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Relatively prime dominating polynomial in graphs

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Abstract

We introduce the concept of relatively prime domination polynomial of a graph G. The relatively prime domination polynomial of a graph G of order n is the polynomial $D_{rpd}(G,x) = \sum_{k=\gamma_{rpd}(G)}^{n} d_{rpd}(G,k)x^k$ where $d_{rpd}(G,k)$ is the number of relatively prime dominating sets of G of size k, and $\gamma_{rpd}(G)$ is the relatively prime domination number of G. We compute this polynomial for path P_n, complete bipartite graph $K_{m,n}$, star $K_{1,n}$, bistar $B_{m,n}$, spider graph $K_{1,n,n}$ and Helm graph H_n.

Keywords

Dominating polynomial, relatively prime dominating polynomial, relatively prime dominating polynomial roots.

AMS Subject Classification

05C69, 11B83.

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1. Introduction

By a graph $G = (V,E)$ we mean a finite undirected graph without loops and multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretical terms, we refer to Harary [5] and for terms related to domination we refer to Haynes [6]. A subset S of V is said to be a dominating set in G if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G.

Berge and Ore [2, 13] formulated the concept of domination in graphs. It was further extended to define many other domination related parameters in graphs. Let G be a non - trivial graph. A set $S \subseteq V$ is said to be a relatively prime dominating set if it is a dominating set and for every pair of vertices u and v in S such that $(d(u),d(v)) = 1$. The minimum cardinality of a relatively prime dominating set is called the relatively prime domination number and it is denoted by $\gamma_{rpd}(G)$ [8]. Switching in graphs was introduced by Lint and Seidel [12]. For a finite undirected graph $G(V,E)$ and a subset $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^{\sigma}(V,E')$ which is obtained from G by removing all edges between σ and its complement $V - \sigma$ and adding as edges all non edges between σ and $V - \sigma$. For $\sigma = \{v\}$, we write G^v instead of $G^{\{v\}}$ and the corresponding switching is called as vertex switching [7]. Bistar is the graph obtained by joining the center of two stars $K_{1,m}$ and $K_{1,n}$ with an edge and it is denoted by $B_{m,n}$ [14]. A spider is a tree with one vertex of degree at least 3, called the center, and all others with degree at most 2 and it is denoted by $K_{1,n,n}$ [4]. A wounded spider is the graph formed by sub dividing at most $n-1$ of the edges of a star $K_{1,n}$ for $n \geq 0$ [11]. For more details about the basic definitions which is not appear here, we refer to Harrary [5].

Graph polynomials are powerful and well-developed tools to express graph parameters. Saeid Alikhani and Peng, Y. H. [1], have introduced the Domination polynomial of a graph. The Domination polynomial of a graph G of order n is the polynomial $D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i, and $\gamma(G)$ is the domination number of G. This motivated us to introduce the relatively prime domination polynomial of a graph. In this paper, we define the relatively prime domination polynomial of a graph G and find the relatively prime domination polynomial of some standard graphs.

2. Definition and Examples

Definition 2.1. *Let $G=(V,E)$ be a graph of order n with relatively prime domination number $\gamma_{rpd}(G)$. The relatively prime domination polynomial of G is,*

$$D_{rpd}(G,x)=\sum_{k=\gamma_{rpd}(G)}^{n} d_{rpd}(G,k)x^k$$

where $d_{rpd}(G,k)$ is the number of relatively prime dominating sets of G of size k and $\gamma_{rpd}(G)$ is the relatively prime domination number of G. The roots of the polynomial $D_{rpd}(G,k)$ are called the relatively prime dominating roots of G.

Example 2.2. *Let G be the graph given in Figure 1. Clearly $\gamma_{rpd}(G)=2$ and there are only two minimum relatively prime dominating sets of size 2, namely $\{v_2,v_4\}$ and $\{v_3,v_5\}$, three relatively prime dominating sets of size 3, namely $\{v_1,v_4,v_5\}$, $\{v_1,v_3,v_5\}$ and $\{v_1,v_2,v_4\}$ and two relatively prime dominating sets of size 4, namely $\{v_1,v_2,v_4,v_5\}$ and $\{v_1,v_3,v_4,v_5\}$. Hence $D_{rpd}(G,x)=2x^2+3x^3+2x^4=x^2(2+3x+2x^2)$.*

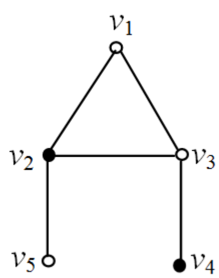

Figure 1. G

Example 2.3. *Consider the graph $G=2K_2$ given in Figure 2. Clearly $\gamma_{rpd}(G)=2$ and there are only four minimum relatively prime dominating sets of size 2, namely $\{v_1,v_3\},\{v_1,v_4\}$, $\{v_2,v_3\}$ and $\{v_2,v_4\}$, four relatively prime dominating sets of size 3, namely $\{v_1,v_2,v_3\},\{v_1,v_2,v_4\},\{v_2,v_3,v_4\}$ and $\{v_1,v_3,v_4\}$ and one relatively prime dominating set of size 4 which is $\{v_1,v_2,v_3,v_4\}$. Hence $D_{rpd}(G,x)=4x^2+4x^3+x^4=x^2(4+4x+x^2)$. Obviously, there are two relatively prime dominating roots of G which are 0 and -2.*

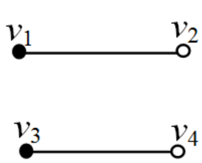

Figure 2. $G=2K_2$

Theorem 2.4. [8] *For a complete bipartite graph $K_{m,n}$, $\gamma_{rpd}(K_{m,n})=2$ if and only if $(m,n)=1$.*

Theorem 2.5. [8] *If $G_1\cong G_2$, then $\gamma_{rpd}(G_1)=\gamma_{rpd}(G_2)$.*

