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Multifunction between bitopologcal spaces via ideals

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Abstract
In this paper, we introduce and study some properties of pairwise upper (lower) semi-$\mathscr{I}$-continuous multifunctions.

Keywords
Ideal bitopological spaces, (i,j)-semi-$\mathscr{I}$-open sets, (i,j)-semi-$\mathscr{I}$-continuous multifunctions.

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Contents

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [8–11]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathaswamy, [13]. An ideal $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathscr{I}$ and $B \subset A$ implies $B \in \mathscr{I}$ and (ii) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$. Given a bitopological space (X,τ_1,τ_2) with an ideal $\mathscr{I}$ on X and if $\mathscr{P}(X)$ is the set of all subsets of X, a set operator $(.)_i^\star : \mathscr{P}(X) \to \mathscr{P}(X)$, called the local function [13] of A with respect to τ_i and $\mathscr{I}$, is defined as follows: for $A \subset X$, $A_i^\star(\tau_i,\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}$ for every $U \in \tau_i(x)\}$, where $\tau_i(x) = \{U \in \tau_i : x \in U\}$. A Kuratowski closure operator $i\mathrm{Cl}^\star(\cdot)$ is defined by $i\mathrm{Cl}^\star(A) = A \cup A_i^\star(\tau_i,\mathscr{I})$ when there is no chance of confusion, $A_i^\star(\mathscr{I})$ is denoted by $A_i^\star$. If $\mathscr{I}$ is an ideal on X, then $(X,\tau_1,\tau_2,\mathscr{I})$ is called an ideal bitopological space. In this paper, we introduce and study pairwise upper (lower) semi-$\mathscr{I}$-continuous multifunctions and obtain several characterizations of such functions.

2. Preliminaries

By a multifunction $F : X \to Y$, following [1], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. A subset S of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ is (i,j)-semi-$\mathscr{I}$-open [5] if $S \subset j\mathrm{Cl}^\star(i\mathrm{Int}(S))$. The complement of an (i,j)-semi-$\mathscr{I}$-closed set is said to be an (i,j)-semi-$\mathscr{I}$-open set. The (i,j)-semi-$\mathscr{I}$-closure and the (i,j)-semi-$\mathscr{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by (i,j)-$s\mathscr{I}\mathrm{Cl}(A)$ and (i,j)-$s\mathscr{I}\mathrm{Int}(A)$, respectively. The family of all (i,j)-semi-$\mathscr{I}$-open (resp. (i,j)-semi-$\mathscr{I}$-closed) sets of $(X,\tau_1,\tau_2,\mathscr{I})$ is denoted by (i,j)-$S\mathscr{I}O(X)$ (resp. (i,j)-$S\mathscr{I}C(X)$). The family of all (i,j)-semi-$\mathscr{I}$-open (resp. (i,j)-semi-$\mathscr{I}$-closed) sets of $(X,\tau_1,\tau_2,\mathscr{I})$ containing a point $x \in X$ is denoted by (i,j)-$S\mathscr{I}O(X,x)$ (resp. (i,j)-$S\mathscr{I}C(X,x)$). A subset A of a bitopolgical space (X,τ_1,τ_2) is said to be $\tau_1\tau_2$-closed [2] if $A = \tau_1$-$\mathrm{Cl}(\tau_2$-$\mathrm{Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is said to be $\tau_1\tau_2$-open. The intersection of all $\tau_1\tau_2$-closed sets containing A is called $\tau_1\tau_2$-closure of A and denoted by $\tau_1\tau_2$-$\mathrm{Cl}(A)$. The union of all $\tau_1\tau_2$-open sets contained in A is called $\tau_1\tau_2$-interior of A and denoted by $\tau_1\tau_2$-$\mathrm{Int}(A)$.

