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Continuous mappings on cubic topological spaces

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Abstract

In this paper we have introduced and investigated some continuous mappings on P - cubic topological spaces and R- cubic topological spaces and obtained interrelations between them. We have proved and analysed some basic properties and characterization of the newly defined continuous mappings.

Keywords

Cubic sets, P-cubic topology, R-cubic topology, P-cubic continuous mappings, R-cubic continuous mappings.

AMS Subject Classification

54A40.

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Contents

1. Introduction

Y.B.Jun [2] introduced the concept of cubic sets, using fuzzy sets [6] and interval valued fuzzy sets [7] in 2012. In 2016, Akhtar [1] has constructed topological structure on cubic set theory called cubic topological space and discussed about two types of cubic topological spaces such as P-cubic topological space and R-cubic topological space. In 2017, Mahmood, et al. [5] introduced cubic hesitant fuzzy set and defined internal (external)cubic hesitant fuzzy set, $P(R)$-union and $P(R)$-intersection of cubic hesitant fuzzy sets. In 2019, Loganayaki and Jayanthi [4] introduced interior and closure in P-cubic topological space and R-cubic topological space and discussed various types of open sets. The objective of our work is to introduce continuous mappings, α-continuous mappings, semi continuous mappings , pre- continuous mappings and β-continuous mappings in both P-cubic topological spaces and R-cubic topological spaces.

2. Preliminaries

In this section some preliminary definitions with references are given.

Definition 2.1. *[2] Let X be a non-empty set. Then $A = \{\langle x, \mu(x), \lambda(x)\rangle \,|x \in X\}$ structure is a cubic set in X in which μ is a an IVFS in X and λ is a fuzzy set in X. Simply a cubic set is denoted by $A = \langle \mu, \lambda\rangle$ and C^x denotes the collection of all cubic sets in X.*

(i) A cubic set $A = \langle \mu, \lambda\rangle$ in which $\mu(x) = 0$ and $\lambda(x) = 1$ (resp. $\mu(x) = 1$ and $\lambda(x) = 0$) $\forall x \in X$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$).

(ii) A cubic set $A = \langle \mu, \lambda\rangle$ in which $\mu(x) = 0$ and $\lambda(x) = 1$ (resp. $\mu(x) = 0$ and $\lambda(x) = 0$) $\forall x \in X$ is denoted by $\hat{0}$ (resp.$\hat{1}$).

Definition 2.2. *[2] Let $A = \langle \mu, \lambda\rangle$ and $B = \langle \beta, \eta\rangle$ be two cubic sets in X, Then we define:*

(a) Equal: $A = B \Leftrightarrow \mu = \beta$ and $\lambda = \eta$

(b) P-order: $A = B \subseteq_P \mu \subseteq \beta$ and $\lambda \leq \eta$

(c) R-order: $A = B \subseteq_R \mu \subseteq \beta$ and $\lambda \geq \eta$

Definition 2.3. *[2] The complement of a cubic set $A = \langle \mu, \lambda\rangle = \{\langle x, [\mu^-(x), \mu^+(x)], \lambda(x)\rangle \,|x \in X\}$ in X is defined to be $A^c = \langle \mu^c, 1-\lambda\rangle = \{\langle x, [1-\mu^+(x), 1-\mu^-(x)], 1-\lambda(x)\rangle \,|x \in X\}$. Obviously, $(A^c)^c = A, \ddot{0}^c = \ddot{1}, \ddot{1}^c = \ddot{c}, \hat{0}^c = \hat{1}$ and $\hat{1}^c = \hat{0}$*

Definition 2.4. *[2] For any cubic set $A_i = \{\langle x, \mu_i(x), \lambda_i(x)\rangle \,|x \in X\}$ where $i \in N$, we define*

(a) *P-Union*

$$\bigcup_{i\in N} {}_P A_i = \{\langle x, \cup_{i\in N}\mu_i(x), \vee_{i\in N}\lambda_i(x)\rangle \,|x\in X\}$$