Theorem 2.6. [9] $\gamma_{rpd}(C_n^v)=\begin{cases}2 & \text{for } 3\le n\le 6\\ 3 & \text{for } n\ge 7\end{cases}$

Theorem 2.7. [10] $\gamma_{rpd}(K_{1,m}\cup K_n)=\begin{cases}2 & \text{if } (m,n-1)=1\\ m+1 & \text{if } (m,n-1)\ne 1\end{cases}$

Theorem 2.8. [10] $\gamma_{rpd}(B_{m,n})=\begin{cases}2 & \text{if } (m+1,n+1)=1\\ r+1 & \text{if } (m+1,n+1)\ne 1\end{cases}$, *where $r=min\{m,n\}$.*

Theorem 2.9. [10] $\gamma_{rpd}(K_m\cup K_n)=\begin{cases}2 & \text{if } (m-1,n-1)=1\\ 0 & \text{otherwise}\end{cases}$

Theorem 2.10. [8] $\gamma_{rpd}(P_n)=\begin{cases}2 & \text{if } 2\le n\le 5\\ 3 & \text{if } n=6,7\\ 0 & \text{otherwise}\end{cases}$

Theorem 2.11. [8] $\gamma_{rpd}\left(\overline{P}_n\right)=\begin{cases}2 & \text{if } n\ge 3\\ 0 & \text{otherwise}\end{cases}$

Theorem 2.12. [9] *For $n\ge 2, \gamma_{rpd}\left(\overline{K}_{m,n}^v\right)=2$, where $m\ne n$ and $m+n$ is odd.*

3. Main Results

Theorem 3.1. *If $G_1\cong G_2$, then $D_{rpd}(G_1,x)=D_{rpd}(G_2,x)$.*

Proof. Let $G_1\cong G_2$. Then by Theorem 2. 5, $\gamma_{rpd}(G_1)=\gamma_{rpd}(G_2)$. This implies that $D_{rpd}(G_1,x)=D_{rpd}(G_2,x)$. □

Theorem 3.2. $D_{rpd}(P_n,x)=\begin{cases}x^2 & \text{if } n=2\\ 3x^2+x^3 & \text{if } n=3\\ 3x^2+2x^3 & \text{if } n=4\\ 2x^2+3x^3 & \text{if } n=5\\ 2x^3 & \text{if } n=6\\ x^3 & \text{if } n=7\\ 0 & \text{otherwise}\end{cases}$

Proof. Let $v_1v_2...v_n$ be the path P_n. By Theorem 2.10, $\gamma_{rpd}(P_n)$ has value 2 for $2\le n\le 5$, 3 for $n=6,7$ and 0 for $n\ge 8$.

We consider the following three cases.

Case 1. $2\le n\le 5$

Clearly $\gamma_{rpd}(P_n)=2$. We consider the following four subcases.

Subcase 1.1. $n=2$

In this case there is only one relatively prime dominating set of size 2, namely $\{v_1, v_2\}$ and hence $D_{rpd}(P_2, x) = x^2$.

Subcase 1.2. $n = 3$

In this case there are three relatively prime dominating sets of size 2, namely $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ and only one relatively prime dominating set of size 3, namely $\{v_1, v_2, v_3\}$. This implies that $d_{rpd}(P_3, 2) = 3$ and $d_{rpd}(P_3, 3) = 1$ and hence $D_{rpd}(P_3, x) = 3x^2 + x^3$.

Subcase 1.3. $n = 4$

Here there are three relatively prime dominating sets of size 2, namely $\{v_1, v_3\}$, $\{v_1, v_4\}$ and $\{v_2, v_4\}$ and two relatively prime dominating sets of size 3, namely $\{v_1, v_2, v_4\}$ and $\{v_1, v_3, v_4\}$. This implies that $d_{rpd}(P_4, 2) = 3$ and $d_{rpd}(P_4, 3) = 2$ and hence $D_{rpd}(P_4, x) = 3x^2 + 2x^3$.

Subcase 1.4. $n = 5$

Here there are two relatively prime dominating sets of size 2, namely $\{v_1, v_4\}$ and $\{v_2, v_5\}$ and three relatively prime dominating sets of size 3, namely $\{v_1, v_2, v_5\}$, $\{v_1, v_3, v_5\}$ and $\{v_1, v_4, v_5\}$. This implies that $d_{rpd}(P_5, 2) = 2$ and $d_{rpd}(P_5, 3) = 3$. Clearly, $d_{rpd}(P_5, 4) = d_{rpd}(P_5, 5) = 0$, since any relatively prime dominating set of size greater than three must contain at least two vertices of same degree 2. Hence $D_{rpd}(P_4, x) = 2x^2 + 3x^3$.

Case 2. $n = 6, 7$

Clearly, $\gamma_{rpd}(P_n) = 3$. We consider the following two subcases.

Subcase 2.1. $n = 6$

In this case there are two relatively prime dominating sets of size 3, namely $\{v_1, v_3, v_6\}$ and $\{v_1, v_4, v_6\}$ and hence $d_{rpd}(P_6, 3) = 2$. Clearly, $d_{rpd}(P_6, 4) = d_{rpd}(P_6, 5) = d_{rpd}(P_6, 6) = 0$, since any relatively prime dominating set of size greater than three must contain at least two vertices of same degree 2. Hence $D_{rpd}(P_6, x) = 2x^3$.

Subcase 2.2. $n = 7$

In this case there is only one relatively prime dominating set of size 3, namely $\{v_1, v_4, v_7\}$ and hence $d_{rpd}(P_7, 3) = 1$. Clearly, $d_{rpd}(P_7, 4) = ... = d_{rpd}(P_7, 7) = 0$, since any relatively prime dominating set of size greater than three must contain at least two vertices of same degree 2. Hence $D_{rpd}(P_7, x) = x^3$.

