Existence of nonoscillatory solutions for fractional neutral functional differential equation

Velu Muthulakshmi[1]* and Subramani Pavithra[2]

Abstract
In this paper, we study the existence of nonoscillatory solutions for a kind of fractional neutral functional differential equation with Liouville fractional derivative of order $\alpha \geq 0$ on the half-axis. Some sufficient conditions are established using Krasnoselskii's and Schauder's fixed point theorems.

Keywords
Nonoscillatory solution, Liouville fractional derivative, neutral equation.

AMS Subject Classification
34C10, 34K11, 34A08.

[1,2] *Department of Mathematics, Periyar University, Salem, Tamil Nadu-636011, India.*
Corresponding author: [1] vmuthupu@gmail.com; [2]pavithrasubramani51@gmail.com

Contents

1. Introduction

Fractional calculus is a subject involving non-integer order derivatives and integrals. It has many applications in different fields such as physics, chemistry, mechanics, engineering etc. There have been many books on fractional differential equations, we refer the reader to [7, 8, 10] and the references cited therein. Many authors have studied some aspects of fractional differential equations, such as the existence and uniqueness of solutions to Cauchy type problems and the stability of solutions.

Oscillation theory is one of the most important qualitative properties of solutions of differential equations. Oscillatory behavior of ordinary differential equations can be found in the monographs [1–3] and the references cited therein. But, in recent years, oscillation theory of fractional differential equations have been developed rapidly, see [4–6, 9, 11, 12].

Recently, in [11], Zhou et al. discussed the existence of nonoscillatory solutions of the following fractional differential equation

$$D_t^\alpha [x(t) - cx(t-\tau)] + \sum_{i=1}^{m} P_i(t)x(t-\sigma_i) = 0, \ t \geq t_0,$$

where D_t^α denotes the Liouville fractional derivative of order $\alpha \in [1, +\infty)$ on the half-axis.

In [12], Zhou et al. studied the existence of nonoscillatory solutions of

$$D_t^\alpha [x(t) + cx(t-\tau)]' + \sum_{i=1}^{m} P_i(t)F_i(x(t-\sigma_i)) = 0, \ t \geq t_0,$$

where D_t^α denotes the Liouville fractional derivatives of order $\alpha \geq 0$ on the half-axis.

Motivated by the above literature, in this paper, we establish some sufficient conditions for the existence of nonoscillatory solutions of neutral functional differential equation of fractional order

$$(r(t)D_t^\alpha [x(t) + C(t)x(t-\tau_1)]')'$$
$$+ \sum_{i=1}^{m} P_i(t)F_i(x(t-\sigma_i)) = g(t), \ t \geq t_0, \qquad (1.1)$$

where D_-^α is Liouville fractional derivative of order $\alpha \geq 0$ on the half-axis, $C, P_i, g \in C([t_0, \infty), \mathbb{R})$, $r \in C([t_0, \infty), \mathbb{R}^+)$, $F_i \in C(\mathbb{R}, \mathbb{R})$, $\tau, \sigma_i \in \mathbb{R}^+$ for $i = 1, 2, ..., m$, $m \geq 1$ is an integer.

Let $l = \max_{1 \leq i \leq m} \{\tau, \sigma_i\}$. By a solution of Eq.(1.1), we mean a function $x \in C([t_1 - l, \infty), \mathbb{R})$ for some $t_1 \geq t_0$ such that $(r(t)D_-^\alpha[x(t) + C_1(t)x(t-\tau_1)]')'$ exists on $[t_1, \infty)$ and Eq.(1.1)

is satisfied for $t \geq t_1$.

A nontrivial solution of Eq.(1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1.1) is called oscillatory if all of its solutions are oscillatory.

In this paper, we obtain some sufficient conditions for the existence of nonoscillatory solution of Eq.(1.1) by using Krasnosel'skii's and Schauder's fixed point theorems.

2. Preliminaries

In this section, we provide some preliminary details which will be used in the subsequent section.

Definition 2.1. *[7] The Liouville fractional integral on the half-axis is defined by*

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s) ds,$$

where $t \in \mathbb{R}$ and $\alpha \in [0, \infty)$.

