Malaya Journal of Matematik 1(1)(2012) 62–66

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On some integral inequalities using Hadamard fractional integral

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Abstract

In this paper, using Hadamard fractional integral, we establish two main new result on fractional integral inequalities by considering the extended Chebyshev functional in case of synchronous function. The first result concerns with some inequalities using one fractional parameter and other with two parameter.

Keywords: Chebyshev functional, Hadamard fractional integral, Hadamard fractional derivative and fractional integral inequality.

2010 MSC: 26D10,26A33.

1 Introduction

In recent years, many authors have worked on fractional integral inequalities and its application which plays important role in classical differential and integral equations, see [3, 5, 6, 7, 8, 9, 10]. Dahmani gave the following fractional integral inequalities, using the Riemann-Liouville fractional integral for extended Chebyshev functional, see for instance [6].

Theorem 1.1. *Let f and g be two synchronous function on $[0,\infty[$ and let $r,p,q : [0,\infty[\to [0,\infty[$ for all $t>0, \alpha>0$ and then*

$$\begin{aligned}&2J^{\alpha}r(t)\left[J^{\alpha}p(t)J^{\alpha}(qfg)(t)+J^{\alpha}q(t)J^{\alpha}(pfg)(t)\right]+2J^{\alpha}p(t)J^{\alpha}q(t)J^{\alpha}(rfg)(t)\geq\\&J^{\alpha}r(t)\left[J^{\alpha}(pf)(t)J^{\alpha}(qg)(t)+J^{\alpha}(qf)(t)J^{\alpha}(pg)(t)\right]J^{\alpha}p(t)[J^{\alpha}(rf)(t)J^{\alpha}(qg)(t)+J^{\alpha}(qf)(t)J^{\alpha}(rg)(t)]\\&+J^{\alpha}q(t)\left[J^{\alpha}(rf)(t)J^{\alpha}(pg)(t)+J^{\alpha}(pf)(t)J^{\alpha}(rg)(t)\right].\end{aligned}\tag{1.1}$$

Theorem 1.2. *Let f and g be two synchronous function on $[0,\infty[$ and let $r,p,q : [0,\infty[\to [0,\infty[$ for all $t>0, \alpha>0$ $\beta>0$ then we have,*

$$\begin{aligned}&J^{\alpha}r(t)\left[J^{\alpha}q(t)J^{\beta}(pfg)(t)+2J^{\alpha}p(t)J^{\beta}(qfg)(t)+J^{\beta}q(t)J^{\alpha}(pfg)(t)\right]\\&+\left[J^{\alpha}p(t)J^{\beta}q(t)+J^{\beta}p(t)J^{\alpha}q(t)\right]J^{\alpha}(rfg)(t)\geq\\&J^{\alpha}r(t)\left[J^{\alpha}(pf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(pg)(t)\right]J^{\alpha}p(t)[J^{\alpha}(rf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(rg)(t)]\\&+J^{\alpha}q(t)\left[J^{\alpha}(rf)(t)J^{\beta}(pg)(t)+J^{\beta}(pf)(t)J^{\alpha}(rg)(t)\right].\end{aligned}\tag{1.2}$$

The main objective of this paper is to establish some inequalities for the extended Chebyshev functional given in [6], using Hadamard fractional integrals. The paper has been organized as follows. In Section 2, we define basic definitions and proposition related to Hadamard fractional derivatives and integrals. In Section 3, we give the main results.

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2 Preliminaries

Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville, Caputo derivative, see [3, 5, 6, 7, 8, 9, 10]. The necessary background details are given in the book A.A. Kilbas [1], and in book of S.G. Samko et al. [4], here we present some definitions of Hadamard derivative and integral as given in [2, p.159-171].

Definition 2.1. *The Hadamard fractional integral of order $\alpha \in R^+$ of function $f(x)$, for all $x > 1$ is defined as,*

$$ {}_HD_{1,x}^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_1^x \ln(\frac{x}{t})^{\alpha-1}f(t)\frac{dt}{t}, \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$.

Definition 2.2. *The Hadmard fractional derivative of order $\alpha \in [n-1, n)$, $n \in Z^+$, of function $f(x)$ is given as follows*

$$ {}_HD_{1,x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}(x\frac{d}{dx})^n\int_1^x \ln(\frac{x}{t})^{n-\alpha-1}f(t)\frac{dt}{t}. \tag{2.2}$$

From the above definitions, we can see obviously the difference between Hadamard fractional and Riemann-Liouville fractional derivative and integrals, which include two aspects. The kernel in the Hadamard integral has the form of $\ln(\frac{x}{t})$ instead of the form of $(x-t)$, which is involves both in the Riemann-Liouville and Caputo integral. The Hadamard derivative has the operator$(x\frac{d}{dx})^n$, whose construction is well suited to the case of the half-axis and is invariant relation to dilation [4, p.330], while the Riemann-Liouville derivative has the operator $(\frac{d}{dx})^n$.

