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Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials

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Abstract
In this paper, we introduce three new subclasses of the function class Σ' of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials. Furthermore, we obtain estimates on the first two coefficients of functions in these new subclasses. Also, Fekete-Szegö inequalities of functions belonging to these subclasses are founded.

Keywords
Analytic functions, Univalent and bi-univalent functions, Fekete-Szegö problem; Horadam polynomials, Poisson distribution, Coefficient bounds, Subordination.

AMS Subject Classification
Primary 11B39. 30C45. 33C45, Secondary 30C50.

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1. Introduction

Let A denote the class of the functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the open unit open disk $\mathbb{U}=\{z:z\in\mathbb{C},|z|<1\}$, and let $\mathbb{S}$ be class of all functions in A which are univalent and normalized by the conditions

$$f(0)=0=f'(0)-1$$

in $\mathbb{U}$. It is well known that every function $f\in\mathbb{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z\ \ (z\in\mathbb{U}),$$

and

$$f^{-1}(f(w))=w\ \ (|w|<r_0(f);r_0(f)\geq\frac{1}{4}),$$

where

$$f^{-1}(w)=w+a_2w^2+(2a_2^2-3a_3)w^3$$

$$-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Several open problems and conjectures involving bounds of the coefficients of the functions in Σ', can be found in the earlier work studied by Lewin [10], he studied the class of bi-univalent functions and derived that $|a_2|<1.51$.
In 2010, Srivastava et al. [14] revived the study of bi-univalent functions by their pioneering work on the study of coefficient problems. Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for

the initial coefficients (see [4], [5], [1], [2], [3]). However, for the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n| (n = 3,4,...)$ is still an open problem.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to the function g in $\mathbb{U}$, written as $f(z) \prec g(z), \quad (z \in \mathbb{U})$, provided that there exists an analytic function (that is, Schwarz function) $w(z)$ defined on $\mathbb{U}$ with

$$w(0) = 0 \text{ and } \quad |w(z)| < 1 \text{ for all } z \in \mathbb{U},$$

such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$.

Indeed, it is known that

$$f(z) \prec g(z) \ (z \in \mathbb{U}) \ \Rightarrow \ f(0) = g(0) \ \text{ and } \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

Recently, Horcum and Kocer [8] considered Horadam polynomials $h_n(x)$, which are given by the following recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in \mathbb{N} > 2), \qquad (1.3)$$

with $h_1 = a$, $h_2 = bx$, and $h_3 = pbx^2 + aq$ where (a,b,p,q are some real constants) .
The characteristic equation of recurrence relation (1.3) is

$$t^2 - pxt - q = 0. \qquad (1.4)$$

This equation has two real roots;

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},$$

and

$$\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Remark 1.1. *for particular values of a,b,p and q, the Horadam polynomial $h_n(x)$ leads to various polynomials, among those, we list a particular cases of Horadam polynomials sequence here*

- *If $a = b = p = q = 1$, the Fibonacci polynomials sequence is obtained*

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ F_1(x) = 1,\ F_2(x) = x.$$

- *If $a = 2, b = p = q = 1$, the Lucas polynomials sequence is obtained*

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x),\ L_0(x) = 2,\ L_1(x) = x.$$

- *If $a = q = 1, b = p = 2$, the Pell polynomials sequence is obtained*

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),\ p_1(x) = 1,\ P_2(x) = 2x.$$

- *If $a = b = p = 2, q = 1$, the Pell-Lucas polynomials sequence is obtained*

$$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x),\ Q_0(x) = 2,\ Q_1(x) = 2x.$$

- *If $a = 1, b = p = 2, q = -1$, the Chebyshev polynomials of second kind sequence is obtained*

$$U_{n-1}(x) = 2xU_{n-2}(x) + U_{n-3}(x),\ U_0(x) = 1,\ U_1(x) = 2x.$$

- *If $a = b = 1, p = 2, q = -1$, the Chebyshev polynomials of First kind sequence is obtained*

$$T_{n-1}(x) = 2xT_{n-2}(x) + T_{n-3}(x),\ T_0(x) = 1,\ T_1(x) = x.$$

- *If $x = 1$, The Horadam numbers sequence is obtained*

$$h_{n-1}(1) = ph_{n-2}(1) + qh_{n-3}(1),\ h_0(1) = a,\ h_1(1) = b.$$

For more information associated with these polynomials sequences can be found in ([6], [7], [9], [11]).
Note that, the above polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in many of sciences, specially in mathematics, statistics and physics.