(b) *R-Union*

$$\bigcup_{i\in N} {}_R A_i = \{\langle x, \cup_{i\in N}\mu_i(x), \wedge_{i\in N}\lambda_i(x)\rangle \,|x\in X\}$$

(c) *P-Intersection*

$$\bigcap_{i\in N} {}_P A_i = \{\langle x, \cap_{i\in N}\mu_i(x), \wedge_{i\in N}\lambda_i(x)\rangle \,|x\in X\}$$

(d) *R-Intersection*

$$\bigcap_{i\in N} {}_P A_i = \{\langle x, \cap_{i\in N}\mu_i(x), \vee_{i\in N}\lambda_i(x)\rangle \,|x\in X\}$$

Definition 2.5. *[1] A P-cubic topology $\mathscr{F}_P$ is the family of cubic sets in X which satisfies the following conditions:*

(i) $\hat{0}, \hat{1} \in \mathscr{F}_P$

(ii) *If $A_i \in \mathscr{F}_P$ then*

$$\bigcup_{i\in N} {}_P A_i \in \mathscr{F}_P$$

(iii) *If $A, B \in \mathscr{F}_P$ then*

$$A \cap_R B \in \mathscr{F}_P$$

The pair $(X, \mathscr{F}_P)$ is called the P-cubic topological space and any cubic set in $\mathscr{F}_P$ is known as R-cubic open set in X. The complement A^c of a P-cubic open set A in P-cubic topological space $(X, \mathscr{F}_P)$ is called a P-cubic closed set in X.

Definition 2.6. *[1] A R-cubic topology $\mathscr{F}_R$ is the family of cubic sets in X which satisfies the following conditions:*

(i) $\hat{0}, \hat{1}, \ddot{0}, \ddot{1} \in \mathscr{F}_R$

(ii) *If $A_i \in \mathscr{F}_R$ then*

$$\bigcup_{i\in N} {}_R A_i \in \mathscr{F}_R$$

(iii) *If $A, B \in \mathscr{F}_R$ then*

$$A \cap_R B \in \mathscr{F}_R$$

The pair $(X, \mathscr{F}_R)$ is called the R-cubic topological space and any cubic set in $\mathscr{F}_R$ is known as R-cubic open set in X. The complement A^c of a R-cubic open set A in R-cubic topological space $(X, \mathscr{F}_R)$ is called a R-cubic closed set in X. Throughout this paper $(X, \mathscr{F}_P)$ or X_P denotes the P-cubic topological space and $(X, \mathscr{F}_R)$ or X_R denotes the R-cubic topological space.

3. Continuous Mappings on P-Cubic Topological Spaces

In this section we have defined and analysed the basic properties and characterization of some continuous mappings like α continuous mappings, semi continuous mappings, pre-continuous mappings and β-continuous mappings in P-cubic topological spaces.

Definition 3.1. *Let $f_P : X \to Y$ be a mapping and let $A = (\mu, \lambda)$ be a cubic set in X. Then the image of A under f_P, denoted by $f_P(A) = (f_P(\mu), f_P(\lambda))$, is defined by*

$$[f_P(\mu)(y)]^- = \begin{cases} \sup_{f_P(x)=y}[\mu(x)]^-, & \text{if } f_P^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$[f_P(\mu)(y)]^+ = \begin{cases} \sup_{f_P(x)=y}[\mu(x)]^+, & \text{if } f_P^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$[f_P(\lambda)](y) = \begin{cases} \sup_{f_P^{-1}(y)}\{\lambda(x)\}, & \text{if } f_P^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

for all y in Y, where $f_P^{-1}(y) = \{x | f_P(x) = y\}$ Let $B = \langle \beta, \eta \rangle$ be an cubic set in Y. Then the inverse image of B under, f_P denoted by $f_P^{-1}(B) = (f_P^{-1}(\beta), f_P^{-1}(\eta))$, is defined by $[f_P^{-1}(\beta)(x)]^- = [\beta(f_P(x))]^-$, $[f_P^{-1}(\beta)(x)]^+ = [\beta(f_P(x))]^+$, $f_P^{-1}(\eta)(x) = \eta(f_P(x))$, for all $x \in X$.