We give some image formulas under the operator (2.1) and (2.2), which would be used in the derivation of our main result.

Proposition 2.1. *[2] If $0 < \alpha < 1$, the following relation hold:*

$$ {}_HD_{1,x}^{-\alpha}(\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln x)^{\beta+\alpha-1}, \tag{2.3}$$

$$ {}_HD_{1,x}^{\alpha}(\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln x)^{\beta-\alpha-1}, \tag{2.4}$$

respectively.

For the convenience of establishing the result, we give the semigroup property,

$$ ({}_HD_{1,x}^{-\alpha})({}_HD_{1,x}^{-\beta})f(x) = {}_HD_{1,x}^{-(\alpha+\beta)}f(x). \tag{2.5}$$

3 Main Results

In this section, we present and prove the main results.

Lemma 3.1. *Let f and g be two synchronous function on $[0,\infty[$. and $x, y : [0,\infty) \to [0,\infty)$. Then for all $t > 0$, $\alpha > 0$, we have,*

$$ \begin{aligned} &{}_HD_{1,t}^{-\alpha}x(t){}_HD_{1,t}^{-\alpha}(yfg)(t) + {}_HD_{1,t}^{-\alpha}y(t){}_HD_{1,t}^{-\alpha}(xfg)(t) \geq \\ &{}_HD_{1,t}^{-\alpha}(xf)(t){}_HD_{1,t}^{-\alpha}(yg)(t) + {}_HD_{1,t}^{-\alpha}(yf)(t){}_HD_{1,t}^{-\alpha}(xg)(t). \end{aligned} \tag{3.1}$$

Proof. Since f and g are synchronous on $[0,\infty[$ for all $\tau \geq 0$, $\rho \geq 0$, we have

$$ (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \tag{3.2}$$

From (3.2),

$$ f(\tau).g(\tau) + f(\rho).g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{3.3}$$

Now, multiplying both side of (3.3) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}x(\tau)}{\tau\Gamma(\alpha)}$, $\tau\in(0,t)$, $t>0$. Then the integrating resulting identity with respect to τ from 1 to t we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\tau).g(\tau)\frac{d\tau}{\tau}+\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\rho).g(\rho)\frac{d\tau}{\tau}\geq\\&\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\tau).g(\rho)\frac{d\tau}{\tau}+\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\rho).g(\tau)\frac{d\tau}{\tau}.\end{aligned}\tag{3.4}$$

Consequently,

$${}_HD_{1,t}^{-\alpha}(xfg)(t)+f(\rho).g(\rho)_HD_{1,t}^{-\alpha}(x)(t)\geq g(\rho)_HD_{1,t}^{-\alpha}(xf)(t)+f(\rho)_HD_{1,t}^{-\alpha}(xg)(t).\tag{3.5}$$

Multiplying both side of (3.5) by $\frac{(\ln(\frac{t}{\rho}))^{\alpha-1}y(\rho)}{\rho\Gamma(\alpha)}$, $\rho\in(0,t)$, $t>0$. Then integrating resulting identity with respect to ρ from 1 to t we obtain

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}(xfg)(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(x)(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)f(\rho)g(\rho)\frac{d\rho}{\rho}\\&\geq_H D_{1,t}^{-\alpha}(xf)(t)\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)g(\rho)\frac{d\rho}{\rho}+_H D_{1,t}^{-\alpha}(xg)(t)\frac{1}{\Gamma(\alpha)},\int_1^t \ln(\frac{t}{\rho})^{\alpha-1}y(\rho)f(\rho)\frac{d\rho}{\rho},\end{aligned}\tag{3.6}$$

and this ends the proof of inequality 3.1. □

Now, we gave our main result here.

Theorem 3.2. *Let f and g be two synchronous function on $[0,\infty[$, and $r,p,q:[0,\infty)\to[0,\infty)$. Then for all $t>0$, $\alpha>0$, we have*

$$\begin{aligned}&2_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_H D_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\right]+\\&2_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t)\right]+\\&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right]+\\&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(pg)(t)+_H D_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right]\end{aligned}\tag{3.7}$$

Proof. To prove above theorem, putting $x=p$, $y=q$, and using lemma 3.1, we get

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_H D_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t).\end{aligned}\tag{3.8}$$