Remark 1.2. *[7] Let $\Omega(x,z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then*

$$\Omega(x,z) = \frac{a + (b - ap)xt}{1 - pxt - qt^2} = \sum_{n=1}^{\infty} h_n(x) z^{n-1}. \qquad (1.5)$$

A variable x is said to be Poisson distributed if it takes the values $0, 1, 2, 3, ...$ with probabilities

$$e^{-m}, m\frac{e^{-m}}{1!}, m\frac{e^{-m}}{2!}, m\frac{e^{-m}}{3!}, ...,$$

respectively, where m is called the parameter. Thus

$$P(x = r) = \frac{m^n e^{-m}}{r!}, r = 0, 1, 2, ...$$

Recently, Porwal [12] introduced a power series whose coefficients are probabilities of Poisson distribution

$$K(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad (m > 0,\ z \in \mathbb{U}).$$

We note that, by ratio test, the radius of convergence of the above series is infinity. Also, by using the above series they obtained some interesting results on certain classes of analytic univalent functions.

In 2016, Porwal and Kumar [13] introduced a new linear operator $I(m,z) : A \to A$ by using the convolution (or Hadamard product), and defined as follows

$$\begin{aligned} I(m,z)f &= K(m,z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad (m > 0,\ z \in \mathbb{U}), \end{aligned}$$

where $*$ denote the convolution (or Hadamard product) of two series.

Lemma 1.3. *If $h \in p$ then $|c_k| < 1$, for each k, where p is the family of all functions h analytic in $\mathbb{U}$ for which $\Re\{h(z)\} > 0$, then*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, z \in \mathbb{U}.$$

The object of the present paper is to introduce new subclasses of Σ' involving the Poisson distribution associated with Horadam polynomials $h_n(x)$. Estimates on the initial coefficients and the Fekete-Szegö inequalities for certain subclasses of bi-univalent functions defined by means of Horadam polynomials are obtained.

2. Coefficient Bounds for the Function Class $S^*_\Sigma(\mu, m, x)$

We begin with introducing the function class $S^*_\Sigma(\mu, m, x)$ corresponding to $I(m,z)f(z)$ by means of the following definition.

Definition 2.1. *A function $f \in \Sigma'$ given by (1.1) is said to be in the class $S^*_\Sigma(\mu, m, x)$, if the following conditions are satisfied:*

$$\frac{z[I(m,z)f(z)]'}{I(m,z)f(z)} \prec \Omega(x,z) + 1 - \alpha \tag{2.1}$$

and

$$\frac{w[I(m,w)g(w)]'}{I(m,w)g(w)} \prec \Omega(x,w) + 1 - \alpha \tag{2.2}$$

where the real constants a, b, p and q are some real constants, $\Omega(x,z)$ is given by (1.5) and $g(w) = f^{-1}(z)$ is given by (1.2).

We first state and prove the following result.

Theorem 2.2. *Let the function $f \in \Sigma'$ given by (1.1) be in the class $S^*_\Sigma(\mu, m, x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{e^m|bx|}}{\sqrt{|m[(b - me^{-m}p)bx^2 - me^{-m}aq]|}} \tag{2.3}$$

$$|a_3| \leq \frac{2}{m}\left(\left[\frac{e^m bx}{m}\right]^2 + \frac{e^m|bx|}{2m}\right), \tag{2.4}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq$$

$$\begin{cases} \frac{e^m|bx|}{m^2} & , if\ |1-\mu| \leq \frac{|(b-me^{-m}p)bx^2 - me^{-m}aq|}{mb^2x^2} \\ \frac{e^m|bx|^3|1-\mu|}{|(b-me^{-m}p)bx^2 - me^{-m}aq|} & , if\ |1-\mu| \geq \frac{|(b-me^{-m}p)bx^2 - me^{-m}aq|}{mb^2x^2} \end{cases}. \tag{2.5}$$

Proof. Let $f \in \Sigma'$ be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Psi(0) = \Phi(0) = 0$, $|\psi(z)| < 1$ and $|\Phi(w)| < 1$, $z, w \in \mathbb{U}$ and using Definition 2.1, we can write

$$\frac{z[I(m,z)f(z)]'}{I(m,z)f(z)} = \omega(x, \Phi(z)) + 1 - \alpha$$

and

$$\frac{w[I(m,w)g(w)]'}{I(m,w)g(w)} = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$\frac{z[I(m,z)f(z)]'}{I(m,z)f(z)}$$

$$= 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \cdots \tag{2.6}$$

and

$$\frac{w[I(m,w)g(w)]'}{I(m,w)g(w)}$$

$$= 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \cdots. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\frac{z[I(m,z)f(z)]'}{I(m,z)f(z)}$$

$$= 1 + h_2(x)p_1 z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \cdots \tag{2.8}$$

and

$$\frac{w[I(m,w)g(w)]'}{I(m,w)g(w)}$$

$$= 1 + h_2(x)p_1 w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \cdots. \tag{2.9}$$

