions f, g and h defined on $[0,1]_q$ by

$$f(u) = u^{p-m}, \quad g(u) = (1-q^n u^k)_{q,k}^{\frac{p'-n}{k}} \quad \textit{and} \quad h(u) = u^{m-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1}.$$

From the relations

$$D_q f(u) = [p-m]_q u^{p-m-1} \tag{3.21}$$

and

$$D_q g(u) = [n-p']_q \, q^{p'} \, u^{k-1}(1-q^{n+k}u^k)_{q,k}^{\frac{p'-n}{k}-1}, \tag{3.22}$$

one can see that f and g are q-synchronous (q-asynchronous) on $I = [0,1]_q$.
So, by using the relation (2.15) and Proposition 3.1,
we obtain

$$\int_0^1 u^{m-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1} d_q u \int_0^1 u^{p-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1}(1-q^n u^k)_{q,k}^{\frac{p'-n}{k}} d_q u \geq$$
$$(\leq) \int_0^1 u^{p-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1} d_q u \int_0^1 u^{m-1}(1-q^k u^k)_{q,k}^{\frac{n}{k}-1}(1-q^n u^k)_{q,k}^{\frac{p'-n}{k}} d_q u,$$

which implies that

$$B_{q,k}(m,n)B_{q,k}(p,p') \geq (\leq) B_{q,k}(p,n)B_{q,k}(m,p'). \tag{3.23}$$

Now, according to the relations (2.16) and (3.19), we obtain

$$\frac{\Gamma_{q,k}(m)\Gamma_{q,k}(n)}{\Gamma_{q,k}(m+n)} \frac{\Gamma_{q,k}(p)\Gamma_{q,k}(p')}{\Gamma_{q,k}(p+p')} \geq (\leq) \frac{\Gamma_{q,k}(p)\Gamma_{q,k}(n)}{\Gamma_{q,k}(p+n)} \frac{\Gamma_{q,k}(m)\Gamma_{q,k}(p')}{\Gamma_{q,k}(m+p')}. \tag{3.24}$$

Therefore

$$\Gamma_{q,k}(p+n)\Gamma_{q,k}(p'+m) \geq (\leq) \Gamma_{q,k}(p+p')\Gamma_{q,k}(m+n). \tag{3.25}$$

□

Corollary 3.1. *For all $p, m > 0$, we have*

$$B_{q,k}(p,m) \geq \Big[B_{q,k}(p,p)B_{q,k}(m,m)\Big]^{1/2} \tag{3.26}$$

and

$$\Gamma_{q,k}(p+m) \leq \Big[\Gamma_{q,k}(2p)\Gamma_{q,k}(2m)\Big]^{1/2}. \tag{3.27}$$

Proof. A direct application of Theorem 3.1, with $p' = p$ and $n = m$, gives the results. □

Corollary 3.2. *For all $u, v > 0$, we have*

$$\Gamma_{q,k}\Big(\frac{u+v}{2}\Big) \leq \sqrt{\Gamma_{q,k}(u)\Gamma_{q,k}(v)}. \tag{3.28}$$

Proof. The inequality follows from (3.27), by taking $p = \frac{u}{2}$ and $m = \frac{v}{2}$. □

Theorem 3.2. *Let m, p and r be real numbers satisfying $m, p > 0$ and $p > r > -m$ and let n be a nonnegative integer.*
If

$$r(p-m-r) \geq (\leq) 0 \tag{3.29}$$

then

$$\Gamma_{q,k}^{(2n)}(p)\Gamma_{q,k}^{(2n)}(m) \geq (\leq) \Gamma_{q,k}^{(2n)}(p-r)\Gamma_{q,k}^{(2n)}(m+r). \tag{3.30}$$

Proof. Let f, g and h be the functions defined on $I = [0, (\frac{[k]_q}{(1-q^k)})^{\frac{1}{k}}]_q$ by

$$f(x) = x^{p-m-r}, \quad g(x) = x^r \quad and \quad h(x) = x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n}.$$