Now, multiplying both side of (3.8) by ${}_HD_{1,t}^{-\alpha}r(t)$, we have

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_H D_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(pfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(pg)(t)\right],\end{aligned}\tag{3.9}$$

putting $x=r,y=q$, and using lemma 3.1, we get

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}r(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\&{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t),\end{aligned}\tag{3.10}$$

multiplying both side of (3.10) by ${}_HD_{1,t}^{-\alpha}p(t)$, we have

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}r(t)_HD_{1,t}^{-\alpha}(qfg)(t)+_{1,t}^{-\alpha}q(t)_HD_{1,t}^{-\alpha}(rfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(qg)(t)+_H D_{1,t}^{-\alpha}(qf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right].\end{aligned}\tag{3.11}$$

With the same arguments as before, we can write

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}r(t)_{1,t}^{-\alpha}(pfg)(t)+_H D_{1,t}^{-\alpha}p(t)_HD_{1,t}^{-\alpha}(rfg)(t)\right]\geq\\&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)_HD_{1,t}^{-\alpha}(pg)(t)+_H D_{1,t}^{-\alpha}(pf)(t)_HD_{1,t}^{-\alpha}(rg)(t)\right].\end{aligned}\tag{3.12}$$

Adding the inequalities (3.9), (3.11) and (3.12), we get required inequality (3.7). □

Lemma 3.3. *Let f and g be two synchronous function on $[0,\infty[$. and $x, y : [0,\infty[\to [0,\infty[$. Then for all $t > 0$, $\alpha > 0$, we have*

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}x(t)\,{}_HD^{-\beta}_{1,t}(yfg)(t) + {}_HD^{-\beta}_{1,t}y(t)\,{}_HD^{-\alpha}_{1,t}(xfg)(t) \geq \\ &{}_HD^{-\alpha}_{1,t}(xf)(t)\,{}_HD^{-\beta}_{1,t}(yg)(t) + {}_HD^{-\beta}_{1,t}(yf)(t)\,{}_HD^{-\alpha}_{1,t}(xg)(t).\end{aligned} \tag{3.13}$$

Proof. Now multiplying both side of (3.5) by $\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}$, $\rho \in (0,t)$, $t > 0$ we obtain:

$$\begin{aligned}&\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.{}_HD^{-\alpha}_{1,t}(xfg)(t) + \frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.f(\rho)g(\rho)\,{}_HD^{-\alpha}_{1,t}x(t) \geq \\ &\frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.g(\rho)\,{}_HD^{-\alpha}_{1,t}(xf)(t) + \frac{(\ln(\frac{t}{\rho}))^{\beta-1}y(\rho)}{\rho\Gamma(\beta)}.f(\rho)\,{}_HD^{-\alpha}_{1,t}(xg)(t),\end{aligned} \tag{3.14}$$

then integrating (3.14) over (1,t), we obtain

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}(xfg)(t)\frac{1}{\Gamma(\beta)}\int_1^t \ln(\frac{t}{\rho})^{\beta-1}y(\rho)\frac{d\rho}{\rho} + {}_HD^{-\alpha}_{1,t}(x)(t)\frac{1}{\Gamma(\beta)}\int_1^t \ln(\frac{t}{\rho})^{\beta-1}y(\rho)f(\rho)g(\rho)\frac{d\rho}{\rho} \\ &\geq {}_HD^{-\alpha}_{1,t}(xf)(t)\frac{1}{\Gamma(\beta)}\int_1^t \ln(\frac{t}{\rho})^{\beta-1}y(\rho)g(\rho)\frac{d\rho}{\rho} + {}_HD^{-\alpha}_{1,t}(xg)(t)\frac{1}{\Gamma(\beta)},\int_1^t \ln(\frac{t}{\rho})^{\beta-1}y(\rho)f(\rho)\frac{d\rho}{\rho},\end{aligned} \tag{3.15}$$

this ends the proof of inequality (3.13). □

Theorem 3.4. *Let f and g be two synchronous function on $[0,\infty[$, and $r, p, q : [0,\infty) \to [0,\infty)$. Then for all $t > 0$, $\alpha > 0$, we have*

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}r(t)\left[{}_HD^{-\alpha}_{1,t}q(t)\,{}_HD^{-\beta}_{1,t}(pfg)(t) + 2\,{}_HD^{-\alpha}_{1,t}p(t)\,{}_HD^{-\beta}_{1,t}(qfg)(t) + {}_HD^{-\beta}_{1,t}q(t)\,{}_HD^{-\alpha}_{1,t}(pfg)(t)\right] \\ &+\left[{}_HD^{-\alpha}_{1,t}p(t)\,{}_HD^{-\beta}_{1,t}q(t) + {}_HD^{-\beta}_{1,t}p(t)\,{}_HD^{-\alpha}_{1,t}q(t)\right]_H D^{-\alpha}_{1,t}(rfg)(t) \geq \\ &{}_HD^{-\alpha}_{1,t}r(t)\left[{}_HD^{-\alpha}_{1,t}(pf)(t)\,{}_HD^{-\beta}_{1,t}(qg)(t) + {}_HD^{-\beta}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(pg)(t)\right] + \\ &{}_HD^{-\alpha}_{1,t}p(t)\left[{}_HD^{-\alpha}_{1,t}(rf)(t)\,{}_HD^{-\beta}_{1,t}(qg)(t) + {}_HD^{-\beta}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(rg)(t)\right] + \\ &{}_HD^{-\alpha}_{1,t}q(t)\left[{}_HD^{-\alpha}_{1,t}(rf)(t)\,{}_HD^{-\beta}_{1,t}(pg)(t) + {}_HD^{-\beta}_{1,t}(pf)(t)\,{}_HD^{-\alpha}_{1,t}(rg)(t)\right].\end{aligned} \tag{3.16}$$