3. On (i,j)-semi-$\mathscr{I}$-continuous multifunctions

Definition 3.1. *A function $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is said to be*

1. *upper (i,j)-semi-$\mathscr{I}$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists $U \in (i,j)$-$S\mathscr{I}O(X,x)$ such that $F(U) \subset V$.*
2. *lower (i,j)-semi-$\mathscr{I}$-continuous if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ such that $F(x)\cap V \neq \emptyset$, there exists $U \in (i,j)$-$S\mathscr{I}O(X,x)$ such that $F(u)\cap V \neq \emptyset$ for every $u \in U$.*
3. *upper (lower) (i,j)-semi-$\mathscr{I}$-continuous if it has the property at each point of X.*

Theorem 3.2. *Let $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ be a multifuntion, the following statements are equivalent*

1. *F is upper (i,j)-semi-$\mathscr{I}$-continuous.*
2. *$F^+(V) \in (i,j)$-$S\mathscr{I}O(X)$ for any $\sigma_1\sigma_2$-open set V of Y.*
3. *$F^-(V)$ is (i,j)-semi-$\mathscr{I}$-closed in X for any $\sigma_1\sigma_2$-closed set V of Y.*
4. *$F((i,j)\text{-}s\mathscr{I}\mathrm{Cl}(A)) \subset \sigma_1\sigma_2\text{-}\mathrm{Cl}(F(A))$ for each subset A in X.*
5. *$(i,j)\text{-}s\mathscr{I}\mathrm{Cl}(F^+(B)) \subset F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))$ for each subset B in Y.*

Proof. The proof is clear. □

Theorem 3.3. *The following are equivalent for a multifunction $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$*

1. *F is lower (i,j)-semi-$\mathscr{I}$-continuous.*
2. *$F^-(V) \in (i,j)$-$S\mathscr{I}O(X)$ for any $\sigma_1\sigma_2$-open set V of Y.*
3. *$F^+(V)$ is (i,j)-semi-$\mathscr{I}$-closed in X for any $\sigma_1\sigma_2$-closed set V of Y.*
4. *$(i,j)\text{-}s\mathscr{I}\mathrm{Cl}(F^+(B)) \subset F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))$ for any $B \subset Y$.*
5. *$F((i,j)\text{-}s\mathscr{I}\mathrm{Cl}(A)) \subset \sigma_1\sigma_2\text{-}\mathrm{Cl}(F(A))$ for any $A \subset X$.*

Proof. Straightforward. □

For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Lemma 3.4. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following hold:*

1. *$G_F^+(A\times B) = A \cap F^+(B)$*
2. *$G_F^-(A\times B) = A \cap F^-(B)$ for any subsets $A \subset X$ and $B \subset Y$.*

Definition 3.5. *[2] A bitopological space (X,τ_1,τ_2) is said to be $\tau_1\tau_2$-compact if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.*

Theorem 3.6. *Let $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$. Then F is upper (i,j)-semi-$\mathscr{I}$-continuous if and only if $G_F : X \to X\times Y$ is upper (i,j)-semi-$\mathscr{I}$-continuous.*

Proof. Let $x \in X$ and W be any $\sigma_1\sigma_2$-open set of $X\times Y$ containing $G_F(x)$. For each $y \in F(x)$ there exist $\sigma_1\sigma_2$-open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x,y) \in U(y)\times V(x) \subset W$. The family of $\{V(y) : y \in F(x)\}$ is a $\sigma_1\sigma_2$-open cover of $F(x)$ and $F(x)$ is $\sigma_1\sigma_2$-compact. Then there exist a finite number of points, say $y_1,y_2...,y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \le i \le n\}$. Set $U = \cap\{U(y_i) : 1 \le i \le n\}$ and $V = \cup\{V(y_i) : 1 \le i \le n\}$. Then U and V are $\sigma_1\sigma_2$-open in X and Y respectively and $\{x\}\times F(x) \subset U\times V \subset W$. Since F is upper (i,j)-semi-$\mathscr{I}$-continuous, there exists $U_0 \in (i,j)$-$S\mathscr{I}O(X,x)$ such that $F(U_0) \subset V$. By Lemma 3.4, we have $U\cap U_0 \subset U\cap F^+(V) = G_F^+(U\times V) \subset G_F^+(W)$. Therefore, we obtain $U\cap U_0 \in (i,j)$-$S\mathscr{I}O(X,x)$ and $G_F(U\cap U_0) \subset W$. This shows that G_F is upper (i,j)-semi-$\mathscr{I}$-continuous. Conversely, suppose that $G_F : X \to X\times Y$ is upper (i,j)-semi-$\mathscr{I}$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $X\times V$ is $\sigma_1\sigma_2$-open in $X\times Y$ and $G_F(x) \subset X\times V$, there exists $U \in (i,j)$-$S\mathscr{I}O(X,x)$ such that $G_F(U) \subset X\times V$. By Lemma 3.4 we have $U \subset G_F^+(X\times V) = F^+(V)$ and $F(U) \subset V$. This shows that F is upper (i,j)-semi-$\mathscr{I}$-continuous. □