Definition 3.2. *Let X_P and Y_P be any two P-cubic topological spaces. A mapping $f_P : X_P \to Y_P$ said to be a*

(i) *P-cubic continuous mapping if $f_P^{-1}(A)$ is a P-cubic open set in X_P for each P-cubic open set A in Y_P.*

(ii) *P-cubic semi continuous (resp. α-continuous, pre-continuous, β-continuous) mapping if $f_P^{-1}(A)$ is a P-cubic semi open $(resp.\alpha open, pre-open, \beta-open)$ set [4] in X_P for each P-cubic open set A in Y_P.*

Proposition 3.3. *The identity mapping $f_P : X_P \to X_P$ is a P-cubic continuous mapping.*

Proof. Straightforward. □

Proposition 3.4. *The composition of two P-cubic continuous mappings is again a P-cubic continuous mapping in general.*

Proof. Straightforward. □

Proposition 3.5. *Every P-cubic continuous mapping is a P-cubic semi continuous (resp. α-continuous, pre-continuous, β-continuous) mapping but the converses are not true in general.*

Proof. Since every P-cubic open set is a P-cubic semi open (resp. α-open, pre-open, β-pen) set,the result follows obviously. □

The following examples show that the converses of proposition 3.5 is not true.

Example 3.6. *Let $X \neq \phi, \mathbb{F}_P = \{\hat{0}, A_1, A_2, A_3, \hat{1}\}$ and $\mathbb{F}'_P = \{\hat{0}, B, \hat{1}\}$ be two P-cubic topologies on X where $A_1 = \langle[0.2,0.4],0.2\rangle, A_2 = \langle[0.3,0.4],0.3\rangle, A_3 = \langle[0.4,0.6],0.5\rangle$ and $B = \langle[0.2,0.5],0.25\rangle$. Define a mapping $f_p : (X, \mathbb{F}_P) \to (X, \mathbb{F}'_P)$ by $f_P(x) = x$, then f_P is a P-cubic α continuous mapping but not a P-cubic continuous mapping.*

Example 3.7. *Let $X \neq \phi, \mathbb{F}_P = \{\hat{0}, A_1, A_2, A_3, A_4, \hat{1}\}$ and $\mathbb{F}'_P = \{\hat{0}, C, \hat{1}\}$ be two P-cubic topologies on X where $A_1 = \langle[0.2,0.4],0.2\rangle, A_2 = \langle[0.3,0.4],0.3\rangle$, $A_3 = \langle[0.4,0.6],0.5\rangle, A_4 = \langle[0.5,0.7],0.6\rangle$ and $C = \langle[0.3,0.5],0.4\rangle$. Define a mapping $g_P : (X, \mathbb{F}_P) \to (X, \mathbb{F}'_P)$ by $g_P(x) = x$, then g_P is a P-cubic semi continuous mapping but not a P-cubic continuous mapping.*

Example 3.8. *Let $X \neq \phi, \mathbb{F}_P = \{\hat{0}, A_1, A_2, A_3, \hat{1}\}$ and $\mathbb{F}'_P = \{\hat{0}, D, \hat{1}\}$ be two P-cubic topologies on X where $A_1 = \langle[0.2,0.4],0.2\rangle, A_2 = \langle[0.3,0.4],0.3\rangle, A_3 = \langle[0.4,0.6],0.5\rangle$ and $D = \langle[0.1,0.5],0.3\rangle$. Define a mapping $h_p : (X, \mathbb{F}_P) \to (X, \mathbb{F}'_P)$ by $h_P(x) = x$, then is a P-cubic pre-continuous mapping and a P-cubic β-continuous mapping, but not a P-cubic continuous mapping.*

Theorem 3.9. *A mapping $f_P : X_P \to Y_P$ is a P-cubic α-continuous mapping if and only if it is both a P-cubic semi continuous mapping and P-cubic pre-continuous mapping.*

Proof. Let f_P be both a P-cubic semi continuous mapping and a P-cubic pre-continuous mapping. Let A be a P-cubic open set in Y_P, then by hypothesis f_P^{-1} is a P-cubic semi open set and P-cubic pre-open set. Hence by proposition 3.25[6], f_P^{-1} is a P-cubic α open set and hence it is a P-cubic α-continuous mapping . The converse is immediate. □

Theorem 3.10. *Let $f_P : X_P \to Y_P$ be a mapping. Then*

(i) $f_P^{-1}(B^c) = [f_P^{-1}(B)]^c$ for all cubic sets B in Y.

