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Induced magic labeling of some graphs

K.B. Libeeshkumar[1]* and V. Anil Kumar[2]

Abstract

Let $G=(V,E)$ be a graph and let $(A,+)$ be an Abelian group with identity element0. Let $f:V\to A$ be a vertex labeling and $f^*:E\to A$ be the induced labeling of f, defined by $f^*(v_1v_2)=f(v_1)+f(v_2)$ for all $v_1v_2\in E$. Then f^* again induces a labeling say $f^{**}:V\to A$ defined by $f^{**}(v)=\sum\limits_{vv_1\in E} f^*(vv_1)$. A graph $G=(V,E)$ is said to be an Induced A-Magic Graph (IAMG) if there exists a non zero labeling $f:V\to A$ such that $f\equiv f^{**}$. The function f, so obtained is called an Induced A-Magic Labeling (IAML) of G and a graph which has no such Induced Magic Labeling is called a Non-induced magic graph. In this paper we discuss the existence of Induced Magic Labeling of some special graphs like P_n, C_n, K_n and $K_{m,n}$.

Keywords

Induced A-Magic Labeling of Graphs, Induced A-Magic graphs.

AMS Subject Classification

05C78, 05C25.

[1,2] *Department of Mathematics University of Calicut, Malappuram, Kerala-670007, India.*

*__Corresponding author__: [1] libeesh123@gmail.com; [2]anil@uoc.ac.in

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1. Introduction

This paper deals with only finite, un directed simple and connected graphs. We refer [3] for the phrasing and standard notations related to graph theory. A *graph* is a pair $G=(V,E)$, where V,E are the vertex set and edge set respectively. The *degree* of a vertex v in G is the number of edges incident at v and it is denoted as $deg(v)$. Let $(A,+)$ be an Abelian group with identity element0. Let $f:V\to A$ be a vertex labeling and $f^*:E\to A$ be the induced edge labeling of f, defined by $f^*(v_1v_2)=f(v_1)+f(v_2)$ for all $v_1v_2\in E$. Then f^* again induces a vertex labeling say $f^{**}:V\to A$ defined by $f^{**}(v)=\sum\limits_{vv_1\in E} f^*(vv_1)$. A graph $G=(V,E)$ is said to be an an Induced A-Magic Graph (IAMG) if there exists a non zero labeling $f:V\to A$ such that $f\equiv f^{**}$. The function f, so obtained is called an Induced A-Magic Labeling (IAML) of G and a graph which has no such Induced Magic Labeling is called a Non-induced magic graph. If an induced magic labeling f where $f(v)=k$ for all verex v in G,then f is called k-induced magic labeling of G and G,a k-induced magic graph.This paper discuss some special Induced magic graphs that belongs to the following sets:

(i) $\Gamma(A):=$ Set of all induced A-magic graphs.

(ii) $\Gamma(A,f):=$Set of all induced A-magic graphs with IAML f.

(iii) $\Gamma_k(A):=$ Set of all induced A-magic graphs with k-induced magic labeling.

2. Main Results

Lemma 2.1. *Let $G=(V,E)$ be a graph and f is an IAML of G. If $v_1\in V$ is a pendant vertex adjacent to $v\in V$, then $f(v_1)=0$.*

Proof. Let f be an IAML of a graph G and v_1 be a pendant vertex adjacent to v. Then $f^*(vv_1)=f(v)+f(v_1)$ and v_1 is a pendant vertex implies that $f^{**}(v_1)=f(v)+f(v_1)$. Also f is an induced magic labeling of G implies that $f(v_1)=f^{**}(v_1)=f(v)+f(v_1)$. Thus $f(v)=0$. □

Corollary 2.2. *If G has a pendant vertex, then $G\notin\Gamma_k(A)$ for any Abelian group A.*

Proof. Proof is indisputable from the lemma 2.1. □

Lemma 2.3. *Let f be an IAML of a graph G and $wuvz$ be a path in G with w and z are pendant vertices in G, then $f^*(uv) = 0$.*