Notice that if

$$|\Phi(z)| = |p_1 z + p_2 z^2 + p_3 z^3 + ...| < 1 \quad (z \in \mathbb{U})$$

and

$$|\psi(w)| = |q_1 w + q_2 w^2 + q_3 w^3 + ...| < 1 \quad (w \in \mathbb{U}),$$

then

$$|p_i| \leq 1 \quad and \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (2.8) and (2.9), we have

$$me^{-m}a_2 = h_2(x)p_1 \tag{2.10}$$

$$me^{-m}\left[ma_3 - a_2^2\right] = h_2(x)p_2 + h_3(x)p_1^2 \tag{2.11}$$

$$-me^{-m}a_2 = h_2(x)q_1 \tag{2.12}$$

$$me^{-m}\left[3a_2^2 - ma_3\right] = h_2(x)q_2 + h_3(x)q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we have

$$p_1 = -q_1 \tag{2.14}$$

and

$$2m^2e^{-2m}a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \tag{2.15}$$

If we add (2.11) and (2.13), we get

$$2me^{-m}a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2). \tag{2.16}$$

By substituting (2.15) in (2.16), we reduce that

$$2me^{-m}\left[1 - \frac{me^{-m}h_3(x)}{[h_2(x)]^2}\right]a_2^2 = h_2(x)(p_2 + q_2), \tag{2.17}$$

$\Rightarrow$

$$a_2^2 = \frac{h_2^3(x)(p_2 + q_2)}{2me^{-m}[b^2x^2 - me^{-m}(pbx^2 + aq)]}, \tag{2.18}$$

which yields

$$|a_2| \leq \frac{|bx|\sqrt{e^m|bx|}}{\sqrt{|m[(b - me^{-m}p)bx^2 - me^{-m}aq]|}}.$$

By subtracting (2.13) from (2.11) and in view of (2.14), we obtain

$$2m^2e^{-m}a_3 - 4me^{-m}a_2^2 = h_2(x)(p_2 - q_2) + h_3(x)(p_1^2 - q_1^2). \tag{2.19}$$

In view of (2.14) and (2.15), equation (2.19) becomes

$$a_3 = \frac{4me^{-m}a_2^2}{2m^2e^{-m}} + \frac{h_2(x)(p_2 - q_2)}{2m^2e^{-m}}. \tag{2.20}$$

Hence using (2.14) and applying (1.3), we get desired inequality (2.4).
From (2.20), for $\eta \in \mathbb{R}$, we get,

$$a_3 - \eta a_2^2 = \frac{[h_2(x)]^3(1-\eta)(p_2+q_2)}{2me^{-m}[(h_2(x))^2 - me^{-m}h_3(x)]} + \frac{h_2(x)(p_2 - q_2)}{2m^2e^{-m}}$$

$$= h_2(x)\left[\left(\Theta(\eta,x) + \frac{1}{2m^2e^{-m}}\right)p_2 + \left(\Theta(\eta,x) - \frac{1}{2m^2e^{-m}}\right)q_2\right],$$

where

$$\Theta(\eta,x) = \frac{[h_2(x)]^2(1-\eta)}{2me^{-m}[(h_2(x))^2 - me^{-m}h_3(x)]}.$$

So, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{e^m|h_2(x)|}{m^2} & , |\Theta(\eta,x)| \leq \frac{e^m}{2m^2} \\ 2|h_2(x)||\Theta(\eta,x)| & , |\Theta(\eta,x)| \geq \frac{e^m}{2m^2} \end{cases}.$$

This proves Theorem 2.2.