(ii) $[f_P(A)]^c \subseteq_P f_p(A^c)$ for all cubic sets A in X.

(iii) $B_1 \subset_P B_2$ implies $f_P^{-1}(B_1) \subset_P f_P^{-1}(B_2)$, where B_1 and B_2 are cubic sets in Y.

(iv) $A_1 \subset_P A_2$ implies $f_P(A_1) \subset_P f_P(A_2)$, where A_1 and A_2 are cubic sets in Y.

(v) $f_P(f_P^{-1}(B)) \subset_P B$ the equality holds if f_P is surjective, for all cubic sets B in Y.

(vi) $A \subset_P f_P^{-1}(f_P(A))$ the equality holds if f_P is injective, for all cubic sets A in X.

(vii)

$$f_P^{-1}(\bigcup_{i\in\Lambda} {}_P B_i) = \bigcup_{i\in\Lambda} {}_P f_P^{-1}(B_i)$$

for all cubic sets B_i in Y.

(viii)

$$f_P^{-1}(\bigcap_{i\in\Lambda} {}_P B_i) = \bigcap_{i\in\Lambda} {}_P f_P^{-1}(B_i)$$

for all cubic sets B_i in Y.

Proof. (i) Let $B = \langle \beta, \eta \rangle$ be a cubic set in Y. Then

$$\begin{aligned} f_P^{-1}(B^c)(x) &= \langle x, f_P^{-1}(\beta^c)(x), f_P^{-1}(\eta^c)(x)\rangle \\ &= \langle x, \beta^c(f_P)(x), \eta^c(f_P)(x)\rangle \\ &= \langle x, 1-\beta(f_P)(x), 1-\eta(f_P)(x)\rangle \\ &= [f_P^{-1}(B)]^c \end{aligned}$$

(ii) $A = \langle \mu, \lambda \rangle$ be a cubic set in X and $f_P^{-1}(y) \neq \phi$. Then $A^c = \{\langle x, [1-\mu^+(x), 1-\mu^-(x)], 1-\lambda(x)\rangle\}$, we have

$$\begin{aligned} &[f_P(A)]^c(y) \\ &= 1 - f_P(A)(y) \\ &= 1 - \langle y, [\sup\ (\mu^-(x)), \sup\ (\mu^+(x))], \\ &\qquad \sup(\lambda(x))\rangle \\ &= \langle 1-y, [1-\sup(\mu^-(x)), 1-\sup(\mu^+(x))], \\ &\qquad 1-\sup(\lambda(x))\rangle \end{aligned}$$

$$[f_P(A)]^c(y) = \langle y, [\sup\{1-\mu^-(x)\}, \sup\{1-\mu^+(x)\}], \sup\{1-\lambda(x)\}\rangle$$