Case 3. $n \geq 8$

In this case $\gamma_{rpd}(P_n) = 0$ and hence P_n has no relatively prime dominating set. This implies that $D_{rpd}(P_n, x) = 0$.

The theorem follows from cases 1, 2 and 3. □

Theorem 3.3. *$D_{rpd}(K_{1,n}, x) = x[(1+x)^n - 1]$.*

Proof. Let u be the centre and let $u_1, u_2, ..., u_n$ be the end vertices of $K_{1,n} = G$. Then $V(G) = \{u, u_i / 1 \leq i \leq n\}$ and $E(G) = \{uv_i / 1 \leq i \leq n\}$. Let $A = \{u\}$ and $B = \{u_1, u_2, ..., u_n\}$. We know that $\gamma(K_{1,n}) = 1$ and $\gamma_{rpd}(K_{1,n}) = 2$. To find the number of minimum relatively prime dominating sets each with size 2, we take the vertex u and one vertex from B. This can be done in $\binom{n}{1}$ ways and hence $d_{rpd}(G, 2) = \binom{n}{1}$. In a similar way we can prove that $d_{rpd}(\mathrm{G}, 3) = \binom{n}{2}$ and so on. Hence $D_{rpd}(K_{1,n}, x) = d_{rpd}(G, 2)x^2 + d_{rpd}(G, 3)x^3 + ... + d_{rpd}(G, n)x^n = \binom{n}{1}x^2 + \binom{n}{2}x^3 + ... + \binom{n}{n}x^{n+1} = x[(1+x)^n - 1]$. □

Theorem 3.4. *For $m, n \geq 2, D_{rpd}(K_{m,n}, x) = mnx^2$ if $(m, n) = 1$.*

Proof. Let (V_1, V_2) be the bipartition of the vertex set of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$ and $(m, n) = 1$. By Theorem 2. 4, $\gamma_{rpd}(K_{m,n}) = 2$. There are mn minimum relatively prime dominating sets of size 2. Any dominating set that contains more than two vertices also must contain at least two vertices of same degree and hence $d_{rpd}(G, 3) = d_{rpd}(G, 4) = ... = d_{rpd}(G, mn) = 0$. Therefore, $D_{rpd}(K_{m,n}, x) = mnx^2$. □

Theorem 3.5. *Let G be the bistar $B_{m,n}$.*

(i) If $(m+1, n+1) = 1, m = 1$ and $n \neq 1$, then $D_{rpd}(G, x) =$

$$2x^2 + \left[\binom{n}{0} + 2\binom{n}{1}\right]x^3 + \left[\binom{n}{1} + \binom{n}{2}\right]x^4 + ... + \left[\binom{n}{n-1} + \binom{n}{n}\right]x^{n+2} + \binom{n}{n}x^{n+3}.$$

(ii) If $(m+1, n+1) = 1, n = 1$ and $m \neq 1$, then $D_{rpd}(G, x) =$

$$2x^2 + \left[\binom{m}{0} + 2\binom{m}{1}\right]x^3 + \left[\binom{m}{1} + \binom{m}{2}\right]x^4 + ... + \left[\binom{m}{m-1} + \binom{m}{m}\right]x^{m+2} + \binom{m}{m}x^{m+3}.$$

(iii) If $(m+1, n+1) = 1$ and both m and n not equal to 1, then $D_{rpd}(B_{m,n}, x) = x^2[(1+x)^{m+n}]$

(iv) If $(m+1, n+1) \neq 1$, then $D_{rpd}(B_{m,n}, x) = x^{r+1}[(1+x)^s]$, where $r = min\{m, n\}, s = max\{m, n\}$ and $r + s = m + n$.

Proof. Let u and v be the vertices of P_2. Let $u_1, u_2, ..., u_m$ be the vertices attached with u and let $v_1, v_2, ..., v_n$ be the vertices attached with v. The resultant graph G is $B_{m,n}$ with $V(G) = \{u, v, u_i, v_j\}, 1 \leq i \leq m, 1 \leq j \leq n$ and $E(G) = \{uv, uu_i, vv_j / 1 \leq i \leq m, 1 \leq j \leq n\}$. Clearly, $d(u) = m+1, d(v) = n+1, d(u_i) = 1$ and $d(v_j) = 1, 1 \leq i \leq m, 1 \leq j \leq n$. Let $A = \{u, v\}, B = \{u_1, u_2, ..., u_m\}$ and $C = \{v_1, v_2, ..., v_n\}$.

Case 1. $(m+1, n+1) = 1$

Subcase 1.1. $m = 1$ and $n \neq 1$

Then $m+1$ is even. Since $(m+1, n+1) = 1, n+1$ is odd and hence n is even. Now $\{u, v\}$ and $\{u_1, v\}$ are the two relatively prime dominating set of order 2 and hence $d_{rpd}(G, 2) = 2$. To find the number of relatively prime dominating sets of size 3, we take either two vertices from A and one vertex from either B or C or by selecting the vertices v and u_1 and a vertex from C. This can be done in $n+1+n = 2n+1$ ways and hence $d_{rpd}(G, 3) = 2n+1$. Similarly,
$d_{rpd}(G,4) = n + \binom{n}{2}, d_{rpd}(G,5) = \binom{n}{2} + \binom{n}{3}, \ldots,$
$d_{rpd}(G, n+2) = \binom{n}{n-1} + \binom{n}{n}, d_{rpd}(G, n+3) = \binom{n}{n}.$

Hence $D_{rpd}(B_{m,n}, x) =$
$d_{rpd}(G,2)x^2 + d_{rpd}(G,3)x^3 + d_{rpd}(G,4)x^4 +$
$d_{rpd}(G,5)x^5 + \ldots + d_{rpd}(G, n+3)x^{n+3} =$
$2x^2 + \left[\binom{n}{0} + 2\binom{n}{1}\right]x^3 + \left[\binom{n}{1} + \binom{n}{2}\right]x^4 + \ldots$
$+ \left[\binom{n}{n-1} + \binom{n}{n}\right]x^{n+2} + \binom{n}{n}x^{n+3}.$