Definition 2.2. *[7] The Liouville fractional derivative on the half-axis is defined by*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (-1)^n \frac{d^n}{dt^n} \left(\int_t^\infty (s-t)^{n-\alpha-1} f(s) ds \right),$$

where $n = [\alpha] + 1$, $\alpha \in (0, \infty)$, $[\alpha]$ denotes the integer part of α and $t \in \mathbb{R}$.

In particular, if $\alpha = n \in \mathbb{N}$, then $D_t^n f(t) = (-1)^n f^{(n)}(t)$ and it has the property $D_t^\alpha (D_t^{-\alpha} f(t)) = f(t)$ for $\alpha > 0$.

Lemma 2.3. *[1] (Arzela-Ascoli theorem) A subset E in $C([a,b], \mathbb{R})$ with norm $\|f\| = \sup_{t \in [a,b]} |f(t)|$ is relatively compact if and only if it is uniformly bounded and equicontinuous on [a,b].*

Lemma 2.4. *[2] (Krasnosel'skii's fixed point theorem) Suppose X is a Banach space and Ω is a bounded, convex and closed subset of X. Let T_1, $T_2 : \Omega \to X$ satisfy the following conditions:*

(i) *$T_1 x + T_2 y \in \Omega$ for any $x, y \in \Omega$,*

(ii) *T_1 is a contraction mapping,*

(iii) *T_2 is completely continuous.*

Then $T_1 + T_2$ has a fixed point in Ω, i.e., the equation $T_1 x + T_2 x = x$ has a solution in Ω.

Lemma 2.5. *[2] (Schauder's fixed point theorem) Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $T : \Omega \to \Omega$ be a continuous mapping such that $T\Omega$ is a relatively compact subset of X. Then T has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Tx = x$.*

3. Main Results

In this section, some sufficient conditions are established for the existence of nonoscillatory solutions of Eq.(1.1) for the following cases:

$$-1 < c_1 \leq C(t) \leq 0, \ -\infty < C(t) \leq c_2 < -1,$$
$$0 \leq C(t) \leq c_3 < 1, \ 1 < c_4 \leq C(t) < \infty \text{ and } C(t) \equiv -1.$$

Theorem 3.1. *Assume that $-1 < c_1 \leq C(t) \leq 0$ and that*

$$\int_{t_0}^\infty \int_{t_0}^s \frac{s^\alpha}{r(s)} |P_i(u)| \, du \, ds < \infty, \ i = 1, 2, ..., m \qquad (3.1)$$

and

$$\int_{t_0}^\infty \int_{t_0}^s \frac{s^\alpha}{r(s)} |g(u)| \, du \, ds < \infty. \qquad (3.2)$$

Then Eq.(1.1) has a bounded nonoscillatory solution.

Proof. Because of conditions (3.1) and (3.2), we can choose a $t_1 > t_0$ sufficiently large such that

$$\frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$
$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| M_1 + |g(u)| \right) du \, ds \leq \frac{1+c_1}{3},$$
$$(3.3)$$

where $M_1 = \max_{\frac{2(1+c_1)}{3} \leq x \leq \frac{4}{3}} \{ |F_i(fx)| : 1 \leq i \leq m \}$.

Let $C([t_0, \infty), \mathbb{R})$ be the set of all continuous functions with the norm $\|x\| = \sup_{t \geq t_0} |x(t)| < \infty$. Then $C([t_0, \infty), \mathbb{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbb{R})$ as follows:

$$\Omega = \left\{ x = x(t) \in C([t_0, \infty), \mathbb{R}) : \frac{2(1+c_1)}{3} \leq x \leq \frac{4}{3}, \ t \geq t_0 \right\}.$$