Therefore, $[f_P(A)]^c \subseteq_P f_p(A^c)$
(iii) $f_P^{-1}(B_1) = B_1(f_P(x))$ and $f_P^{-1}(B_2) = B_2(f_P(x))$ for all $x \in X$. Since $B_1 \subseteq_P B_2, B_1 f_P(x) \leq B_2 f_P(x)$ for all $x \in X$ Therefore, $f_P^{-1}(B_1)(x) \leq f_P^{-1}(B_2)(x)$. Hence $f_P^{-1}(B_1) \subset_P f_P^{-1}(B_2)$
(iv) Let $A_1 = \langle \mu_1, \lambda_1 \rangle$ and $A_2 = \langle \mu_2, \lambda_2 \rangle$ be any two cubic sets in X. Then $f_P(A_1)(x) = \langle x, [\sup\ (\mu_1^-(x)), \sup\ (\mu_1^+(x))], \sup(\lambda_1(x))\rangle$ and $f_P(A_2)(x) = \langle x, [\sup\ (\mu_2^-(x)), \sup\ (\mu_2^+(x))], \sup(\lambda_2(x))\rangle$. Since $A_1 \subseteq_P A_2$, $\sup\ (\mu_1^-(x)) \leq \sup\ (\mu_2^-(x))$, $\sup\ (\mu_1^+(x)) \leq \sup\ (\mu_2^+(x))$ and $\sup(\lambda_1(x) \leq \sup(\lambda_2(x)$. Hence $f_P(A_1) \subseteq_P f_P(A_2)$.
(v) Let $B = \langle \beta, \eta \rangle$ be a cubic set in Y and $y \in Y$.
Case I: $f_P^{-1}(y) \neq \phi$

$$\begin{aligned} [f_P(f_P^{-1}(\beta))(y)]^- &= \sup_{y=f_P(x)} [f_P^{-1}(\beta)(x)]^- \\ &= \sup_{y=f_P(x)} [\beta f_P(x)]^- = [\beta(y)]^- \end{aligned}$$

Similarly,

$$[f_P(f_P^{-1}(\beta))(y)]^+ = [\beta(y)]^+$$

and

$$[f_P(f_P^{-1}(\eta))(y)] = [\eta(y)]$$

Case I:$f_P^{-1}(y) = 0$

$$[f_P(f_P^{-1}(\beta))(y)]^- = 0$$
$$[f_P(f_P^{-1}(\beta))(y)]^+ = 0$$

and

$$[f_P(f_P^{-1}(\eta))(y)] = 0$$

Therefore, by cases I and II we get $f_P(f_P^{-1}(B) \subseteq_P B$ when f_P is surjective, for all $y \in Y$. So by case I the equality holds. (i) Let $A = \langle \mu, \lambda \rangle$ be a cubic set in X. Then

$$\begin{aligned} &f_P^{-1}(f_P(A))(x) \\ &= \langle x, [f_P(\mu(f_P(x)))^-, f_P(\mu(f_P(x)))^+], \\ &\qquad f_P(\lambda(f_P(x))\rangle \\ &= \langle z, [\sup_{z=f_P^{-1}(f_P(x))} \mu^-(z), \sup_{z=f_P^{-1}(f_P(x))} \mu^+(z)], \\ &\qquad \sup_{z=f_P^{-1}(f_P(x))} \lambda(z)\rangle \\ &\geq \langle x, [\mu^-(x), \mu^+(x)], \lambda(x)\rangle \forall x \in X \end{aligned}$$

Therefore $A \subset_P f_P^{-1}(f_P(A))$
(ii) Let $B_i = \langle \beta_i, \eta_i \rangle$ be a cubic set in Y and $y \in Y$. Then

$$\begin{aligned} f_P^{-1}(\bigcup_{i\in\Lambda} {}_P B_i)(y) &= \left(\bigcup_{i\in\Lambda} {}_P B_i\right) f_P(y) \\ &= \bigcup_{i\in\Lambda} {}_P B_i(f_P(y)) \\ &= \bigcup_{i\in\Lambda} {}_P f_P^{-1}(B_i(y)) \end{aligned}$$

(iii) Let $B_i = \langle \beta_i, \eta_i \rangle$ be a cubic set in Y and $y \in Y$. Then

$$\begin{aligned} f_P^{-1}(\bigcap_{i\in\Lambda} {}_P B_i)(y) &= \left(\bigcap_{i\in\Lambda} {}_P B_i\right) f_P(y) \\ &= \bigcap_{i\in\Lambda} {}_P B_i(f_P(y)) \\ &= \bigcap_{i\in\Lambda} {}_P f_P^{-1}(B_i(y)) \end{aligned}$$

□

Theorem 3.11. *If X_P and Y_P are any two P-cubic topological spaces and f_P is a mapping from X_P to Y_P, then the following statements are equivalent:*

(i) The mapping f_P is continuous

(ii) The inverse image of every P-cubic closed set is P-cubic closed

(iii) For each cubic point P_x [3] in X the inverse image of every neighbourhood of $f_P(P_x)$ under f_P is a neighbourhood of P_x.