Proof. Suppose f is an IAML of a graph $G = (V,E)$ and $wuvz$ is any path in G with w and z are pendant vertices. Then by the lemma 2.1, we have $f(u) = 0 = f(v)$. Hence $f^*(uv) = 0$. □

Theorem 2.4. *Let f be a vertex labeling of a graph G. Then f is an IAML of G, if and only if $[deg(u)-1]f(u)+\sum f(v) = 0$, for any vertex $u \in V(G)$, where the summation is taken over all the vertices v which are adjacent to u.*

Proof. Let f be an IAML of G and u be a vertex in G with $deg(u) = m$. Let $v_1, v_2, v_3, \dots, v_m$ be those vertices adjacent to u in G. Now f is an IAML if and only if $f(u) = f^{**}(u) = f^*(uv_1)+f^*(uv_2)+f^*(uv_3)+\cdots+f^*(uv_m) = mf(u)+f(v_1) + f(v_2)+f(v_3)+\cdots+f(v_m)$.
That is if and only if $(m-1)f(u)+\sum f(v) = 0$, where v is adjacent to u. □

Theorem 2.5. *$P_n \in \Gamma(A)$ if and only if n is a multiple of 3.*

Proof. Suppose $n = 3m$, for some integer m. Let P_n be the path with vertex set $V = \{v_1, v_2, v_3, \dots, v_{n-1}, v_n\}$. For any $a \neq 0$ in A, define $f : V \to A$ as :

$$f(v_i) = \begin{cases} a & \text{if} \quad i = 1,4,7,\cdots,3m-2 \\ 0 & \text{if} \quad i = 2,5,8,\cdots,3m-1 \\ a^{-1} & \text{if} \quad i = 3,6,9,\cdots,3m. \end{cases}$$

Then, f is an IAML of P_n. Conversely suppose n is not a multiple of 3, then $n = 3m+1$ or $n = 3m+2$ for some positive integer m. Let $f : V \to A$ be a vertex labeling function with $f \equiv f^{**}$.Then for $1 \leq k \leq n-3$ and any path $v_k v_{k+1} v_{k+2} v_{k+3}$ in P_n, we have $f(v_{k+1}) = f^{**}(v_{k+1})$ implies that $f(v_k)+f(v_{k+1})+f(v_{k+2}) = 0$. Also $f(v_{k+2}) = f^{**}(v_{k+2})$ implies that $f(v_{k+1})+f(v_{k+2})+f(v_{k+3}) = 0$.Therefore we should have $f(v_k) = f(v_{k+3})$. Let us deal with the following cases:

Case 1 : $n = 3m+1$

In this context, from the above discussion we have, $0 = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3m-1}) = f(v_{n-2})$ and $0 = f(v_{n-1}) = f(v_{n-4}) = \cdots = f(v_6) = f(v_3) = 0$. Thus $f(v_3) = 0$ and $f(v_1)+f(v_3) = 0$ implies that $f(v_1) = 0$, which again implies that $0 = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3m+1}) = f(v_n.)$ Hence$f \equiv 0$, Therefore f is not an IAML.

Case 2 : $n = 3m+2$

In this context from the above discussion we have, $0 = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3m+2}) = f(v_n)$ and $0 = f(v_{n-1}) = f(v_{n-4}) = \cdots = f(v_4) = f(v_1)$. Thus $f(v_1) = 0$ and $f(v_1)+f(v_3) = 0$ implies that $f(v_3) = 0$, which implies $0 = f(v_3) = f(v_6) = f(v_9) = \cdots = f(v_{3m}) = f(v_{n-2}.)$ Hence $f \equiv 0$. Therefore, f is not an IAML.