Theorem 3.7. *A multifunction $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is lower (i,j)-semi-$\mathscr{I}$-continuous if and only if $G_F : X \to X\times Y$ is lower (i,j)-semi-$\mathscr{I}$-continuous.*

Proof. Suppose that F is lower (i,j)-semi-$\mathscr{I}$-continuous. Let $x \in X$ and W be any $\sigma_1\sigma_2$-open set of $X\times Y$ such that $x \in G_F^-(W)$. Since $W\cap(\{x\}\times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x,y) \in W$ and hence $(x,y) \in U\times V \subset W$ for some $\sigma_1\sigma_2$-open sets $U \subset X$ and $V \subset Y$. Since $F(x)\cap V \neq \emptyset$, there exits $G \in (i,j)$-$S\mathscr{I}O(X,x)$ such that $G \subset F^-(V)$. By Lemma 3.4, we have $U\cap G \subset U\cap F^-(V) = G_F^-(U\times V) \subset G_F^-(W)$. Moreover, $x \in U\cap G \in (i,j)$-$S\mathscr{I}O(X)$ and hence G_F is a lower (i,j)-semi-$\mathscr{I}$-continuous. Conversely, suppose that G_F is a lower (i,j)-semi-$\mathscr{I}$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $x \in F^-(V)$. Then $X\times V$ is open in $X\times Y$ and $G_F(x)\cap(X\times V) = (\{x\}\times F(x))\cap(X\times V) = \{x\}\times(F(x)\cap V) \neq \emptyset$. Since G_F is a lower (i,j)-semi-$\mathscr{I}$-continuous, there exists $U \in (i,j)$-$S\mathscr{I}O(X,x)$ such that $U \subset G_F^-(X\times V)$. By Lemma 3.4, we obtain $U \subset F^-(V)$. This shows that F is lower (i,j)-semi-$\mathscr{I}$-continuous. □

Theorem 3.8. *If $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is an upper (i,j)-semi-$\mathscr{I}$-continuous multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$ and Y is a $\sigma_1\sigma_2$-Housdorff* *space, then the multigraph $G(F)$ of F is (i,j)-semi-$\mathscr{I}$-closed in $X \times Y$.*

Proof. Let $(x,y) \notin G(F)$, that is, $y \notin F(x)$. Since Y is $\sigma_1\sigma_2$-Housdorff, for each $z \in F(x)$, there exist disjoint $\sigma_1\sigma_2$-open sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(z)$. Then $\{U(z) : z \in F(x)\}$ is a $\sigma_1\sigma_2$-open cover of $F(x)$ and since $F(x)$ is $\sigma_1\sigma_2$-compact, there exits a finite number of points say, $s, z_1, z_2, ...z_n$ in $F(x)$ such that $F(x) \subset \cup\{U(z_i) : i = 1,2,...n\}$. Put $U = \cup\{U(z_i) : i = 1,2,...n\}$ and $V = \cap\{V(y_i) : i = 1,2,...n\}$. Then U and V are $\sigma_1\sigma_2$-open in Y such that $F(x) \subset U, y \in V$ and $U \cap V = \emptyset$. Since F is upper (i,j)-semi-$\mathscr{I}$-continuous, there exist $W \in (i,j)\text{-}S\mathscr{I}O(X,x)$ such that $F(W) \subset U$. Since V is $\sigma_1\sigma_2$-open, $W \times V \in (i,j)\text{-}S\mathscr{I}O(X \times Y)$ and $(x,y) \in W \times V \subset X \times Y \backslash G(F)$. Then $X \times Y \backslash G(F) = \bigcup_{(x,y) \in X \times Y \backslash G(F)} W \times V \in (i,j)\text{-}S\mathscr{I}O(X \times Y)$ and hence $G(F)$ is semi-$\mathscr{I}$-closed in $X \times Y$. □