(iv) For each cubic point P_x in X and each neighbourhood V of $f_P(P_x)$, there is a neighbourhood W of P_x such that $f_P(W) \subset_P V$.

Proof. (i) ⇔ (ii) : The result is obvious as $f_P^{-1}(B^c) = [f_P^{-1}(B)]^c$ for any cubic set B.
(i) ⇔ (iii) Assume that the mapping f_P is continuous and let B be a neighbourhood of $f_P(P_x)$. Then there exists a P-cubic open set U such that $f_P(P_x) \in U \subseteq_P B$. Now $P_x \in f_P^{-1}(f_P(P_x)) \in\subseteq P$, where $f_P^{-1}(B)$ is a P-cubic open set in X implying that the inverse of every neighbourhood of $f_P(P_x)$ under f_P is a neighbourhood of P_x.
(iii) ⇔ (i): Let $f_P(P_x)$ be an arbitrary P-cubic point of a P-cubic open set B of Y_P. Then B is a neighbourhood of $f_P(P_x)$. By hypothesis, $f_P^{-1}(B)$ is a neighbourhood of P_x, then there is a P-cubic open set U_x such that $P_x \in U_x \subseteq_P f_P^{-1}(B)$. Then $B = \bigcup_{P_x \in B} U_x$ is a P-union of P-cubic open set of X_P which implies $f_P^{-1}(B)$ is a P-cubic open set of X_P. (iii) ⇔ (iv): Let P_x be a cubic point in X and V be a neighbourhood of $f_P(P_x)$, then by (iii) $f_P^{-1}(V)$ is a neighbourhood of P_x, we have $f_P(W) = f_P[f_P^{-1}(V)] \subseteq_P V$ where $W = f_P^{-1}(V)$
(iv) ⇔ (iii): Let V be a neighbourhood of $f_P(P_x)$. Then there is a neighbourhood W of P_x such that $f_P(W) \subseteq_P V$. Hence $f_P^{-1}[f_P(W)] \subseteq_P f_P^{-1}(V)$. Futhermore, since $W \subset_P f_P[f_P^{-1}(W)], f_P^{-1}(V)$ is a neighbourhood of P_x.

□

4. Continuous mappings on R-cubic topological spaces

In this section we have defined and analysed the basic properties and characterization of some continuous mappings like α-continuous mappings, semi continuous mappings, pre-continuous mappings and β-continuous mappings in R-cubic topological spaces.

Definition 4.1. *Let $f_R : X \to Y$ be a mapping and let $A = (\mu, \lambda)$ be a cubic set in X. Then the image of A under f_R, denoted by $f_R(A) = (f_R(\mu), f_R(\lambda))$, is defined by*

$$[f_R(\mu)(y)]^- = \begin{cases} \sup_{f_R(x)=y}[\mu(x)]^-, & \text{if } f_R^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$[f_R(\mu)(y)]^+ = \begin{cases} \sup_{f_R(x)=y}[\mu(x)]^+, & \text{if } f_R^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$[f_R(\lambda)](y) = \begin{cases} \inf_{x\in f_R^{-1}(y)}\{\lambda(x)\}, & \text{if } f_R^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

for all y in Y, where $f_R^{-1}(y) = \{x|f_R(x) = y\}$ Let $B = \langle \beta, \eta \rangle$ be an cubic set in Y. Then the inverse image of B under, f_R denoted by $f_R^{-1}(B) = (f_R^{-1}(\beta), f_R^{-1}(\eta))$, is defined by $[f_R^{-1}(\beta)(x)]^- = [\beta(f_R(x))]^-$, $[f_R^{-1}(\beta)(x)]^+ = [\beta(f_R(x))]^+$, $f_R^{-1}(\eta)(x) = \eta(f_R(x))$, for all $x \in X$.