We have

$$D_q f(x) = [p-m-r]_q x^{p-m-r-1} \quad and \quad D_q g(x) = [r]_q x^{r-1}.$$

If the condition (3.29) holds, one can show that the functions f and g are q-synchronous (q-asynchronous) on I and Proposition 3.1 gives

$$\begin{aligned}&\int_I x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^{p-m-r} x^r x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \\ &\geq (\leq) \int_I x^{p-m-r} x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^r x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_I x^{m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^{p-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \\ &\geq (\leq) \int_I x^{p-r-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x \int_I x^{r+m-1} E_{q,k}^{-q^k \frac{x^k}{[k]_q}} (lnx)^{2n} d_q x.\end{aligned}$$

Hence, the relation

$$\Gamma_{q,k}^{(i)}(x) = \int_I t^{x-1} (lnt)^i E_{q,k}^{-q^k \frac{t^k}{[k]_q}} d_q t, \quad x > 0, \ i \in \mathbb{N},$$

gives

$$\Gamma_{q,k}^{(2n)}(m)\Gamma_{q,k}^{(2n)}(p) \geq (\leq) \Gamma_{q,k}^{(2n)}(p-r)\Gamma_{q,k}^{(2n)}(m+r). \tag{3.31}$$

□

Taking $n = 0$ in the previous theorem, we obtain the following result.

Corollary 3.3. *Let m, p and r be some real numbers under the conditions of Theorem3.2, we have*

$$\Gamma_{q,k}(p)\Gamma_{q,k}(m) \geq (\leq) \Gamma_{q,k}(p-r)\Gamma_{q,k}(m+r) \tag{3.32}$$

and

$$B_{q,k}(p,m) \geq (\leq) B_{q,k}(p-r, m+r). \tag{3.33}$$

Corollary 3.4. *Let n be a nonnegative integer, $p > 0$ and $p' \in \mathbb{R}$ such that $\mid p' \mid < p$. Then*

$$\left[\Gamma_{q,k}^{(2n)}(p)\right]^2 \leq \Gamma_{q,k}^{(2n)}(p-p')\Gamma_{q,k}^{(2n)}(p+p'). \tag{3.34}$$

Proof. By choosing $m = p$ and $r = p'$, we obtain

$$r(p-m-r) = -(p')^2 \leq 0$$

and the result turns out from Theorem 3.2. □

Taking in the previous result $p = \frac{u+v}{2}$ and $p' = \frac{u-v}{2}$, we obtain the following result:

Corollary 3.5. *Let u, v be two positive real numbers and n be a nonnegative integer. Then*

$$\Gamma_{q,k}^{(2n)}(\frac{u+v}{2}) \leq \sqrt{\Gamma_{q,k}^{(2n)}(u)\Gamma_{q,k}^{(2n)}(v)}. \tag{3.35}$$

Corollary 3.6. *Let $p > 0$ and $p' \in \mathbb{R}$ such that $\mid p' \mid < p$.*
Then

$$\Gamma_{q,k}^2(p) \leq \Gamma_{q,k}(p-p')\Gamma_{q,k}(p+p') \tag{3.36}$$

and

$$B_{q,k}(p,p) \leq B_{q,k}(p-p', p+p'). \tag{3.37}$$

Proof. For $n = 0$, the inequality (3.34) becomes

$$\Gamma_{q,k}^2(p) \leq \Gamma_{q,k}(p-p')\Gamma_{q,k}(p+p').$$

The inequality (3.37) follows from (2.16). □

Theorem 3.3. *Let a and b be two positive real numbers such*

$$(a-k)(b-k) \geq (\leq) 0$$

and n a nonnegative integer. Then

$$\Gamma_{q,k}^{(2n)}(2k)\Gamma_{q,k}^{(2n)}(a+b) \geq (\leq) \Gamma_{q,k}^{(2n)}(a+k)\Gamma_{q,k}^{(2n)}(b+k). \tag{3.38}$$