Define two maps T_1 and $T_2 : \Omega \to C([t_0, \infty), \mathbb{R})$ as follows:

$$(T_1 x)(t) = \begin{cases} 1 + c_1 - C(t) x(t - \tau_1), & t \geq t_1, \\ (T_1 x)(t_1), & t_0 \leq t \leq t_1, \end{cases}$$

$$(T_2 x)(t) = \begin{cases} \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)} \\ \times \int_{t_1}^s \left(\sum_{i=1}^m P_i(u) F_i(x(u - \sigma_i)) \right. \\ \left. \quad - g(u) \right) du \, ds, & t \geq t_1, \\ (T_2 x)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

For every $x, y \in \Omega$ and $t \geq t_1$, we get

$$(T_1 x)(t) + (T_2 y)(t)$$

$$\leq 1 + c_1 - C(t)x(t-\tau) + \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x(u-\sigma_i))| + |g(u)| \right) du\, ds$$

$$\leq 1 + c_1 - \frac{4}{3}c_1 + \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)|M_1 + |g(u)| \right) du\, ds$$

$$\leq 1 + c_1 - \frac{4}{3}c_1 + \frac{1+c_1}{3} = \frac{4}{3}$$

and

$$(T_1 x)(t) + (T_2 y)(t)$$

$$\geq 1 + c_1 - C(t)x(t-\tau) - \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x(u-\sigma_i))| + |g(u)| \right) du\, ds$$

$$\geq 1 + c_1 - \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)|M_1 + |g(u)| \right) du\, ds$$

$$\geq 1 + c_1 - \frac{1+c_1}{3} = \frac{2(1+c_1)}{3}.$$

Thus, $(T_1 x)(t) + (T_2 y)(t) \in \Omega$ for any $x, y \in \Omega$. Now, for $x, y \in \Omega$ and $t \geq t_1$, we have

$$|(T_1 x)(t) - (T_1 y)(t)| \leq -C(t)|x(t-\tau) - y(t-\tau)|$$
$$\leq -c_1 \|x-y\|.$$

This implies with the supremum norm that

$$\|(T_1 x)(t) - (T_2 y)(t)\| \leq -c_1 \|x-y\|.$$

Since $0 < -c_1 < 1$, which shows that T_1 is a contraction mapping on Ω. Next, we have to show that T_2 is completely continuous. It is enough to show that T_2 is continuous and $T_2 \Omega$ is relatively compact.

First, we will show that T_2 is continuous. Let $x_k(t)$ be a sequence in Ω such that $x_k(t) \to x(t)$ as $k \to \infty$. Since Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$, we have

$$|(T_2 x_k)(t) - (T_2 x)(t)|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x_k(s-\sigma_i)) - F_i(x(s-\sigma_i))| \right) du\, ds$$

Since $|F_i(x_k(t-\sigma_i)) - F_i(x(t-\sigma_i))| \to 0$ as $k \to \infty$ for $i = 1, 2, ..., m$, by Lebesgue-dominated convergence theorem, we

get that
$\lim_{t \to \infty} \|(T_2 x_k)(t) - (T_2 x)(t)\| = 0$. Thus, T_2 is continuous.

In order to show that $T_2 \Omega$ is relatively compact, it is enough to show that the family of functions $\{T_2 x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniformly boundedness is obvious. For the equicontinuity, by Levitan's result [3], it is enough to show that, for any given $\varepsilon > 0$, $[t_1, \infty)$ can be decomposed into finite subintervals in such a way that all functions of the family have oscillation less than ε on each subinterval. By (3.1) and (3.2), for any $\varepsilon > 0$, choose $T \geq t_1$ sufficiently large such that

$$\frac{1}{\Gamma(\alpha+1)} \int_T^\infty \frac{(s-t)^\alpha}{r(s)} \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)|M_1 + |g(u)| \right) du\, ds < \frac{\varepsilon}{2}.$$