For any two multifunctions $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to Y_2$, the following hold:

1. $(F_1 \times F_2)^+(A \times B) = F_1^+(A) \times F_2^+(B)$
2. $(F_1 \times F_2)^-(A \times B) = F_1^-(A) \times F_2^-(B)$ for any $A \subset X_1$ and $B \subset X_2$.

Lemma 3.9. *If $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to Y_2$ are upper (i,j)-semi-$\mathscr{I}$-continuous (resp. lower (i,j)-semi-$\mathscr{I}$-continuous) multifunctions, then $F_1 \times F_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is upper (i,j)-semi-$\mathscr{I}$-continuous (resp. lower (i,j)-semi-$\mathscr{I}$-continuous).*

Proof. We prove only the case for F upper (i,j)-semi-$\mathscr{I}$-continuous. Let $(x_1,x_2) \in X_1 \times X_2$ and W be any $\sigma_1\sigma_2$-open set of $Y_1 \times Y_2$ containing $F_1(x_1) \times F_2(x_2)$. There exist $\sigma_1\sigma_2$-open sets U and V of Y_1 and Y_2 respectively, such that $F_1(x_1) \times F_2(x_2) \subset U \times V \subset W$. Since F_1 and F_2 are upper (i,j)-semi-$\mathscr{I}$-continuous, there exist $U_0 \in (i,j)\text{-}S\mathscr{I}O(X_1.x_1)$ and $V_0 \in (i,j)\text{-}S\mathscr{I}O(X_2,x_2)$ such that $F_1(U_0) \subset U$ and $F_2(V_0) \subset V$. Then we have $U_0 \times V_0 \subset F_1^+(U) \times F_2^+(V) = (F_1 \times F_2)^+(U \times V) \subset (F_1 \times F_2)^+(W)$. Therefore, we obtain $U_0 \times V_0 \in (i,j)\text{-}S\mathscr{I}O(X_1 \times X_2,(x_1,x_2))$ and $(F_1 \times F_2)^+(U_0 \times V_0) \subset W$. This shows that $F_1 \times F_2$ is upper (i,j)-semi-$\mathscr{I}$-continuous. □

Definition 3.10. *[2] A collection $\mathscr{U}$ of subsets of a bitopological space (X,τ_1,τ_2) is said to be $\tau_1\tau_2$-locally finite if every $x \in X$ has a $\tau_1\tau_2$-neighborhood which intersects only finitely many elements of $\mathscr{U}$.*

Definition 3.11. *[2] A subset A of a bitopological space (X,σ_1,σ_2) is said to be*

1. *$\sigma_1\sigma_2$-paracompact if every cover of A by $\sigma_1\sigma_2$-open sets of X is refined by a cover of A which consists of $\sigma_1\sigma_2$-open set of X and is $\sigma_1\sigma_2$-locally finite in X.*
2. *$\sigma_1\sigma_2$-regular if for each $a \in A$ and each $\sigma_1\sigma_2$-open set U of X containing a, there exists a $\sigma_1\sigma_2$-open set G of X such that $a \in G \subset \sigma_1\sigma_2\text{-Cl}(G) \subset U$.*

Lemma 3.12. *[2] If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompacat set of a bitopological space (X,τ_1,τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set G of X such that $A \subset G \subset \tau_1\tau_2\text{-Cl}(G) \subset U$.*