Proof. In Theorem 3.2, set $m = 2k$, $p = a+b$ and $r = b-k$. The condition (3.29) becomes

$$r(p-m-r) = (a-k)(b-k) \geq (\leq) 0. \tag{3.39}$$

So,

$$\Gamma_{q,k}^{(2n)}(2k)\Gamma_{q,k}^{(2n)}(a+b) \geq (\leq) \Gamma_{q,k}^{(2n)}(a+k)\Gamma_{q,k}^{(2n)}(b+k). \tag{3.40}$$

□

Corollary 3.7. *If $a, b > 0$ such $(a-k)(b-k) \geq (\leq) 0$. Then*

$$\Gamma_{q,k}(a+b) \geq (\leq) \frac{[a]_q[b]_q}{[k]_q}\Gamma_{q,k}(a)\Gamma_{q,k}(b) \tag{3.41}$$

and

$$B_{q,k}(a,b) \leq (\geq) \frac{[k]_q}{[a]_q[b]_q}. \tag{3.42}$$

Proof. The inequality (3.41) follows from the previous theorem by taking $n = 0$ and using the facts that $\Gamma_{q,k}(2k) = [k]_q$, $\Gamma_{q,k}(a+k) = [a]_q\Gamma_{q,k}(a)$ and
$\Gamma_{q,k}(b+k) = [b]_q\Gamma_{q,k}(b)$. (2.16) together with (3.41) give (3.42). □

Corollary 3.8. *The function $\ln \Gamma_{q,k}$ is superadditive for $x \geq k$ and $k \geq 1$, in the sense that*

$$\ln \Gamma_{q,k}(a+b) \quad \geq \quad \ln \Gamma_{q,k}(a) + \ln \Gamma_{q,k}(b).$$

Proof. For all $a, b \geq k$, we have

$$\begin{aligned} \ln \Gamma_{q,k}(a+b) &\geq \ln \frac{[a]_q[b]_q}{[k]_q} + \ln \Gamma_{q,k}(a) + \ln \Gamma_{q,k}(b) \\ &\geq \ln \Gamma_{q,k}(a) + \ln \Gamma_{q,k}(b), \end{aligned}$$

which completes the proof. □

Corollary 3.9. *For $a \geq k$ and $n = 1, 2, \ldots$, we have*

$$\Gamma_{q,k}(na) \geq \frac{[n-1]_{q^a}![a]_q^{2(n-1)}}{[k]_q^{n-1}}[\Gamma_{q,k}(a)]^n. \tag{3.43}$$

Proof. We proceed by induction on n.
It is clear that the inequality is true for $n = 1$.
Suppose that (3.43) holds for an integer $n \geq 1$ and let us prove it for $n+1$.
By (3.41), we have

$$\Gamma_{q,k}((n+1)a) = \Gamma_{q,k}(na + a) \geq \frac{[na]_q[a]_q}{[k]_q}\Gamma_{q,k}(na)\Gamma_{q,k}(a) \tag{3.44}$$

and by hypothesis, we have

$$\Gamma_{q,k}(na) \geq \frac{[n-1]_{q^a}![a]_q^{2(n-1)}}{[k]_q^{n-1}}[\Gamma_{q,k}(a)]^n. \tag{3.45}$$

The use of the fact that $[na]_q = [n]_{q^a}[a]_q$, gives

$$\begin{aligned}\Gamma_{q,k}((n+1)a) &\geq \frac{[na]_q[a]_q[n-1]_{q^a}![a]_q^{2(n-1)}}{[k]_q^n}[\Gamma_{q,k}(a)]^n\Gamma_{q,k}(a) \\ &\geq \frac{[n]_{q^a}![a]_q^{2n}}{[k]_q^n}\left[\Gamma_{q,k}(a)\right]^{n+1}.\end{aligned}$$

The inequality (3.43) is then true for $n+1$. □

For a given real $m > 0$ and a nonnegative integer n, consider the mapping

$$\Gamma_{q,k,m,n}(x) = \frac{\Gamma_{q,k}^{(2n)}(x+m)}{\Gamma_{q,k}^{(2n)}(m)}.$$

We have the following result.