For $t_3 > t_2 \geq T$ and $x \in \Omega$, we obtain

$$|(T_2 x)(t_3) - (T_2 x)(t_2)|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \int_{t_3}^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x(s-\sigma_i))| + |g(u)| \right) du\, ds$$

$$+ \frac{1}{\Gamma(\alpha+1)} \int_{t_2}^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x(s-\sigma_i))| + |g(u)| \right) du\, ds$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \int_{t_3}^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)|M_1 + |g(u)| \right) du\, ds$$

$$+ \frac{1}{\Gamma(\alpha+1)} \int_{t_2}^\infty \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)|M_1 + |g(u)| \right) du\, ds$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $t_1 \leq t_2 < t_3 \leq T$ and $x \in \Omega$, we have

$$|(T_2 x)(t_3) - (T_1 x)(t_2)|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \int_{t_2}^{t_3} \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x(s-\sigma_i))| + |g(u)| \right) du\, ds$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \int_{t_2}^{t_3} \frac{(s-t)^\alpha}{r(s)}$$

$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)|M_1 + |g(u)| \right) du\, ds$$

$$\leq \quad \frac{1}{\Gamma(\alpha+1)} \max_{t_1 \leq s \leq T} \left\{ \frac{(s-t)^\alpha}{r(s)} \right.$$
$$\left. \times \int_{t_1}^{s} \left(\sum_{i=1}^{m} |P_i(u)|M_1 + |g(u)| \right) du \right\} (t_3 - t_2),$$

and hence there exists a $\delta > 0$ such that

$$|(T_2x)(t_3) - (T_2x)(t_2)| < \varepsilon, \text{ whenever } 0 < t_3 - t_2 < \delta.$$

Also, for $t_0 \leq t_2 < t_3 \leq t_1$ and $x \in \Omega$, we can easily see that

$$|(T_2x)(t_3) - (T_2x)(t_2)| = 0 < \varepsilon.$$

Therefore, $\{T_2x : x \in \Omega\}$ is equicontinuous on $[t_0, \infty)$ and hence $T_2\Omega$ is relatively compact. Thus, by Krasnosel'skii's fixed point theorem, $T_1x + T_2x = x$ has a solution in Ω, which is a bounded positive solution of Eq.(1.1). Hence the proof is complete.

Theorem 3.2. *Assume that* $-\infty < C(t) \equiv c_2 < -1$ *and that conditions* (3.1) *and* (3.2) *hold. Then Eq.*(1.1) *has a bounded nonoscillatory solution.*

Proof. Because of conditions (3.1) and (3.2), we can choose a $t_1 > t_0$,

$$t_1 + \tau \geq t_0 + \max\{\sigma_i\}$$

sufficiently large such that

$$\frac{1}{c_2\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^\alpha}{r(s)}$$
$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)|M_2 + |g(u)| \right) duds \leq \frac{-c_2+1}{2},$$
$$(3.4)$$

where $M_2 = \max_{\frac{-(c_2+1)}{2} \leq x \leq -2c_2} \{|F_i(x)| : 1 \leq i \leq m\}$.

Let $C([t_0, \infty), \mathbb{R})$ be the set of all continuous functions with the norm $\|x\| = \sup_{t \geq t_0} |x(t)| < \infty$. Then $C([t_0, \infty), \mathbb{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbb{R})$ as follows:

$$\Omega = \left\{ x = x(t) \in C([t_0, \infty), \mathbb{R}) : \begin{array}{c} -\frac{(c_2+1)}{2} \leq x \leq -2c_2, \\ t \geq t_0 \end{array} \right\}.$$

Define two maps T_1 and $T_2 : \Omega \to C([t_0, \infty), \mathbb{R})$ as follows:

$$(T_1x)(t) = \begin{cases} -c_2 - 1 - \frac{1}{C(t)}x(t+\tau), & t \geq t_1, \\ (T_1x)(t_1), & t_0 \leq t \leq t_1, \end{cases}$$

$$(T_2x)(t) = \begin{cases} \frac{1}{C(t)\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^\alpha}{r(s)} \\ \times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} P_i(u)F_i(x(u-\sigma_i)) \right. \\ \left. -g(u) \right) duds, & t \geq t_1, \\ (T_2x)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