Hence if n is not a multiple of a 3, then $P_n \notin \Gamma(A)$ □

Theorem 2.6. *Let $\{v_1, v_2, v_3 \cdots, v_{n-1}, v_n = v_0\}$ be the vertex set of C_n. Then for any path $v_{k-1}v_k v_{(k+1) mod\ n}$, f is an IAML of C_n if and only if $f(v_{k-1})+f(v_k)+f(v_{(k+1) mod\ n}) = 0$, where $1 \leq k \leq n$. Moreover any IAML f of C_n satisfies $f(v_k) = f(v_{(k+3) mod\ n})$ for $1 \leq k \leq n$.*

Proof. For $k = 1,2,3,\cdots,n$, consider the path $v_{k-1}v_k$ $v_{(k+1) mod\ n}$ in C_n. Observe that f is an IAML of C_n if and only if $f(v_k) = f^{**}(v_k)$, which holds if and only if $f(v_{k-1})+f(v_k) + f(v_{(k+1) mod\ n}) = 0$.
Also for any $0 \leq k \leq n-1$, let $v_k v_{k+1} v_{[(k+2)\ mod\ n]} v_{[(k+3)\ mod\ n]}$, is a path in C_n, we have $f(v_k)+f(v_{k+1})+f(v_{(k+2)\ mod\ n}) = 0$ and $f(v_{k+1})+f(v_{(k+2)\ mod\ n})+f(v_{(k+3)\ mod\ n}) = 0$.
Thus $f(v_k) = f(v_{(k+3) mod\ n})$. □

Corollary 2.7. *$C_n \in \Gamma_k(A)$ if and only if $O(k) = 3$, where $O(k)$ denotes the order of k in A.*

Proof. Consider C_n with $V(C_n) = \{v_1, v_2, \cdots, v_{n-1}, v_n = v_0.\}$. Suppose $C_n \in \Gamma_k(A)$,that is there exist an IAML f of C_n with $f(v_i) = k$ for $i = 1,2,3,\dots,n$.Then by theorem 2.6 we have $3k = 0$ in A,which implies $O(k) = 3$.Conversely suppose $O(k) = 3$. Then consider the vertex label $f(v_i) = k$ for $i = 1,2,3,\cdots,n$. Since $f(v_i) = k$ for all i and $O(k) = 3$, we have, $f^*(v_i v_{i+1}) = 2k$ for all i, and which implies $f^{**}(v_i) = f^*(v_i v_{i+1})+f^*(v_{i-1}v_i) = 4k = k = f(v_i)$, for all i. Thus f is an IAML of C_n, that is $C_n \in \Gamma_k(A)$. Hence the proof. □

Corollary 2.8. *C_n has a non-constant IAML if and only if n is a multiple of 3.*

Proof. Consider C_n with vertex set $\{v_1, v_2, \dots, v_{n-1}, v_n = v_0.\}$. Suppose $n = 3k$, for some integer k. Let a,b,c be any three distinct elements in A, such that $a+b+c = 0$, then define $f : V(C_n) \to A$ as follows:

$$f(v_i) = \begin{cases} a & \text{if} \quad i = 1,4,7,\cdots,3k-2 \\ b & \text{if} \quad i = 2,5,8,\cdots,3k-1 \\ c & \text{if} \quad i = 3,6,9,\cdots,3k. \end{cases}$$

Then clearly f is a non constant IAML of C_n. Conversely assume that n is not a multiple of 3. Then either $n = 3k+1$ or $3k+2$ for some integer k. Let f be an IAML of C_n and $f(v_1) = w$.

Case 1: $n = 3k+1$

In this context, by the theorem 2.6 we have:
$w = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3k+1}) = f(v_n) = f(v_3) = f(v_6) = f(v_9) = \cdots = f(v_{3k}) = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3k-1})$.
Thus $f(v_i) = w$, for $i = 1,2,3,\cdots,n$.

Case 2: $n = 3k+2$

In this context, by the theorem 2.6 we have:
$w = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3k+1}) = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3k-1}) = f(v_{3k+2}) = f(v_n) =$

$f(v_0) = f(v_3) = f(v_6) = f(v_9) = \cdots f(v_{3k})$.
Thus in this case also $f(v_i) = w$, for $i = 1, 2, 3, \cdots, n$.