Subcase 1.2. $n = 1$ and $m \neq 1$

As in subcase 1. 1, $D_{rpd}(B_{m,n}, x) =$
$2x^2 + \left[\binom{m}{0} + 2\binom{m}{1}\right]x^3 + \left[\binom{m}{1} + \binom{m}{2}\right]x^4 + \ldots$
$+ \left[\binom{m}{m-1} + \binom{m}{m}\right]x^{m+2} + \binom{m}{m}x^{m+3}.$

Subcase 1.3. $m \neq 1$ and $n \neq 1$

By Theorem 2.8, $\gamma_{rpd}(B_{m,n}) = 2$. Clearly, there is only one minimal relatively prime dominating set of size 2, namely $\{u, v\}$. To find the number of relatively prime dominating sets each with size 3, we take two vertices from A and one vertex from either B or C. This can be done in $\binom{m+n}{1}$ ways and hence $d_{rpd}(G,3) = \binom{m+n}{1}$. By a similar way, we can prove that $d_{rpd}(G,4) = \binom{m+n}{2}$ and so on. Hence $D_{rpd}(B_{m,n}, x) =$
$d_{rpd}(G,2)x^2 + d_{rpd}(G,3)x^3 + d_{rpd}(G,4)x^4 + \ldots$
$+ d_{rpd}(G,n)x^n = x^2 + \binom{m+n}{1}x^3 + \binom{m+n}{2}x^4 + \ldots$
$+ \binom{m+n}{m+n-1}x^{m+n+1} + \binom{m+n}{m+n}x^{m+n+2} = x^2\left[(1+x)^{m+n}\right].$

Case 2. $(m+1, n+1) \neq 1$

By Theorem 2. 8, $\gamma_{rpd}(B_{m,n}) = r+1$, where $r = \min\{m, n\}$. Clearly, there is only one minimal relatively prime dominating set of size $r+1$. To find a relatively prime dominating set of size $r+2$, first we choose the minimal cardinality set from the sets B and C, the maximum degree vertex from u and v and a vertex from the maximum cardinality set from the set B and C. This can be done in $\binom{s}{1}$ ways and hence $d_{rpd}(G, r+2) = \binom{s}{1}$, where $r = \min\{m, n\}$ and $s = \max\{m, n\}$. By a similar way, we can prove that $d_{rpd}(G, r+3) = \binom{s}{2}$ and so on. Hence, $D_{rpd}(B_{m,n}, x) =$
$d_{rpd}(G, r+1)x^2 + d_{rpd}(G, r+2)x^3 + \ldots + d_{rpd}(G, n)x^n =$
$x^{r+1} + \binom{s}{1}x^{r+2} + \binom{s}{2}x^{r+3} + \ldots + \binom{s}{s}x^{r+s+1} =$
$x^{r+1}\left[(1+x)^s\right]$
where $r = \min\{m, n\}, s = \max\{m, n\}$ and $r+s = m+n$.

The theorem follows from case 1 and case 2. ☐

Theorem 3.6. *Let G be the spider graph $K_{1,n,n}$ with centre v. Then,*

$$D_{rpd}(G, x) = \begin{cases} n^2x^n + (n+1)x^{n+1}, & \text{if } d(v) \text{ is even} \\ n^2x^n + (n+1)x^{n+1} + nx^{n+2}, & \text{if } d(v) \text{ is odd.} \end{cases}$$

Proof. Let v be the center and let $v_1, v_2, \ldots, v_n$ be the end vertices of $K_{1,n}$. Let $u_1, u_2, \ldots, u_n$ be the vertices attached with $v_1, v_2, \ldots, v_n$, respectively. The resultant graph G is the spider graph with $V(G) = \{v, v_i, u_j / 1 \leq i, j \leq n\}$ and $E(G) = vv_i, v_iu_j / 1 \leq i, j \leq n$. Now $d_G(v) = n, d_G(v_i) = 2$ and $d_G(u_j) = 1, 1 \leq i, j \leq n$. Clearly, $\gamma_{rpd}(G) = \gamma(G) = n$. Let $A = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ and $B = \{u_1, u_2, \ldots, u_{n-1}, u_n\}$.

Case 1. $d(v)$ is even

A minimal relatively prime dominating set of size n is obtained by selecting a vertex from set A and $n-1$ vertices from set B. This can be done in $\binom{n}{1}\binom{n}{n-1} = \binom{n}{1}\binom{n}{1} = n^2$ ways. Therefore, $d_{rpd}(G, n) = n^2$. A relatively prime dominating set of size $n+1$ is obtained by selecting the set B and any one of the vertex from $A \cup \{v\}$. This can be done in $n+1$ ways. Therefore, $d_{rpd}(G, n+1) = n+1$. Clearly, $d_{rpd}(G, n+2) = d_{rpd}(G, n+3) = \ldots = d_{rpd}(G, 2n+1) = 0$, since any relatively prime dominating set of size more than $n+1$ vertices must contain at least two vertices of even degrees. Therefore, $D_{rpd}(G, x) = d_{rpd}(G, n)x^n + d_{rpd}(G, n+1)x^{n+1} = n^2x^n + (n+1)x^{n+1}$.

Case 2. $d(v)$ is odd

Clearly, $d_{rpd}(G, n) = n^2$. A relatively prime dominating set of size $n+1$ is obtained by selecting either the vertex v and the set B or the vertex v, a vertex v_i from A and the set $B - \{u_i\}, 1 \leq i \leq n$. Therefore, $d_{rpd}(G, n+1) = n+1$. A relatively prime dominating set of size $n+2$ is obtained by selecting a vertex from A, the set B and the vertex v. This can be done in $\binom{n}{1}$ ways. Therefore, $d_{rpd}(G, n+$

$2) = n$. Clearly, $d_{rpd}(G, n+3) = \ldots = 0$, since any relatively prime dominating set of size more than $n+2$ vertices must contain at least two vertices of same degree. Hence $D_{rpd}(G,x) = d_{rpd}(G,n)x^n + d_{rpd}(G,n+1)x^{n+1} + d_{rpd}(G,n+2)x^{n+2} = n^2x^n + (n+1)x^{n+1} + nx^{n+2}$.