3. Coefficient Bounds for the Function Class $K_\Sigma^*(\mu,m,x)$

Definition 3.1. *A function $f \in \Sigma'$ given by (1.1) is said to be in the class $K_\Sigma^*(\mu,m,x)$, if the following conditions are satisfied:*

$$1 + \frac{z[I(m,z)f(z)]''}{[I(m,z)f(z)]'} \prec \Omega(x,z) + 1 - \alpha \tag{3.1}$$

and

$$1 + \frac{w[I(m,w)g(w)]''}{[I(m,w)g(w)]'} \prec \Omega(x,w) + 1 - \alpha \tag{3.2}$$

where the real constants a, b, p and q are some real constants, $\Omega(x,z)$ is given by (1.5) and $g(w) = f^{-1}(z)$ is given by (1.2).

Theorem 3.2. *Let the function $f \in \Sigma'$ given by (1.1) be in the class $K_\Sigma^*(\mu,m,x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{e^m|bx|}}{\sqrt{|2m[(b - 2me^{-m}p)bx^2 - 2me^{-m}aq]|}} \tag{3.3}$$

$$|a_3| \leq \frac{1}{m}\left(\left[\frac{e^m bx}{2m}\right]^2 + \frac{e^m|bx|}{6m}\right), \tag{3.4}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq$$

$$\begin{cases} \frac{e^m|bx|}{6m^2} & , if\ |1-\mu| \leq \frac{|b^2x^2 - 2me^{-m}(pbx^2+aq)|}{3mb^2x^2} \\ \frac{e^m|bx|^3|1-\mu|}{|b^2x^2 - 2me^{-m}(pbx^2+aq)|} & , if\ |1-\mu| \geq \frac{|b^2x^2 - 2me^{-m}(pbx^2+aq)|}{3mb^2x^2} \end{cases}. \tag{3.5}$$

Proof. Let $f \in \Sigma'$ be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Psi(0) = \Phi(0) = 0$, $|\psi(z)| < 1$ and $|\Phi(w)| < 1$, $z, w \in \mathbb{U}$ and using Definition 3.1, we can write

$$1 + \frac{z[I(m,z)f(z)]''}{[I(m,z)f(z)]'} = \omega(x, \Phi(z)) + 1 - \alpha$$

and

$$1 + \frac{w[I(m,w)g(w)]''}{[I(m,w)g(w)]'} = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$1 + \frac{z[I(m,z)f(z)]''}{[I(m,z)f(z)]'}$$

$$= 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \cdots \quad (3.6)$$

and

$$1 + \frac{w[I(m,w)g(w)]''}{[I(m,w)g(w)]'}$$

$$= 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \cdots . \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$1 + \frac{z[I(m,z)f(z)]''}{[I(m,z)f(z)]'}$$

$$= 1 + h_2(x)p_1 z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \cdots \quad (3.8)$$

and

$$1 + \frac{w[I(m,w)g(w)]''}{[I(m,w)g(w)]'}$$

$$= 1 + h_2(x)p_1 w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \cdots . \quad (3.9)$$

Notice that if'

$$|\Phi(z)| = |p_1 z + p_2 z^2 + p_3 z^3 + ...| < 1 \quad (z \in \mathbb{U})$$

and

$$|\psi(w)| = |q_1 w + q_2 w^2 + q_3 w^3 + ...| < 1 \quad (w \in \mathbb{U}),$$

then

$$|p_i| \leq 1 \quad and \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (3.8) and (3.9), we have

$$2me^{-m}a_2 = h_2(x)p_1 \quad (3.10)$$

$$me^{-m}\left[6ma_3 - 4a_2^2\right] = h_2(x)p_2 + h_3(x)p_1^2 \quad (3.11)$$

$$-2me^{-m}a_2 = h_2(x)q_1 \quad (3.12)$$

$$2me^{-m}\left[4a_2^2 - 3ma_3\right] = h_2(x)q_2 + h_3(x)q_1^2. \quad (3.13)$$

From (3.10) and (3.12), we have

$$p_1 = -q_1 \quad (3.14)$$

and

$$8m^2e^{-2m}a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \quad (3.15)$$

If we add (3.11) and (3.13), we get

$$4me^{-m}a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2). \quad (3.16)$$

By substituting (3.15) in (3.16), we reduce that

$$4me^{-m}\left[1 - \frac{2me^{-m}h_3(x)}{[h_2(x)]^2}\right]a_2^2 = h_2(x)(p_2 + q_2), \quad (3.17)$$

$\Rightarrow$

$$a_2^2 = \frac{h_2^3(x)(p_2 + q_2)}{4me^{-m}[b^2x^2 - 2me^{-m}(pbx^2 + aq)]}, \quad (3.18)$$

which yields

$$|a_2| \leq \frac{|bx|\sqrt{e^m|bx|}}{\sqrt{|2m[(b - 2me^{-m}p)bx^2 - 2me^{-m}aq]|}}.$$