For every $x, y \in \Omega$ and $t \geq t_1$, we get

$$(T_1x)(t) + (T_2y)(t)$$
$$\leq \quad -c_2 - 1 - \frac{1}{C(t)}x(t+\tau)$$
$$-\frac{1}{C(t)\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^\alpha}{r(s)}$$
$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)||F_i(x(u-\sigma_i))| + |g(u)| \right) duds$$
$$\leq \quad -c_2 - 1 + 2 - \frac{1}{c_2\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^\alpha}{r(s)}$$
$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)|M_2 + |g(u)| \right) duds$$
$$\leq \quad -c_2 + 1 - \frac{c_2+1}{2} = -2c_2$$

and

$$(T_1x)(t) + (T_2y)(t)$$
$$\geq \quad -c_2 - 1 - \frac{1}{C(t)}x(t+\tau)$$
$$+\frac{1}{C(t)\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^\alpha}{r(s)}$$
$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)||F_i(x(u-\sigma_i))| + |g(u)| \right) duds$$
$$\geq \quad -c_2 - 1 + \frac{1}{c_2\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^\alpha}{r(s)}$$
$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)|M_2 + |g(u)| \right) duds$$
$$\geq \quad -c_2 - 1 + \frac{c_2+1}{2}$$
$$= \quad -\frac{c_2+1}{2}.$$

Thus, $(T_1x)(t) + (T_2y)(t) \in \Omega$ for any $x, y \in \Omega$. Now, for $x, y \in \Omega$ and $t \geq t_1$, we have

$$|(T_1x)(t) - (T_1y)(t)| \leq -\frac{1}{C(t)} |x(t-\tau) - y(t-\tau)|$$
$$\leq -\frac{1}{c_2} \|x - y\|.$$

This implies with the supremum norm that

$$\|(T_1x)(t) - (T_2y)(t)\| \leq -\frac{1}{c_2} \|x - y\|.$$

Since $0 < -\frac{1}{c_2} < 1$, which shows that T_1 is a contraction mapping on Ω.

Proceeding similarly as in the proof of Theorem 3.1, we obtain that the mapping T_2 is completely continuous. Thus, by Krasnosel'skii's fixed point theorem, $T_1x + T_2x = x$ has a solution in Ω, which is a bounded positive solution of Eq.(1.1). Hence the proof is complete.

15

Theorem 3.3. *Assume that $0 \leq C(t) \leq c_3 < 1$ and that conditions* (3.1) *and* (3.2) *hold. Then Eq.*(1.1) *has a bounded nonoscillatory solution.*

Proof. Because of (3.1) and (3.2), we can choose a $t_1 > t_0$ sufficiently large such that

$$\frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$
$$\times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| M_3 + |g(u)| \right) duds \leq 1 - c_3,$$
(3.5)

where $M_3 = \max\limits_{2(1-c_3) \leq x \leq 4} \{|F_i(x)| : 1 \leq i \leq m\}$.

Let $C([t_0, \infty), \mathbb{R})$ be the set of all continuous functions with the norm $\|x\| = \sup\limits_{t \geq t_0} |x(t)| < \infty$. Then $C([t_0, \infty), \mathbb{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbb{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), \mathbb{R}) : 2(1-c_3) \leq x \leq 4, \ t \geq t_0\}.$$

Define two maps T_1 ans $T_2 : \Omega \to C([t_0, \infty), \mathbb{R})$ as follows:

$$(T_1 x)(t) = \begin{cases} 3 + c_3 - C(t)x(t - \tau_1), & t \geq t_1, \\ (T_1 x)(t_1), & t_0 \leq t \leq t_1, \end{cases}$$

$$(T_2 x)(t) = \begin{cases} \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)} \\ \times \int_{t_1}^s \left(\sum_{i=1}^m P_i(u) F_i(x(u - \sigma_i)) \\ \quad - g(u) \right) duds, & t \geq t_1, \\ (T_2 x)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