Remark 3.13. *[2] For a multifunction $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, by $\mathrm{Cl}\,F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ we denote a multifunction defined as follows $\mathrm{Cl}\,F(x) = \tau_1\tau_2\text{-Cl}(F(x))$ for each $x \in X$.*

Lemma 3.14. *[2] If $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is a multifunction such that $F(x)$ is $\tau_1\tau_2$-paracompact $\tau_1\tau_2$-regular for each $x \in X$, then for each $\sigma_1\sigma_2$-open set V of Y, $G^+(V) = F^+(V)$, where G denotes $\mathrm{Cl}\,F$.*

Theorem 3.15. *Let $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ be a multifunction such that $F(x)$ is a $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then the following statements are equivalent.*

1. *F is upper (i,j)-semi-$\mathscr{I}$-continuous.*
2. *$(i,j)\text{-}s\mathscr{I}\,\mathrm{Cl}\,F$ is upper (i,j)-semi-$\mathscr{I}$-continuous.*
3. *$\mathrm{Cl}\,F$ is upper (i,j)-semi-$\mathscr{I}$-continuous.*

Proof. We put $G = (i,j)\text{-}s\mathscr{I}\,\mathrm{Cl}\,F$ or $\mathrm{Cl}\,F$ in the sequel. Suppose that F is upper (i,j)-semi-$\mathscr{I}$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$. By Lemma 3.14, we have $x \in G^+(V) = F^+(V)$ and hence there exists $U \in (i,j)\text{-}S\mathscr{I}O(X,x)$ such that $F(U) \subset V$. Since $F(u)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $u \in U$ by Lemma 3.12, there exists a $\sigma_1\sigma_2$-open set W such that $F(u) \subset W \subset \sigma_1\sigma_2\text{-Cl}(W) \subset V$; hence $G(u) \subset \sigma_1\sigma_2\text{-Cl}(W) \subset V$ for each $u \in U$. Therefore, we obtain $G(U) \subset V$. This shows that G is upper (i,j)-semi-$\mathscr{I}$-continuous. Conversely, suppose that G is upper (i,j)-semi-$\mathscr{I}$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ By Lemma 3.14, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$ there exist $U \in (i,j)\text{-}S\mathscr{I}O(X,x)$ such that $F(U) \subset V$. Therefore, we obtain $U \subset G^+(V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper (i,j)-semi-$\mathscr{I}$-continuous. □

Lemma 3.16. *[2] If $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is a multifunction then for each $\sigma_1\sigma_2$-open set V of Y. $G^-(V) = F^-(V)$ where G denotes $(i,j)\text{-}s\mathscr{I}\,\mathrm{Cl}F$ or $\mathrm{Cl}\,F$.*

Theorem 3.17. *The following are equivalent for a multifunction $F : (X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$:*

1. *F is lower (i,j)-semi-$\mathscr{I}$-continuous.*
2. *$(i,j)\text{-}s\mathscr{I}\,\mathrm{Cl}\,F$ is lower (i,j)-semi-$\mathscr{I}$-continuous.*
3. *$\mathrm{Cl}\,F$ is lower (i,j)-semi-$\mathscr{I}$-continuous.*

Proof. By using Lemma 3.16 this is shown similarly to that of Theorem 3.15. □

Definition 3.18. *The (i,j)-semi-$\mathscr{I}$-frontier of a subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$, denote by (i,j)-$s\mathscr{I}Fr(A)$, is defined by (i,j)-$s\mathscr{I}Fr(A) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(A) \cap (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(X\backslash A) = (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(A)\backslash(i,j)$-$s\mathscr{I}\,\mathrm{Int}(A)$.*

Theorem 3.19. *The set of all points of X at which a multifunction $F:(X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is not upper (lower) (i,j)-semi-$\mathscr{I}$-continuous is identical with the union of the semi-$\mathscr{I}$-frontier of the upper (lower) inverse images of open sets containing (meeting) $F(x)$.*