Proof. To prove above theorem, putting $x = p$, $y = q$, and using lemma 3.3 we get

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}p(t)\,{}_HD^{-\beta}_{1,t}(qfg)(t) + {}_HD^{-\beta}_{1,t}q(t)\,{}_HD^{-\alpha}_{1,t}(pfg)(t) \geq \\ &{}_HD^{-\alpha}_{1,t}(pf)(t)\,{}_HD^{-\beta}_{1,t}(qg)(t) + {}_HD^{-\beta}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(pg)(t).\end{aligned} \tag{3.17}$$

Now, multiplying both side of (3.16) by ${}_HD^{-\alpha}_{1,t}r(t)$, we have

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}r(t)\left[{}_HD^{-\alpha}_{1,t}p(t)\,{}_HD^{-\beta}_{1,t}(qfg)(t) + {}_HD^{-\beta}_{1,t}q(t)\,{}_HD^{-\alpha}_{1,t}(pfg)(t)\right] \geq \\ &{}_HD^{-\alpha}_{1,t}r(t)\left[{}_HD^{-\alpha}_{1,t}(pf)(t)\,{}_HD^{-\beta}_{1,t}(qg)(t) + {}_HD^{-\beta}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(pg)(t)\right],\end{aligned} \tag{3.18}$$

putting $x = r$, $y = q$, and using lemma 3.3, we get

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}r(t)\,{}_HD^{-\beta}_{1,t}(qfg)(t) +^{-\beta}_{1,t} q(t)\,{}_HD^{-\alpha}_{1,t}(rfg)(t) \geq \\ &{}_HD^{-\alpha}_{1,t}(rf)(t)\,{}_HD^{-\beta}_{1,t}(qg)(t) + {}_HD^{-\beta}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(rg)(t),\end{aligned} \tag{3.19}$$

multiplying both side of (3.19) by ${}_HD^{-\alpha}_{1,t}p(t)$, we have

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}p(t)\left[{}_HD^{-\alpha}_{1,t}r(t)\,{}_HD^{-\beta}_{1,t}(qfg)(t) + {}_HD^{-\beta}_{1,t}q(t)\,{}_HD^{-\alpha}_{1,t}(rfg)(t) \geq\right] \\ &{}_HD^{-\alpha}_{1,t}p(t)\left[{}_HD^{-\alpha}_{1,t}(rf)(t)\,{}_HD^{-\beta}_{1,t}(qg)(t) + {}_HD^{-\beta}_{1,t}(qf)(t)\,{}_HD^{-\alpha}_{1,t}(rg)(t)\right].\end{aligned} \tag{3.20}$$

With the same argument as before, we obtain

$$\begin{aligned}&{}_HD^{-\alpha}_{1,t}q(t)\left[{}_HD^{-\alpha}_{1,t}r(t)^{-\beta}_{1,t}(pfg)(t) + {}_HD^{-\beta}_{1,t}p(t)\,{}_HD^{-\alpha}_{1,t}(rfg)(t)\right] \geq \\ &{}_HD^{-\alpha}_{1,t}q(t)\left[{}_HD^{-\alpha}_{1,t}(rf)(t)\,{}_HD^{-\beta}_{1,t}(pg)(t) + {}_HD^{-\beta}_{1,t}(pf)(t)\,{}_HD^{-\alpha}_{1,t}(rg)(t)\right].\end{aligned} \tag{3.21}$$

Adding the inequalities (3.18), (3.20) and (3.21), we follows the inequality (3.16). □

Remark 3.1. *Applying theorem 3.4 for $\alpha = \beta$, we obtain Theorem 3.2.*

Remark 3.2. *If f, g, r, p and q satisfies the following condition,*

1. *The function f and g is asynchronous on $[0, \infty)$.*
2. *The function r,p,q are negative on $[0, \infty)$.*
3. *Two of the function r,p,q are positive and the third is negative on $[0, \infty)$.*

then the inequality 3.7 and 3.16 are reversed.

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Received: October 7, 2012; *Accepted*: October 25, 2012

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