For every $x, y \in \Omega$ and $t \geq t_1$, we get

$$(T_1 x)(t) + (T_2 y)(t)$$
$$\leq 3 + c_3 - C(t)x(t - \tau)$$
$$\quad + \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$
$$\quad \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x(u - \sigma_i))| \\ \quad + |g(u)| \right) duds$$
$$\leq 3 + c_3 + \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$
$$\quad \times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| M_3 + |g(u)| \right) duds$$
$$\leq 3 + c_3 + 1 - c_3 = 4$$

and

$$(T_1 x)(t) + (T_2 y)(t)$$
$$\geq 3 + c_3 - C(t)x(t - \tau)$$
$$\quad - \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$
$$\quad \times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| |F_i(x(u - \sigma_i))| \\ \quad + |g(u)| \right) duds$$
$$\geq 3 + c_3 - 4c_3 - \frac{1}{\Gamma(\alpha+1)} \int_t^\infty \frac{(s-t)^\alpha}{r(s)}$$
$$\quad \times \int_{t_1}^s \left(\sum_{i=1}^m |P_i(u)| M_3 + |g(u)| \right) duds$$
$$\geq 3 + c_3 - 4c_3 - (1 - c_3)$$
$$= 2(1 - c_3).$$

Thus, $(T_1)x(t) + (T_2)y(t) \in \Omega$ for any $x, y \in \Omega$.

Proceeding similarly as in the proof of Theorem 3.1, we obtain that the mapping T_1 is contraction on Ω and T_2 is completely continuous. Thus, by Krasnosel'skii's fixed point theorem, $T_1 x + T_2 x = x$ has a solution in Ω, which is a bounded positive solution of Eq.(1.1). Hence the proof is complete.

Theorem 3.4. *Assume that $1 < c_4 \equiv C(t) < \infty$ and that conditions* (3.1) *and* (3.2) *hold. Then Eq.*(1.1) *has a bounded nonoscillatory solution.*

Proof. Because of conditions (3.1) and (3.2), we can choose a $t_1 > t_0$,

$$t_1 + \tau \geq t_0 + \max\{\sigma_i\}$$

sufficiently large such that

$$\frac{1}{c_4 \Gamma(\alpha+1)} \int_{t+\tau}^\infty \frac{(s-t-\tau)^\alpha}{r(s)}$$
$$\times \int_{t_1+\tau}^s \left(\sum_{i=1}^m |P_i(u)| M_4 + |g(u)| \right) duds \leq c_4 - 1,$$
(3.6)

where $M_4 = \max\limits_{2(c_4-1) \leq x \leq 4c_4} \{|F_i(x)| : 1 \leq i \leq m\}$.

Let $C([t_0, \infty), \mathbb{R})$ be the set of all continuous functions with the norm $\|x\| = \sup\limits_{t \geq t_0} |x(t)| < \infty$. Then $C([t_0, \infty), \mathbb{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbb{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), \mathbb{R}) : 2(c_4 - 1) \leq x \leq 4c_4, \ t \geq t_0\}.$$

Define two maps T_1 and $T_2 : \Omega \to C([t_0, \infty), \mathbb{R})$ as follows:

$$(T_1 x)(t) = \begin{cases} 3c_4 + 1 - \frac{1}{C(t)}x(t + \tau), & t \geq t_1, \\ (T_1 x)(t_1), & t_0 \leq t \leq t_1, \end{cases}$$

$$(T_2x)(t) = \begin{cases} \frac{1}{C(t)\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{\alpha}}{r(s)} \\ \times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} P_i(u)F_i(x(u-\sigma_i)) \right. \\ \left. -g(u) \right) duds, & t \geq t_1, \\ (T_2x)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

For every $x,y \in \Omega$ and $t \geq t_1$, we get

$$(T_1x)(t) + (T_2y)(t)$$

$$\leq 3c_4 + 1 - \frac{1}{C(t)}x(t+\tau)$$

$$+ \frac{1}{C(t)\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| |F_i(x(u-\sigma_i))| \right.$$

$$\left. + |g(u)| \right) duds$$

$$\leq 3c_4 + 1 + \frac{1}{c_4\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| M_4 + |g(u)| \right) duds$$

$$\leq 3c_4 + 1 + c_4 - 1 = 4c_4$$

and

$$(T_1x)(t) + (T_2y)(t)$$

$$\geq 3c_4 + 1 - \frac{1}{C(t)}x(t+\tau)$$

$$- \frac{1}{C(t)\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| |F_i(x(u-\sigma_i))| + |g(u)| \right) duds$$

$$\geq 3c_4 + 1 - 4 - \frac{1}{c_4\Gamma(\alpha+1)} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| M_4 + |g(u)| \right) duds$$

$$\geq 3c_4 + 1 - (c_4 - 1) = 2(c_4 - 1).$$

Thus, $(T_1)x(t) + (T_2)y(t) \in \Omega$ for any $x,y \in \Omega$.