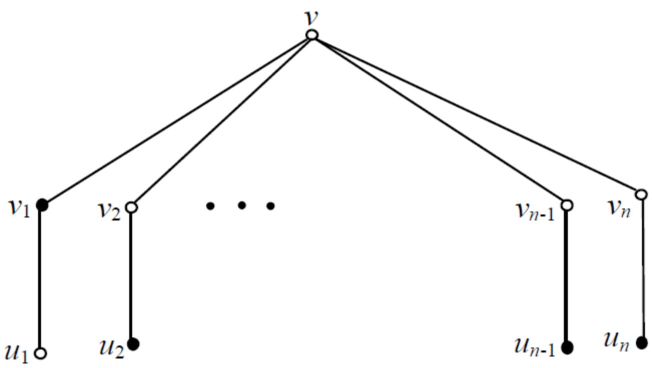

Figure 3. $K_{1,n,n}$

□

Theorem 3.7. *For the wounded spider graph G with centre v,*
$D_{rpd}(G,x) =$

$$\begin{cases} x^{s+1} + \binom{n-s}{1}x^{s+2} + \binom{n-s}{2}x^{s+3} + \ldots \\ + \binom{n-s}{n-s-2}x^{n-1} + \left(n+1\right)x^n \\ + \left(s+1\right)x^{n+1} & \text{if } d(v) \text{ is even} \\ \\ x^{s+1} + \binom{n}{1}x^{s+2} \\ + \left[\binom{s}{1}\binom{n-s}{1} + \binom{n-s}{2}\right]x^{s+3} + \ldots \\ + \left[\binom{n-s}{1} + s\binom{n-s}{2} + s^2 + 1\right]x^n \\ + \left[s + s\binom{n-s}{1} + s^2 + 1\right]x^{n+1} + sx^{n+2} & \text{if } d(v) \text{ is odd.} \end{cases}$$

where s is the number of sub dividing edges of a star and $s < n$.

Proof. Let v be the centre and let $v_1, v_2, \ldots, v_n$ be the end vertices of $K_{1,n}$. Attach $u_1, u_2, \ldots, u_s$ with $v_1, v_2, \ldots, v_s$, respectively where $s < n$. The resultant graph G is the wounded spider with $V(G) = \{v, v_i, u_j / 1 \le i \le n, 1 \le j \le s\}$ and $E(G) = \{vv_i, v_ju_j / 1 \le i \le n, 1 \le j \le s\}$. Now, $d_G(v) = n, d_G(v_i) = 2$, $1 \le i \le n$ and $d_G(u_j) = 1$, $1 \le j \le s$. Clearly, $\gamma_{rpd}(G) = \gamma(G) = s+1$. Let $A = \{v_1, v_2, \ldots, v_s, v_{s+1}, \ldots, v_n\}$, $B = \{v_1, v_2, \ldots, v_s\}$, $C = \{v_{s+1}, \ldots, v_n\}$, and $D = \{u_1, u_2, \ldots, u_s\}$.

Case 1. $d(v)$ is even

The only minimal relatively prime dominating set of size $s+1$ is obtained by selecting the vertex set D and the vertex v. Therefore, $d_{rpd}(G, s+1) = 1$. A relatively prime dominating set of size $s+2$ is obtained by selecting the vertex set D, a vertex from C and the vertex v. This can be done in $\binom{n-s}{1}$ ways and hence $d_{rpd}(G, s+2) = \binom{n-s}{1}$. A relatively prime dominating set of size $s+3$ is obtained by selecting the vertex set D, two vertices from C and the vertex v. This can be done in $\binom{n-s}{2}$ ways and hence $d_{rpd}(G, s+3) = \binom{n-s}{2}$. Similarly, $d_{rpd}(G, s+4) = \binom{n-s}{3}, \ldots, d_{rpd}(G, n-1) = \binom{n-s}{n-s-2} = \binom{n-s}{2}$. A relatively prime dominating set of size n is obtained by selecting either the vertex v, the vertex set D and $n-(s+1)$ vertices from C and this can be done in $\binom{n-s}{1} = n-s$ ways or the vertex set C, a vertex v_j from B and $s-1$ vertices from $D - \{u_j\}$ and this can be done in s ways or the vertex sets C and D. Therefore, $d_{rpd}(G,n) = n-s+s+1 = n+1$. A relatively prime dominating set of size $n+1$ is obtained by selecting the vertex sets C and D and the vertex v and the vertex sets C and D, any one of the vertex from $B \cup \{v\}$. This can be done in $s+1$ ways. Therefore, $d_{rpd}(G, n+1) = s+1$. Clearly, $d_{rpd}(G, n+2) = \ldots = d_{rpd}(G, n+s+1) = 0$, since any dominating set that contains more than $n+1$ vertices must contain at least two vertices of same degree. Hence, $D_{rpd}(G,x) = d_{rpd}(G,s+1)x^{s+1} + d_{rpd}(G,s+2)x^{s+2} + d_{rpd}(G,s+3)x^{s+3} + \ldots + d_{rpd}(G,n)x^n + d_{rpd}(G,n+1)x^{n+1} = x^{s+1} + \binom{n-s}{1}x^{s+2} + \binom{n-s}{2}x^{s+3} + \ldots + \binom{n-s}{n-s-2}x^{n-1} + (n+1)x^n + (s+1)x^{n+1}$.