By subtracting (3.13) from (3.11) and in view of (3.14), we obtain

$$12m^2e^{-m}a_3 - 12me^{-m}a_2^2 = h_2(x)(p_2 - q_2) + h_3(x)(p_1^2 - q_1^2). \quad (3.19)$$

In view of (3.14) and (3.15), equation (3.19) becomes

$$a_3 = \frac{12me^{-m}a_2^2}{12m^2e^{-m}} + \frac{h_2(x)(p_2 - q_2)}{12m^2e^{-m}}. \quad (3.20)$$

Hence using (3.14) and applying (1.3), we get desired inequality (3.4).
From (3.20), for $\eta \in \mathbb{R}$, we get,

$$a_3 - \eta a_2^2 = \frac{[h_2(x)]^3(1-\eta)(p_2 + q_2)}{4me^{-m}[h_2(x) - 2me^{-m}h_3(x)]} + \frac{h_2(x)(p_2 - q_2)}{12m^2e^{-m}}$$

$$= h_2(x)\left[\left(\Theta(\eta,x) + \frac{e^m}{12m^2}\right)p_2 + \left(\Theta(\eta,x) - \frac{e^m}{12m^2}\right)q_2\right],$$

where

$$\Theta(\eta,x) = \frac{[h_2(x)]^2(1-\eta)}{4me^{-m}[h_2(x) - 2me^{-m}h_3(x)]}.$$

So, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{e^m h_2(x)}{6m^2} & , |\Theta(\eta,x)| \leq \frac{e^m}{12m^2} \\ 2h_2(x)|\Theta(\eta,x)| & , |\Theta(\eta,x)| \geq \frac{e^m}{12m^2} \end{cases}.$$

This proves Theorem 3.2.

4. Coefficient Bounds for the Function Class $SK^*_\Sigma(\mu,m,x)$

Definition 4.1. *A function $f\in\Sigma'$ given by (1.1) is said to be in the class $SK^*_\Sigma(\mu,m,x)$, if the following conditions are satisfied:*

$$[I(m,z)f(z)]'\prec\Omega(x,z)+1-\alpha \tag{4.1}$$

and

$$[I(m,w)g(w)]'\prec\Omega(x,w)+1-\alpha \tag{4.2}$$

where the real constants a,b,p and q are some real constants, $\Omega(x,z)$ is given by (1.5) and $g(w)=f^{-1}(z)$ is given by (1.2).

Theorem 4.2. *Let the function $f\in\Sigma'$ given by (1) be in the class $SK^*_\Sigma(\mu,m,x)$. Then*

$$|a_2|\le\frac{|bx|\sqrt{e^m|bx|}}{\sqrt{|m[(3b-4me^{-m}p)bx^2-4me^{-m}aq]|}} \tag{4.3}$$

$$|a_3|\le\frac{1}{2m}\left(\left[\frac{e^mbx}{m}\right]^2+\frac{4e^m|bx|}{3m}\right), \tag{4.4}$$

and for some $\eta\in\mathbb{R}$,

$$|a_3-\eta a_2^2|\le \begin{cases}\frac{2e^m|bx|}{3m^2} & ,if\ |1-\mu|\le\frac{2|[(3b-4me^{-m}p)bx^2-4me^{-m}aq]|}{3mb^2x^2}\\ \frac{e^m|bx|^3|1-\mu|}{|(3b-4me^{-m}p)bx^2-4me^{-m}aq|} & ,if\ |1-\mu|\ge\frac{2|[(3b-4me^{-m}p)bx^2-4me^{-m}aq]|}{3mb^2x^2}\end{cases} \tag{4.5}$$

Proof. Let $f\in\Sigma'$ be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Psi(0)=\Phi(0)=0$, $|\psi(z)|<1$ and $|\Phi(w)|<1$, $z,w\in\mathbb{U}$ and using Definition 4.1, we can write

$$[I(m,z)f(z)]'=\omega(x,\Phi(z))+1-\alpha$$

and

$$[I(m,w)g(w)]'=\omega(x,\psi(w))+1-\alpha$$

or, equivalently,

$$[I(m,z)f(z)]'$$

$$=1+h_1(x)-a+h_2(x)\Phi(z)+h_3(x)[\Phi(z)]^3+\cdots \tag{4.6}$$

and

$$[I(m,w)g(w)]'$$

$$=1+h_1(x)-a+h_2(x)\psi(w)+h_3(x)[\psi(w)]^3+\cdots. \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$[I(m,z)f(z)]'$$