Thus in either case, we have $f(v_i) = w$ for $i = 1, 2, 3, \cdots, n$. Thus if $n \not\equiv 0(mod\ 3)$ then every IAML of C_n is a constant IAML of C_n. □

Theorem 2.9. *The complete graph $K_n \in \Gamma(A, f)$ if and only if $(n-3)f(v_1) = (n-3)f(v_2) = (n-3)f(v_3) = \cdots = (n-3)f(v_n) = -[f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n)]$ where $v_1, v_2, v_3 \ldots, v_n$ are the vertices of K_n.*

Proof. For $1 \le i, j \le n$, we have $f(v_i) = f^{**}(v_i)$ holds if and only if $f(v_1) + f(v_2) + f(v_3) + \cdots f(v_{i-1}) + (n-2)f(v_i) + f(v_{i+1}) + \cdots + f(v_n) = 0$, similarly the condition $f(v_j) = f^{**}(v_j)$ is equivalent to the condition $f(v_1) + f(v_2) + f(v_3) + \cdots f(v_{j-1}) + (n-2)f(v_j) + f(v_{j+1}) + \cdots + f(v_n) = 0$. Thus we have f is an IAML if and only if $(n-3)f(v_i) = (n-3)f(v_j) = -[f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n)]$, for $1 \le i, j \le n$. Hence the proof. □

Corollary 2.10. *$K_n \in \Gamma_k(A)$ if and only if $O(k)$ divides $2n-3$, where $O(k)$ denotes the order of k in A.*

Proof. Let K_n be the complete graph with vertex set $\{v_1, v_2, v_3 \cdots, v_n\}$. We have $K_n \in \Gamma_k(A)$, means there exist an IAML f with $f(v) = k$, for all $v \in V(K_n)$. Also by the theorem 2.9, we have f is an IAML of K_n if and only if $(n-3)f(v) = -[f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n)]$, for all $v \in V(K_n)$. Thus $K_n \in \Gamma_k(A)$ if and only if $(n-3)k = -nk$, that is if and only if $(2n-3)k = 0$, that is if and only if $O(k)$ divides $2n-3$ in A. Completes the proof. □

Theorem 2.11. *$K_{m,n} \in \Gamma_k(A)$ if and only if $O(k)$ divides $2m-1$ and $O(k)$ divides $2n-1$,where $O(k)$ denotes the order of k in A.*

Proof. Let $V(K_{m,n}) = \{v_1, v_2, v_3, \cdots, v_m, u_1, u_2, u_3, \cdots, u_n\}$ with each $(v_i u_j) \in E(K_{m,n})$, for $1 \le i \le m, 1 \le j \le n$. Suppose $K_{m,n} \in \Gamma_k(A)$, then we have there exist an IAML f with $f(v_i u_j) = k$, for $1 \le i \le m, 1 \le j \le n$.Now f is an IAML of $K_{m,n}$ implies $k = f(v_1) = f^{**}(v_1) = 2nk$, since $f^*(v_1 u_j) = 2k$ for $1 \le j \le n$, that is $(2n-1)k = 0$ in A, which implies $O(k)$ divides $2n-1$. similarly by considering the equation $f(u_1) = f^{**}(u_1)$ we get $k = f(u_1) = f^{**}(u_1) = 2mk$, that is $(2m-1)k = 0$ in A, which implies $O(k)$ divides $2m-1$. Conversely suppose that $O(k)$ divides $2m-1$ and $O(k)$ divides $2n-1$. Consider the vertex label $f(v_i) = k = f(u_j)$, for $v_i, u_j \in V(K_{m,n}), 1 \le i \le m, 1 \le j \le n$. Then $f^*(v_i, u_j) = 2k$ for $1 \le i \le m, 1 \le j \le n$. There for $i = 1, 2, 3, \cdots, m$, $f^{**}(v_i) = \Sigma_{j=1}^{n} f^*(v_i u_j) = 2nk = k$, since $O(k)$ divides $2n-1$. Thus we have $f^{**}(v_i) = f(v_i) = k$ for $i = 1, 2, 3, \cdots, m$. In a similar way, we have $f^{**}(u_j) = f(u_j) = k$ for $j = 1, 2, 3, \cdots, n$. Hence we have $f = f^{**}$, Thus we get $K_{m,n} \in \Gamma_k(A)$. This concludes the proof. □

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