Case 2. $d(v)$ is odd

The only minimal relatively prime dominating set of size $s+1$ is obtained by selecting the vertex set D and the vertex v. Therefore, $d_{rpd}(G, s+1) = 1$. A relatively prime dominating set of size $s+2$ is obtained by selecting the vertex set D, the vertex v and a vertex from A. This can be done in $\binom{n}{1}$ ways. This implies that $d_{rpd}(G, s+2) = \binom{n}{1}$. A relatively prime dominating set of size $s+3$ is obtained by selecting the vertex set D, the vertex v and either one vertex from B and one vertex from C or two vertices from C. This can be done in $\binom{s}{1}\binom{n-s}{1} + \binom{n-s}{2}$ ways and hence $d_{rpd}(G, s+3) = \binom{s}{1}\binom{n-s}{1} + \binom{n-s}{2}$. A relatively prime dominating set of size $s+4$ is obtained by selecting the vertex set D, the vertex v and either one vertex from B and two vertices from C or three vertices from C. This

can be done in $\binom{s}{1}\binom{n-s}{2}+\binom{n-s}{3}$ ways and hence $d_{rpd}(G,s+4)=\binom{s}{1}\binom{n-s}{2}+\binom{n-s}{3}$. Proceeding like this, we get $d_{rpd}(G,s+5)=\binom{s}{1}\binom{n-s}{3}+\binom{n-s}{4}$. A relatively prime dominating set of size n is obtained by selecting either the vertex v, the vertex set D and $n-(s+1)$ vertices from C which can be done in $\binom{n-s}{n-(s+1)}=\binom{n-s}{1}$ ways or the vertex v, the vertex set D, a vertex from B and $n-(s+2)$ vertices from C which can be done in $\binom{s}{1}\binom{n-s}{n-(s+2)}=s\binom{n-s}{2}$ ways or the vertex set C, a vertex from B and $s-1$ vertices from D and this can be done in $\binom{s}{1}\binom{s}{s-1}=s^2$ ways or the vertex sets C and D. Therefore, $d_{rpd}(G,n)=\binom{n-s}{1}+s\binom{n-s}{2}+s^2+1$. A relatively prime dominating set of size $n+1$ is obtained by selecting either the vertex sets C and D and the vertex v or the vertex sets C and D and one vertex from B which can be done in $\binom{s}{1}=s$ ways or the vertex v, the vertex set D, a vertex from B and $n-(s+1)$ vertices from C and which can be done in $\binom{s}{1}\binom{n-s}{n-(s+1)}=s\binom{n-s}{1}$ ways or the vertex v, the vertex set C, a vertex from B and $s-1$ vertices from D and this can be done in $\binom{s}{1}\binom{s}{s-1}=s^2$ ways. Therefore, $d_{rpd}(G,n+1)=s+s\binom{n-s}{1}+s^2+1$. A relatively prime dominating set of size $n+2$ is obtained by selecting the vertex sets C and D and one vertex from B and the vertex v. This can be done in s ways. Therefore, $d_{rpd}(G,n+2)=s$. Clearly, $d_{rpd}(G,n+3)=\ldots=d_{rpd}(G,n+s+1)=0$, since any dominating set that contains more than $n+2$ vertices must contain at least two vertices of same degree. Hence, $D_{rpd}(G,x)=$
$d_{rpd}(G,s+1)x^{s+1}+d_{rpd}(G,s+2)x^{s+2}+$
$d_{rpd}(G,s+3)x^{s+3}+\ldots+d_{rpd}(G,n)x^n+$
$d_{rpd}(G,n+1)x^{n+1}+d_{rpd}(G,n+2)x^{n+2}=$

$$x^{s+1}+\binom{n}{1}x^{s+2}+\left[\binom{s}{1}\binom{n-s}{1}+\binom{n-s}{2}\right]x^{s+3}+\ldots$$
$$+\left[\binom{n-s}{1}+s\binom{n-s}{2}+s^2+1\right]x^n$$
$$+\left[s+s\binom{n-s}{1}+s^2+1\right]x^{n+1}+sx^{n+2}.$$

The theorem follows from cases 1 and 2. □

Theorem 3.8. *Let $G=K_{1,m}\cup K_n$.*

(i) If $(m,n-1)=1$, then $D_{rpd}(G,x)=n\sum_{i=0}^{m}mC_ix^{i+2}$.

(ii) If $(m,n-1)\neq 1$, then $D_{rpd}(G,x)=nx^{m+1}$.

Proof. Let u be the central vertex and $v_i, 1\leq i\leq m$ be the end vertices of $K_{1,m}$. Let $u_1,u_2,\ldots,u_n$ be the vertices of K_n. Let $A=\{v_i/1\leq i\leq m\}$ and $B=\{u_j/1\leq j\leq n$.

Case 1. $(m,n-1)=1$

By Theorem 2. 7, $\gamma_{rpd}(K_{1,m}\cup K_n)=2$. There are n ways to choose a minimal relatively prime dominating set of size 2 by choosing the central vertex of $K_{1,n}$ and a vertex from K_n. Hence $d_{rpd}(G,2)=n$. A relatively prime dominating set of size 3 is obtained by selecting the central vertex u, a vertex from A and vertex from B. There are $\binom{m}{1}$ ways to choose a vertex from A and $\binom{n}{1}$ ways to choose a vertex from B. This implies that number of relatively prime dominating set each of size 3 is $\binom{n}{1}\binom{m}{1}$. Hence $d_{rpd}(G,3)=n\binom{m}{1}$. Since we can't choose two vertices from B, the number of relatively prime dominating set of size 4 is $n\binom{m}{2}$ and hence $d_{rpd}(G,4)=n\binom{m}{2}$. Continuing this, we see that $d_{rpd}(G,m+2)=n\binom{m}{m}$. Clearly, $d_{rpd}(G,m+3)=d_{rpd}(G,m+4)=\ldots=0$, since there do not exist relatively prime dominating sets of size $k\geq m+3$. Hence $D_{rpd}(G,x)=$
$d_{rpd}(G,2)x^2+d_{rpd}(G,3)x^3+d_{rpd}(G,4)x^4+\ldots$

$$+d_{rpd}(G,m+2)x^{m+2}=nx^2+n\binom{m}{1}x^3+n\binom{m}{2}x^4+\ldots$$
$$+n\binom{m}{m}x^{m+2}=n\sum_{i=0}^{m}mC_ix^{i+2}.$$