Corollary 3.10. *The mapping $\Gamma_{q,k,m,n}(.)$ is suppermultiplicative on $[0, \infty)$, in the sense*

$$\Gamma_{q,k,m,n}(x+y) \geq \Gamma_{q,k,m,n}(x)\Gamma_{q,k,m,n}(y).$$

Proof. Fix x, y in $[0, \infty)$ and put $p = x + y + m$ and $r = y$. We have

$$y(x + y + m - m - y) = xy \geq 0.$$

So, the theorem 3.2 leads to

$$\Gamma_{q,k}^{(2n)}(m)\Gamma_{q,k}^{(2n)}(x+y+m) \geq \Gamma_{q,k}^{(2n)}(x+m)\Gamma_{q,k}^{(2n)}(y+m), \tag{3.46}$$

which is equivalent to

$$\Gamma_{q,k,m,n}(x+y) \geq \Gamma_{q,k,m,n}(x)\Gamma_{q,k,m,n}(y). \tag{3.47}$$

This achieves the proof. □

4 Inequalities via the q- Hölder's one

We begin this section by recalling the q-analogue of the Hölder's integral inequality [3].

Lemma 4.1. *Let p and p' be two positive reals satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, f and g be two functions defined on I. Then*

$$\left|\int_I f(x)g(x)d_qx\right| \leq \left(\int_I |\ f(x)\ |^p\ d_qx\right)^{\frac{1}{p}} \left(\int_I |\ g(x)\ |^{p'}\ d_qx\right)^{\frac{1}{p'}}. \tag{4.48}$$

Owing this lemma, one can establish some new inequalities involving the q,k-Gamma and q,k-Beta functions.

Theorem 4.4. *Let n be a nonnegative integer, x,y be two positive real numbers and a,b be two nonnegative real numbers such that $a+b=1$. Then*

$$\Gamma_{q,k}^{(2n)}(ax+by) \leq \left[\Gamma_{q,k}^{(2n)}(x)\right]^a \left[\Gamma_{q,k}^{(2n)}(y)\right]^b, \tag{4.49}$$

that is, the mapping $\Gamma_{q,k}^{(2n)}$ is logarithmically convex on $(0,\infty)$.

Proof. Consider the following functions defined on $I = [0, (\frac{[k]_q}{(1-q^k)})^{\frac{1}{k}}]_q$,

$$f(t) = t^{a(x-1)} \left(E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n}\right)^a \quad \text{and} \quad g(t) = t^{b(y-1)} \left(E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n}\right)^b.$$

By application of the q-Hölder's integral inequality, with $p = \dfrac{1}{a}$, we get

$$\begin{aligned}\int_I t^{a(x-1)} t^{b(y-1)} E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n} d_q t \quad &\leq \quad \left[\int_I t^{a(x-1).(1/a)} E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n} d_q t\right]^a \times \\ &\quad \left[\int_I t^{b(y-1).(1/b)} E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n} d_q t\right]^b,\end{aligned}$$

which is equivalent to

$$\int_I t^{ax+by-1} E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n} d_q t \leq \left[\int_I t^{x-1} E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n} d_q t\right]^a \left[\int_I t^{y-1} E_{q,k}^{-q^k \frac{t^k}{[k]_q}} (lnt)^{2n} d_q t\right]^b.$$

Then, (4.49) is a direct consequence of (2.13). □

Corollary 4.11. *Let $(p,p'), (m,m') \in (0,\infty)^2$ such that $p+p' = m+m'$ and $a,b \geq 0$ with $a+b=1$. Then, we have*

$$B_{q,k}(a(p,p') + b(m,m')) \leq \left[B_{q,k}(p,p')\right]^a \left[B_{q,k}(m,m')\right]^b. \tag{4.50}$$