$$=1+h_2(x)p_1z+[h_2(x)p_2+h_3(x)p_1^2]z^2+\cdots \tag{4.8}$$

and

$$[I(m,w)g(w)]'$$

$$=1+h_2(x)p_1w+[h_2(x)q_2+h_3(x)q_1^2]w^2+\cdots. \tag{4.9}$$

Notice that if

$$|\Phi(z)|=|p_1z+p_2z^2+p_3z^3+...|<1 \quad (z\in\mathbb{U})$$

and

$$|\psi(w)|=|q_1w+q_2w^2+q_3w^3+...|<1 \quad (w\in\mathbb{U}),$$

then

$$|p_i|\le1 \quad and \quad |q_i|\le1 \quad (i\in\mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (4.8) and (4.9), we have

.

$$2me^{-m}a_2=h_2(x)p_1 \tag{4.10}$$

$$\frac{3}{2}m^2e^{-m}a_3=h_2(x)p_2+h_3(x)p_1^2 \tag{4.11}$$

$$-2me^{-m}a_2=h_2(x)q_1 \tag{4.12}$$

$$6me^{-m}a_2^2-\frac{3}{2}m^2a_3=h_2(x)q_2+h_3(x)q_1^2. \tag{4.13}$$

From (4.12) and (4.10), we have

$$p_1=-q_1 \tag{4.14}$$

and

$$8m^2e^{-2m}a_2^2=h_2^2(x)(p_1^2+q_1^2). \tag{4.15}$$

If we add (4.11) and (4.13), we get

$$6me^{-m}a_2^2=h_2(x)(p_2+q_2)+h_3(x)(p_1^2+q_1^2). \tag{4.16}$$

By substituting (4.16) in (4.15), we reduce that

$$2me^{-m}\left[3-\frac{4me^{-m}h_3(x)}{[h_2(x)]^2}\right]a_2^2=h_2(x)(p_2+q_2), \quad (4.17)$$

$\Rightarrow$

$$a_2^2=\frac{h_2^3(x)(p_2+q_2)}{2me^{-m}[3b^2x^2-4me^{-m}(pbx^2+aq)]}, \quad (4.18)$$

which yields

$$|a_2|\leq\frac{|bx|\sqrt{e^m|bx|}}{\sqrt{|m[(3b-4me^{-m}p)bx^2-4me^{-m}aq]|}}.$$

By subtracting (2.13) from (2.11) and in view of (4.14), we obtain

$$3m^2e^{-m}a_3-6me^{-m}a_2^2=h_2(x)(p_2-q_2)+h_3(x)(p_1^2-q_1^2). \quad (4.19)$$

In view of (4.14) and (4.15), equation (4.19) becomes

$$a_3=\frac{6me^{-m}a_2^2}{3m^2e^{-m}}+\frac{h_2(x)(p_2-q_2)}{3m^2e^{-m}}. \quad (4.20)$$

Hence using (4.14) and applying (1.3), we get desired inequality (4.4).
From (4.20), for $\eta\in\mathbb{R}$, we get,

$$a_3-\eta a_2^2=\frac{[h_2(x)]^3(1-\eta)(p_2+q_2)}{2me^{-m}[3(h_2(x))^2-4me^{-m}h_3(x)]}+\frac{h_2(x)(p_2-q_2)}{3m^2e^{-m}}$$

$$=h_2(x)\left[\left(\Theta(\eta,x)+\frac{1}{3m^2e^{-m}}\right)p_2+\left(\Theta(\eta,x)-\frac{1}{3m^2e^{-m}}\right)q_2\right],$$

where

$$\Theta(\eta,x)=\frac{[h_2(x)]^2(1-\eta)}{2me^{-m}[3(h_2(x))^2-4me^{-m}h_3(x)]}.$$

So, we conclude that

$$|a_3-\eta a_2^2|\leq\begin{cases}\frac{2e^m|h_2(x)|}{3m^2} & ,|\Theta(\eta,x)|\leq\frac{e^m}{3m^2}\\ 2|h_2(x)||\Theta(\eta,x)| & ,|\Theta(\eta,x)|\geq\frac{e^m}{3m^2}\end{cases}.$$

This proves Theorem 4.2.

Remark 4.3. *By taking all particular cases of Horadam polynomials sequence as shown in remark 1.1, and use the same technique we will have a new results for these subclasses.*

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