Proof. Suppose that $x \in X$ at which F is not upper (i,j)-semi-$\mathscr{I}$-continuous. Then there exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ such that $U \cap (X\backslash F^+(x)) \neq \emptyset$ for every $U \in (i,j)$-$S\mathscr{I}O(X,x)$. Then $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}(X\backslash F^+(V)) = X\backslash(i,j)$-$s\mathscr{I}\,\mathrm{Int}(F^+(V))$ and $x \in F^+(V)$. Hence we obtain $x \in (i,j)$-$s\mathscr{I}Fr(F^+(V))$. Conversely, suppose V is $\sigma_1\sigma_2$-open set of Y containing $F(x)$ such that $x \in (i,j)$-$s\mathscr{I}Fr(F^+(V))$. If F is upper (i,j)-semi-$\mathscr{I}$-continuous at x, there exists $U \in (i,j)$-$S\mathscr{I}O(X,x)$ such that $U \subset F^+(V)$. It is clear that $x \in (i,j)$-$s\mathscr{I}\,\mathrm{Int}(F^+(V))$. This is a contradiction and hence F is not (i,j)-upper semi-$\mathscr{I}$-continuous at x. □

In the following $(D,>)$ is directed set, (F_λ) is a net of multifunction $F_\lambda:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ for every $\lambda \in D$ and F is a multifunction from X into Y.

Definition 3.20. *Let $(F_\lambda)_{\lambda\in D}$ be a net of multifunctions from (X,τ_1,τ_2) to (Y,σ_1,σ_2). A multifunction $F^*:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is defined as for each $x \in X, F^*(x) = \{y \in Y :$ for each $\sigma_1\sigma_2$-open neighbourhood V of y and each $\mu \in D$, there exist $\lambda \in D$ such that $\lambda > \mu$ and $V \cap F_\lambda(x) \neq \emptyset\}$ is called the $\sigma_1\sigma_2$-topological limit of the net $(F_\lambda)_{\lambda\in D}$.*

Definition 3.21. *A net $(F_\lambda)_{\lambda\in D}$ is said to be equally upper (i,j)-semi-$\mathscr{I}$-continuous at $x_0 \in X$ if for every $\sigma_1\sigma_2$-open set V_λ containing $F_\lambda(x_0)$, there exists $U \in (i,j)$-$S\mathscr{I}O(X,x_0)$ such that $F_\lambda(U) \subset V_\lambda$ for all $\lambda \in D$.*

Theorem 3.22. *Let $(F_\lambda)_{\lambda\in D}$ be a net of multifunction from an ideal bitopological space $(X,\tau_1,\tau_2,\mathscr{I})$ into a $\sigma_1\sigma_2$-compact space (Y,σ_1,σ_2). If the following are satisfied.*

1. *$\cup\{F_\mu(x):\mu>\lambda\}$ is $\sigma_1\sigma_2$-closed in Y for each $\lambda \in D$ and each $x \in X$.*
2. *$(F_\lambda)_{\lambda\in D}$ is equally upper (i,j)-semi-$\mathscr{I}$-continuous on X, then F^* is upper (i,j)-semi-$\mathscr{I}$-continuous on X.*