Proof. On the one hand, we have

$$\begin{aligned}B_{q,k}(a(p,p') + b(m,m')) \quad &= \quad B_{q,k}(ap+bm, ap'+bm') = \frac{\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm')}{\Gamma_{q,k}(ap+bm+ap'+bm')} \\ &= \quad \frac{\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm')}{\Gamma_{q,k}(a(p+p')+b(m+m'))}.\end{aligned}$$

Since $p+p' = m+m'$ and $a+b=1$, we have

$$\Gamma_{q,k}(a(p+p')+b(m+m')) = \Gamma_{q,k}(p+p') = \Gamma_{q,k}(m+m'). \tag{4.51}$$

On the other hand, from Theorem 4.4, with $n=0$, we obtain

$$\Gamma_{q,k}(ap+bm) \leq \left[\Gamma_{q,k}(p)\right]^a \left[\Gamma_{q,k}(m)\right]^b \tag{4.52}$$

and

$$\Gamma_{q,k}(ap'+bm') \leq \left[\Gamma_{q,k}(p')\right]^a \left[\Gamma_{q,k}(m')\right]^b. \tag{4.53}$$

Thus

$$\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm') \leq \left[\Gamma_{q,k}(p)\Gamma_{q,k}(p')\right]^a \left[\Gamma_{q,k}(m)\Gamma_{q,k}(m')\right]^b. \tag{4.54}$$

From (4.51), we deduce that

$$\frac{\Gamma_{q,k}(ap+bm)\Gamma_{q,k}(ap'+bm')}{\Gamma_{q,k}(a(p+p')+b(m+m'))} \leq \left[\frac{\Gamma_{q,k}(p)\Gamma_{q,k}(p')}{\Gamma_{q,k}(p+p')}\right]^a \left[\frac{\Gamma_{q,k}(m)\Gamma_{q,k}(m')}{\Gamma_{q,k}(m+m')}\right]^b, \tag{4.55}$$

which completes the proof. □

Now, we recall that the logarithmic derivative of the q,k-Gamma function is defined on $(0,\infty)$, by

$$\Psi_{q,k}(x) = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)}.$$

The following result gives some properties of the function $\Psi_{q,k}$.

Theorem 4.5. *$\Psi_{q,k}$ is monotonic non-decreasing and concave on $(0,\infty)$.*

Proof. By taking $n=0$ in Theorem 4.4, we obtain

$$\Gamma_{q,k}(ax+by) \le \left[\Gamma_{q,k}(x)\right]^a \left[\Gamma_{q,k}(y)\right]^b,$$

for $x,\ y>0$ and $a,\ b\ge 0$ such that $a+b=1$.
So the function $ln\Gamma_{q,k}$ is convex. Then the monotonicity of $\Psi_{q,k}$ follows from the relation

$$\frac{d}{dx}[ln\Gamma_{q,k}(x)] = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)} = \Psi_{q,k}(x), \quad x>0.$$

On the other hand, since

$$\Gamma_{q,k}(x) = \frac{(1-q^k)^{\infty}_{q,k}}{(1-q^x)^{\infty}_{q,k}(1-q)^{\frac{x}{k}-1}}, \tag{4.56}$$

we obtain, for $x>0$,

$$\begin{aligned}
\Psi_{q,k}(x) &= \frac{d}{dx}[ln\Gamma_{q,k}(x)] = -\frac{1}{k}\,ln(1-q) + lnq\sum_{j=0}^{\infty}\frac{q^{x+jk}}{1-q^{x+jk}} \\
&= -\frac{1}{k}\,ln(1-q) + lnq\sum_{j=0}^{\infty}q^{x+jk}\sum_{n=0}^{\infty}q^{(x+jk)n} = -\frac{1}{k}\,ln(1-q) + lnq\sum_{n=0}^{\infty}\frac{q^{(n+1)x}}{1-q^{(n+1)k}} \\
&= -\frac{1}{k}\,ln(1-q) + \frac{lnq}{(1-q)}\int_0^q \frac{t^{x-1}}{1-t^k}d_qt.
\end{aligned}$$