Proceeding similarly as in the proof of Theorem 3.1, we obtain that the mapping T_1 is contraction on Ω and T_2 is completely continuous. Thus, by Krasnosel'skii's fixed point theorem, $T_1x + T_2x = x$ has a solution in Ω, which is a bounded positive solution of Eq.(1.1). Hence the proof is complete.

Theorem 3.5. *Assume that $C(t) \equiv -1$ and that conditions (3.1) and (3.2) hold. Then Eq.(1.1) has a bounded nonoscillatory solution.*

Proof. Because of conditions (3.1) and (3.2), we can choose a $t_1 > t_0$,

$$t_1 + \tau \geq t_0 + \max\{\sigma_i\}$$

sufficiently large such that

$$\frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \right) duds \leq 1, \tag{3.7}$$

where $M_5 = \max_{0 \leq x \leq 1} \{|F_i(x)| : 1 \leq i \leq m\}$.

Let $C([t_0,\infty),\mathbb{R})$ be the set of all continuous functions with the norm $\|x\| = \sup_{t \geq t_0} |x(t)| < \infty$. Then $C([t_0,\infty),\mathbb{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0,\infty),\mathbb{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0,\infty),\mathbb{R}) : 2 \leq x \leq 4, \ t \geq t_0\}.$$

Define a mapping $T : \Omega \to C([t_0,\infty),\mathbb{R})$ as follows:

$$(Tx)(t) = \begin{cases} 3 - \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)} \\ \times \int_{t_1+j\tau}^{s} \left(\sum_{i=1}^{m} P_i(u)F_i(x(u-\sigma_i)) \right. \\ \left. -g(u) \right) duds, & t \geq t_1, \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

For every $x \in \Omega$ and $t \geq t_1$, we get

$$(Tx)(t) \leq 3 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| |F_i(x(u-\sigma_i))| \right.$$

$$\left. + |g(u)| \right) duds$$

$$\leq 3 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \right) duds$$

$$\leq 4$$

and

$$(Tx)(t) \geq 3 - \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| |F_i(x(u-\sigma_i))| \right.$$

$$\left. + |g(u)| \right) duds$$

$$\geq 3 - \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \left(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \right) duds$$

$$\geq 2.$$

Thus, $T\Omega \subset \Omega$. Next, we have to show that T is completely continuous. It is enough to show that T is continuous and $T\Omega$ is relatively compact.

First, we will show that T is continuous. Let $x_k(t)$ be a sequence in Ω such that $x_k(t) \to x(t)$ as $k \to \infty$. Since Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$, we have

$$|(Tx_k)(t) - (Tx)(t)|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| \, |F_i(x_k(s-\sigma_i))|$$

$$-F_i(x(s-\sigma_i))| \Big) du \, ds$$

Since $|F_i(x_k(t-\sigma_i)) - F_i(x(t-\sigma_i))| \to 0$ as $k \to \infty$ for $i = 1, 2, ..., m$, by Lebesgue-dominated convergence theorem, we get that
$\lim_{t \to \infty} \|(Tx_k)(t) - (Tx)(t)\| = 0$. Thus, T is continuous.