Case 2. $(m,n-1)\neq 1$

By Theorem 2. 7, $\gamma_{rpd}(G)=m+1$. A minimal relatively prime dominating set is obtained by choosing m vertices from A and a vertex from B. The m vertices from A can be chosen in one way and a vertex from B can be selected in n ways. Therefore, the number of ways of choosing minimal relatively prime dominating set of size $m+1$ is n. Clearly $d_{rpd}(G,m+2)=d_{rpd}(G,m+3)=\ldots=0$, since there do not exist relatively prime dominating sets of size $k\geq m+2$. Hence $D_{rpd}(G,x)=d_{rpd}(G,m+1)x^{m+1}=nx^{m+1}$.

The theorem follows from cases 1 and 2. □

Theorem 3.9. *Let $G=K_m\cup K_n$ where $m,n\geq 2$. Then, $D_{rpd}(G,x)=mnx^2$ if $(m-1,n-1)=1$.*

Proof. By Theorem 2. 9, $\gamma_{rpd}(K_m\cup K_n)=2$ if $(m-1,n-1)=1$. Any minimal relatively prime dominating set contains a vertex of K_m and a vertex from K_n. Clearly, there are mn minimum relatively prime dominating sets of size 2. Any dominating set contains more than two vertices must contain at least two vertices of same degree and hence there is no relatively prime dominating set exists of size 3 and so on. Therefore, $d_{rpd}(G,3) = d_{rpd}(G,4) = ... = 0$. Hence, $D_{rpd}((K_m \cup K_n),x) = mnx^2$. □

Result 3.10. *For $n \geq 2, D_{rpd}(\overline{K}_{m,n},x) = mnx^2$ if $(m-1,n-1) = 1$.*

Proof. Clearly, $\overline{K}_{m,n} = K_m \cup K_n$. By Theorem 3. 9, $D_{rpd}(\overline{K}_{m,n},x) = mnx^2$ if $(m-1,n-1) = 1$. □

Result 3.11. $D_{rpd}(\overline{P}_3,x) = 2x^2 + x^3$.

Proof. Clearly $\overline{P}_3 = K_1 \cup K_2$, where K_1 is u and v, w are the vertices of K_2. Hence there are only two relatively prime dominating sets of size 2, namely $\{u,v\}$ and $\{u,w\}$ and only one relatively prime dominating set of size 3, namely $\{u,v,w\}$. This implies that $d_{rpd}(G,2) = 2$ and $d_{rpd}(G,3) = 1$. Hence $D_{rpd}(\overline{P}_3,x) = 2x^2 + x^3$. □

Result 3.12. $D_{rpd}(\overline{P}_4,x) = 3x^2 + 2x^3$.

Proof. Clearly $\overline{P}_4 \cong P_4$ and hence by Theorem 3. 2, $D_{rpd}(\overline{P}_4,x) = 3x^2 + 2x^3$. □

Theorem 3.13. *For a path P_n where $n \geq 5, D_{rpd}(\overline{P}_n,x) = 2(n-3)x^2$.*

Proof. Let $v_1v_2...v_n$ be the path P_n. Let $A = \{v_1,v_n\}$ and $B = \{v_2,v_3,...,v_{n-1}\}$. By Theorem 2.11, $\gamma_{rpd}(\overline{P}_n) = 2$. A relatively prime dominating sets of size 2 is obtained by selecting v_1 from A and a vertex from $B - \{v_3\}$ or by selecting v_n from A and a vertex from $B - \{v_{n-2}\}$ this can be done in $2(n-3)$ ways and hence $d_{rpd}(G,2) = 2(n-3)$. Any dominating set that contains more than two vertices must contain at least two vertices of same degree. This implies that $d_{rpd}(G,3) = ... = 0$. Hence $D_{rpd}(\overline{P}_n,x) = d_{rpd}(G,2)x^2 = 2(n-3)x^2$. □

Theorem 3.14. *Let $G = K_n \circ K_1$. Then $D_{rpd}(G,x) = x^n(n+1+x)$.*

Proof. Let $u_1,u_2,...,u_n$ be the vertices of K_n and let v_i be the vertex of i^{th} copy of $K_1, 1 \leq i \leq n$. Join u_i with $v_i, 1 \leq i \leq n$. Let $A = \{u_1,u_2,...,u_n\}$ and $B = \{v_1,v_2,...,v_n\}$. Clearly, $\gamma_{rpd}(G) = n$. A relatively prime dominating set of size n is obtained by selecting either a vertex u_i from A and $n-1$ vertices from $B - \{v_i\}, 1 \leq i \leq n$ and this can be done in n ways or select all the vertices of B. Therefore, $d_{rpd}(G,n) = n+1$. A relatively prime dominating set of size $n+1$ is obtained by selecting a vertex from A and all the vertices of B. This can be done in n ways. Therefore, $d_{rpd}(G,n+1) = n$. Any relatively prime dominating set with more than $n+1$ vertices must contain at least two vertices of same degree. Therefore, $d_{rpd}(G,n+2) = d_{rpd}(G,n+3) = ... = 0$. Hence, $D_{rpd}(G,x) = d_{rpd}(G,n)x^n + d_{rpd}(G,n+1)x^{n+1} = (n+1)x^n + nx^{n+1} = x^n(n+1+x)$. □

Theorem 3.15. *For the star $K_{1,n}$ where $n \geq 2$ is even, $D_{rpd}(K^v_{1,n},x) = 2(n-1)x^2$, if v is an end vertex of $K_{1,n}$.*