Now, let $x,y>0$ and $a,b\ge 0$ such that $a+b=1$. Then

$$\Psi_{q,k}(ax+by) + \frac{1}{k}\,ln(1-q) = \frac{lnq}{(1-q)}\int_0^q \frac{t^{ax+by-1}}{1-t^k}d_qt = \frac{lnq}{(1-q)}\int_0^q \frac{t^{a(x-1)+b(y-1)}}{1-t^k}d_qt. \tag{4.57}$$

Since the mapping $x\mapsto t^x$ is convex on $\mathbb{R}$ for $t\in(0,1)$, we have

$$t^{a(x-1)+b(y-1)} \le at^{x-1}+bt^{y-1}, \quad for \ \ t\in[0,q]_q, \quad x,y>0.$$

Thus,

$$\frac{lnq}{(1-q)}\int_0^q \frac{t^{ax+by-1}}{1-t^k}d_qt \ge a\left(\frac{lnq}{(1-q)}\int_0^q \frac{t^{x-1}}{1-t^k}d_qt\right) + b\left(\frac{lnq}{(1-q)}\int_0^q \frac{t^{y-1}}{1-t^k}d_qt\right). \tag{4.58}$$

According to the relations (4.57) and (4.58), we have

$$\begin{aligned}
\Psi_{q,k}(ax+by) + \frac{1}{k}\,ln(1-q) &\ge a(\Psi_{q,k}(x) + \frac{1}{k}\,ln(1-q)) + b(\Psi_{q,k}(y) + \frac{1}{k}\,ln(1-q)) \\
&\ge a\Psi_{\ q,k}(x) + b\Psi_{q,k}(y) + \frac{1}{k}\,ln(1-q).
\end{aligned}$$

This proves the concavity of the function $\Psi_{q,k}$. □

5 Inequalities via the q-Grüss's one

In [5] H. Gauchman gave a q-analogue of the Grüss' integral inequality namely.

Lemma 5.2. *Assume that* $m \leq f(x) \leq M$, $\varphi \leq g(x) \leq \Phi$, *for each* $x \in [a,b]$, *where* m, M, φ, Φ *are given real constants. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)d_qx - \frac{1}{(b-a)^2} \int_a^b f(x)d_qx \int_a^b g(x)d_qx \right| \leq \frac{1}{4}(M-m)(\Phi-\varphi). \tag{5.59}$$

As application of the previous inequality we state the following result

Theorem 5.6. *Let* $m, n > 0$, *we have*

$$\left| B_{q,k}(m+k, n+k) - \frac{1}{[m+1]_q[n+k]_q} \right| \leq \frac{1}{4}. \tag{5.60}$$

Remark that from the relations (2.16) and (2.11), the inequality (5.60) is equivalent to

$$\mid \Gamma_{q,k}(m+n+2k) - \Gamma_{q,k}(n+2k)\Gamma_{q,k}(m+k)[m+1]_q \mid \leq \frac{1}{4}[m+1]_q[n+k]_q\Gamma_{q,k}(m+n+2k). \tag{5.61}$$

Proof. Consider the functions

$$f(u) = u^m, \quad g(u) = u^{k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}, \quad u \in [0,1]_q, \quad m, n > 0.$$

We have

$$0 \leq f(u) \leq 1 \quad and \quad 0 \leq g(u) \leq 1 \qquad \forall\, u \in [0,1]_q.$$

Then, using the q-Grüss' integral inequality, we obtain

$$\left| \int_0^1 u^{m+k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}\, d_qu - \int_0^1 u^m\, d_qu \int_0^1 u^{k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}\, d_qu \right| \leq \frac{1}{4}. \tag{5.62}$$

The inequality (5.60) follows from the definition of the q,k-Beta function (2.15) and the following facts: $\int_0^1 u^m d_qu = \frac{1}{[m+1]_q}$ and
$\int_0^1 u^{k-1}(1-q^ku^k)_{q,k}^{\frac{n}{k}}\, d_qu = B_{q,k}(k, n+k) = \frac{1}{[n+k]_q}$. □

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