In order to show that $\mathscr{B}\Omega$ is relatively compact, it is enough to show that the family of functions $Tx : x \in \Omega$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniformly boundedness is obvious. For the equicontinuity, by Levitan's result [3], it is enough to show that, for any given $\varepsilon > 0$, $[t_1, \infty)$ can be decomposed into finite subintervals in such a way that all functions of the family have oscillation less than ε on each subinterval. By (3.1) and (3.2), for any $\varepsilon > 0$, choose $K \geq t_1$ sufficiently large such that

$$\frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \Big) du \, ds < \frac{\varepsilon}{2}.$$

For $t_3 > t_2 \geq K$ and $x \in \Omega$, we obtain

$$|(Tx)(t_3) - (Tx)(t_2)|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t_3+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| \, |F_i(x(s-\sigma_i))| + |g(u)| \Big) du \, ds$$

$$+ \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t_2+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| \, |F_i(x(s-\sigma_i))| + |g(u)| \Big) du \, ds$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t_3+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \Big) du \, ds$$

$$+ \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t_2+j\tau}^{\infty} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \Big) du \, ds$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $t_1 \leq t_2 < t_3 \leq K$ and $x \in \Omega$, we choose a sufficiently large $J \in \mathbb{N}$ such that $t_1 + j\tau \geq K$ if $j \geq J$. For $x \in \Omega$

$$|(Tx)(t_3) - (Tx)(t_2)|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{\infty} \int_{t_2+j\tau}^{t_3+j\tau} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| \, |F_i(x(s-\sigma_i))| + |g(u)| \Big) du \, ds$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{J} \int_{t_2+j\tau}^{t_3+j\tau} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \Big) du \, ds$$

$$+ \frac{1}{\Gamma(\alpha+1)} \sum_{j=J+1}^{\infty} \int_{t_2+j\tau}^{t_3+j\tau} \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1+j\tau}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \Big) du \, ds$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \max_{t_1+j\tau \leq s \leq T+j\tau} \Big\{ \frac{(s-t-j\tau)^{\alpha}}{r(s)}$$

$$\times \int_{t_1}^{s} \Big(\sum_{i=1}^{m} |P_i(u)| M_5 + |g(u)| \Big) du \Big\} J(t_3 - t_2),$$

and hence there exists a $\delta > 0$ such that

$$|(Tx)(t_3) - (Tx)(t_2)| < \varepsilon, \text{ whenever } 0 < t_3 - t_2 < \delta.$$

Also, for $t_0 \leq t_2 < t_3 \leq t_1$ and $x \in \Omega$, we can easily see that

$$|(Tx)(t_3) - (Tx)(t_2)| = 0 < \varepsilon.$$

Therefore, $\{Tx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$ and hence $T\Omega$ is relatively compact. Thus, by Schauder's fixed point theorem, $Tx = x$ has a solution in Ω, which is a bounded positive solution of Eq.(1.1). Hence the proof is complete.

Acknowledgements

References

[1] R. P. Agarwal, S. R. Grace and D. O'Regan; *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Academic, Dordrecht, 2002.

[2] R. P. Agarwal, S. R. Grace and D. O'Regan; *Oscillation Theory for Second Order Dynamic Equations*, Taylor and Francis, London and New York, 2003.

[3] R. P. Agarwal, M. Bohner and W. T. Li; *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, Dekker, New York, 2004.

[4] D. Chen; Oscillation criteria of fractional differential equations, *Adv. Differ. Equ.*, 33(2012), 1–10.

[5] Q. Feng and F. Meng; Oscillation of solutions to nonlinear forced fractional differential equations, *Electr. J. Differ. Equ.*, 169(2013), 1–10.

[6] S. R. Grace, R. P. Agarwal, P. J. Y. Wong and A. Zafer; On the oscillation of fractional differential equations, *Fract. Calc. Appl. Anal.*, 15(2012), 222-231.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[8] K. S. Miller and B. Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[9] Y. Pan and R. Xu; Some new oscillation criteria for a class of nonlinear fractional differential equations, *Fractional Differ. Calc.*, 6(2016), 17–33.

[10] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[11] Y. Zhou, B. Ahmed and A. Alsaedi; Existence of nonoscillatory solutions for fractional neutral differential equations, *Appl. Math. Lett.*, 72(2017), 70–74.

[12] Y. Zhou, B. Ahmed and A. Alsaedi; Existence of nonoscillatory solutions for fractional functional differential equations, *B. Malays. Math. Sci. So.*, 2017(2017), 1–16.