Proof. Clearly, $K^v_{1,n} \cong K_{2,n-1}$. By Theorem 2. 4, $\gamma_{rpd}(K_{2,n-1}) = 2$ if and only if $(2,n-1) = 1$. This implies that $n-1 \neq 2r$. Therefore, $n \neq 2r+1$ and hence n is even. Clearly $(2,n-1) = 1$. By Theorems 3. 1 and 3. 4, $D_{rpd}(K^v_{1,n},x) = D_{rpd}(K_{2,n-1},x) = 2(n-1)x^2$. □

Theorem 3.16. *Let $G = K_{m,n} \circ K_1$. Then $D_{rpd}(G,x) = (m+n+1)x^{m+n} + (m+n)x^{m+n+1}$.*

Proof. Let (V_1,V_2) be the bipartition of the vertex set of $K_{m,n}$ with $|V_1| = m$ and $|v_2| = n$. By Theorem 2. 4, $\gamma_{rpd}(K_{m,n}) = 2$. Let $V_1 = \{u_1,u_2,...,u_m\}, V_2 = \{v_1,v_2,...,v_n\}$. Let w_i be the vertex of i^{th} copy of $K_1, 1 \leq i \leq m+n$. Join u_i with $w_i, 1 \leq i \leq m$ and v_j with $w_{m+j}, 1 \leq j \leq n$. The resultant graph G is $K_{m,n} \circ K_1$. Clearly, $\gamma_{rpd}(G) = m+n$. Let $C = \{w_1,w_2,...,w_{m+n}\}$. A relatively prime dominating set of size $m+n$ is obtained by selecting either a vertex u_i from V_1 and $m+n-1$ vertices from $C - \{w_i\}, 1 \leq i \leq m$ or a vertex v_j from V_2 and $m+n-1$ vertices from $C - \{w_{m+j}\}, 1 \leq j \leq n$ or select all vertices of V_2. This can be done in $m+n+1$ ways. Therefore, $d_{rpd}(G,m+n) = m+n+1$. A relatively prime dominating set of size $m+n+1$ is obtained by selecting either a vertex from V_1 and all the vertices of C or a vertex from V_2 and all vertices of C. This can be done in $m+n$ ways. Therefore, $d_{rpd}(G,m+n+1) = m+n$. Any relatively prime dominating set with more than $m+n+1$ vertices must contain at least two vertices of same degree. Therefore, $d_{rpd}(G,m+n+2) = d_{rpd}(G,m+n+3) = ... = 0$. Hence $D_{rpd}(G,x) = d_{rpd}(G,m+n)x^{m+n} + d_{rpd}(G,m+n+1)x^{m+n+1} = (m+n+1)x^{m+n} + (m+n)x^{m+n+1} = x^{m+n}(m+n+1+(m+n)x)$. □

Theorem 3.17. *Let G be a Helm graph and u be its centre. Then,*

$$D_{rpd}(G,x) = \begin{cases} nx^n + (n+1)x^{n+1} & \text{if } d(u) \text{ is even} \\ nx^n + (2n+1)x^{n+1} + nx^{n+2} & \text{if } d(u) \text{ is odd.} \end{cases}$$

Proof. Let u be the centre, $v_1,v_2,...,v_n$ be the vertices of the outer cycle and $u_1,u_2,...,u_n$ be the end vertices of H_n. Let $A = \{u\}, B = \{v_1,v_2,...,v_n\}$ and $C = \{u_1,u_2,...,u_n\}$.

Case 1. $d(u)$ is even

A minimal relatively prime dominating set of size n is obtained by selecting a vertex v_i from B and the set $C - \{u_i\}, 1 \leq i \leq n$. This can be done in n ways and hence $d_{rpd}(G,n) = n$. A relatively prime dominating set of size $n+1$ is obtained either by selecting the vertex u and the vertex set C which can be done in one way or by selecting the vertex set C and one vertex from B which can be done in n ways. Therefore, $d_{rpd}(G,n+1) = 1+n = n+1$. Clearly, $d_{rpd}(G,n+2) = d_{rpd}(G,n+3) = ... = 0$, since any relatively prime dominating set of size more than $n+1$ vertices must contain at least two vertices from B of same degree 4 or one vertex from B and the vertex u both have even degree. Hence $D_{rpd}(G,x) = d_{rpd}(G,n)x^n + d_{rpd}(G,n+1)x^{n+1} = nx^n + (n+1)x^{n+1}$.

Case 2. $d(u)$ is odd

As in case 1, we have $d_{rpd}(G,n) = n$. A relatively prime dominating set of size $n+1$ is obtained either by selecting the vertex u and the vertex set C which can be done in one way or by selecting the vertex u, a vertex v_i from B and the set $C - \{u_i\}, 1 \leq i \leq n$ which can be done in n ways or by selecting the vertex set C and one vertex from B which can be done in n ways. Therefore, $d_{rpd}(G,n+1) = 1+n+n = 2n+1$. A relatively prime dominating set of size $n+2$ is obtained by selecting the vertex set C, a vertex from B and the vertex u. This can be done in n ways and hence $d_{rpd}(G,n+2) = n$. Clearly, $d_{rpd}(G,n+3) = d_{rpd}(G,n+4) = ... = 0$, since any relatively prime dominating set of size more than $n+2$ vertices must contain at least two vertices from B of same degree 4. Hence $D_{rpd}(G,x) = d_{rpd}(G,n)x^n + d_{rpd}(G,n+1)x^{n+1} + d_{rpd}(G,n+2)x^{n+2} = nx^n + (2n+1)x^{n+1} + nx^{n+2}$. □

4. Conclusion

In this paper, we introduced the concept of relatively prime domination polynomial of a graph G. These polynomials establish the relationship between the relatively prime domination number and the relatively prime dominating sets in graphs. Further we compute the relatively prime domination polynomial of some standard graphs.

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