Proof. From definition and (1) we have $F^*(x) = \cap(\{\cup F_\mu(x) : \mu > \lambda\} : \lambda \in D)$. Since the net $(\cup\{F_\mu(x):\mu>\lambda\})_{\lambda\in D}$ is a family of $\sigma_1\sigma_2$-closed sets having the finite intersection property and Y is $\sigma_1\sigma_2$-compact, $F^*(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let V be a proper $\sigma_1\sigma_2$-open subset of Y such that $F^*(x_0) \subset V$. Since $F^*(x_0) \cap (Y\backslash V) = \emptyset, F^*(x_0) \neq \emptyset$ and $(Y\backslash V) \neq \emptyset, \cap\{(\cup\{F_\mu(x_0):\mu>\lambda\}):\lambda\in D\} \cap (Y\backslash V) = \emptyset$ and hence $\cap\{(\cup\{F_\mu(x_0)\cap(Y\backslash V):\mu>\lambda\}):\lambda\in D\} = \emptyset$. Since Y is $\sigma_1\sigma_2$-compact and the family $\{(\cup\{F_\mu(x_0)\cap(Y\backslash V):\mu>\lambda\}):\lambda\in D\}$ is a family of $\sigma_1\sigma_2$-closed sets with the empty intersection, there exit $\lambda \in D$ such that $F_\mu(x_0) \cap (Y\backslash V) = \emptyset$ for each $\mu \in D$ with $\mu > \lambda$. Since the net $(F_\lambda)_{\lambda\in D}$ is equally upper (i,j)-semi-$\mathscr{I}$-continuous on X, there exists $U \in (i,j)$-$S\mathscr{I}O(X,x_0)$ such that $F_\mu(U) \subset V$ for each $\mu > \lambda$ that is, $F_\mu(x) \cap (Y\backslash V) = \emptyset$ for each $x \in U$. Then $\cup\{F_\mu(x)\cap(Y\backslash V) : \mu > \lambda\} = \emptyset$ and hence $\cap\{(\cup\{F_\mu(x)\cap(Y\backslash V):\mu>\lambda\}):\lambda \in D\} \cap (Y\backslash V) = \emptyset$. This implies that $F^*(U) \subset V$. If $V = Y$, then it is clear that for each $U \in (i,j)$-$S\mathscr{I}O(X,x_0)$ we have $F^*(U) \subset V$. Hence F^* is upper (i,j)-semi-$\mathscr{I}$-continuous at x_0. Since x_0 is arbitrary, the proof completes. □

Theorem 3.23. *If $F:(X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is upper (i,j)-semi-$\mathscr{I}$-continuous, point $\sigma_1\sigma_2$-closed and Y is $\sigma_1\sigma_2$-regular, then the graph $G(F)$ of F is (i,j)-semi-$\mathscr{I}$-closed set in the product space $X \times Y$.*

Proof. Suppose $(x,y) \notin G(F)$. Then we have $y \notin F(x)$. Since Y is $\sigma_1\sigma_2$-regular, there exist disjoint $\sigma_1\sigma_2$-open sets V_1 and V_2 of Y such that $y \in V_1$ and $F(x) \subset V_2$. Since F is upper (i,j)-semi-$\mathscr{I}$-continuous, $F^+(V_2)$ is (i,j)-semi-$\mathscr{I}$-closed set in X containing x. Therefore, we obtain $(x,y) \in U \times F^+(V_2) \subset X \times Y\backslash G(F)$ and so $G(F)$ is (i,j)-semi-$\mathscr{I}$-closed. □

Theorem 3.24. *If $F:(X,\tau_1,\tau_2,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is an upper (lower) (i,j)-semi-$\mathscr{I}$-continuous, point $\sigma_1\sigma_2$-closed and Y is $\sigma_1\sigma_2$-normal, then $A = \{(x_1,x_2) : F(x_1) = F(x_2)\}$ is (i,j)-semi-$\mathscr{I}$-closed in the product space $X \times Y$.*

Proof. Let $(x_1,x_2) \notin A$, then $F(x_1) \neq F(x_2)$. Since Y is $\sigma_1\sigma_2$-normal, there exist disjoint $\sigma_1\sigma_2$-open sets V_1 and V_2 of Y such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Since F is upper (i,j)-semi-$\mathscr{I}$-continuous, $F^+(V_1)$ and $F^+(V_2)$ are (i,j)-semi-$\mathscr{I}$-open sets and $x_1 \in F^+(V_1)$ and $x_2 \in F^+(V_2)$. Therefore $[F^+(V_1) \times F^+(V_2)] \cap A = \emptyset$. Since $(x_1,x_2) \in [F^+(V_1) \times F^+(V_2)]$ and $F^+(V_1) \times F^+(V_2)$ is (i,j)-semi-$\mathscr{I}$-open set in $X \times Y$, we obtain $(x_1,x_2) \in (i,j)$-$s\mathscr{I}\,\mathrm{Cl}A$. □

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