Definition 4.2. *Let X_R and Y_R be any two R-cubic topological spaces. A mapping $f_R : X_R \to Y_R$ said to be a*

(i) R-cubic continuous mapping if $f_R^{-1}(A)$ is a R-cubic open set in X_R for each R-cubic open set A in Y_R.

(ii) R-cubic semi continuous (resp. α-continuous, pre-continuous, β-continuous) mapping if $f_R^{-1}(A)$ is a R-cubic semi open $(resp. \alpha open, pre-open, \beta-open)$ set [4] in X_R for each R-cubic open set A in Y_R.

Proposition 4.3. *The identity mapping $f_R : X_R \to X_R$ is a R-cubic continuous mapping.*

Proof. Straightforward. □

Proposition 4.4. *The composition of two R-cubic continuous mappings is again a R-cubic continuous mapping in general.*

Proof. Straightforward. □

Proposition 4.5. *Every R-cubic continuous mapping is a R-cubic semi continuous (resp. α-continuous, pre-continuous, β-continuous) mapping but the converses are not true in general.*

Proof. Since every R-cubic open set is a R-cubic semi open (resp. α-open, pre-open, β-pen) set,the result follows obviously. □

The following examples show that the converses of proposition 4.5 is not true.

Example 4.6. *Let* $X \neq \phi, \mathbb{F}_R = \{\hat{0}, \ddot{0}, A_1, A_2, A_3, A_4, A_5, \hat{1}, \ddot{1}\}$ *and* $\mathbb{F}_R^{'} = \{\hat{0}, B, \hat{1}\}$ *be two R-cubic topologies on X where* $A_1 = \langle [0.2, 0.4], 0.6 \rangle, A_2 = \langle [0, 0], 0.6 \rangle, A_3 = \langle [1, 1], 0.6 \rangle, A_4 = \langle [0.2, 0.4], 0 \rangle, A_5 = \langle [0.2, 0.4], 1 \rangle$ *and* $B = \langle [0.2, 0.3], 0 \rangle$. *Define a mapping* $f_R : (X, \mathbb{F}_R) \to (X, \mathbb{F}_R^{'})$ *by* $f_R(x) = x$, *then* f_R *is a R-cubic* α *continuous mapping but not a R-cubic continuous mapping.*

Example 4.7. *Let* $X \neq \phi, \mathbb{F}_R = \{\hat{0}, \ddot{0}, A_1, A_2, A_3, A_4, A_5, \hat{1}, \ddot{1}\}$ *and* $\mathbb{F}_R^{'} = \{\hat{0}, C, \hat{1}\}$ *be two R-cubic topologies on X where* $A_1 = \langle [0.2, 0.4], 0.6 \rangle, A_2 = \langle [0, 0], 0.6 \rangle, A_3 = \langle [1, 1], 0.6 \rangle, A_4 = \langle [0.2, 0.4], 0 \rangle, A_5 = \langle [0.2, 0.4], 1 \rangle$ *and* $C = \langle [0.6, 0.8], 0.4 \rangle$. *Define a mapping* $g_R : (X, \mathbb{F}_R) \to (X, \mathbb{F}_R^{'})$ *by* $g_R(x) = x$, *then* g_R *is a R-cubic semi continuous mapping but not a R-cubic continuous mapping.*

Example 4.8. *Let* $X \neq \phi, \mathbb{F}_R = \{\hat{0}, \ddot{0}, A_1, A_2, A_3, A_4, A_5, \hat{1}, \ddot{1}\}$ *and* $\mathbb{F}_R^{'} = \{\hat{0}, D, \hat{1}\}$ *be two R-cubic topologies on X where* $A_1 = \langle [0.2, 0.4], 0.6 \rangle, A_2 = \langle [0, 0], 0.6 \rangle, A_3 = \langle [1, 1], 0.6 \rangle, A_4 = \langle [0.2, 0.4], 0 \rangle, A_5 = \langle [0.2, 0.4], 1 \rangle$ *and* $D = \langle [0.7, 0.8], 0.7 \rangle$. *Define a mapping* $h_R : (X, \mathbb{F}_R) \to (X, \mathbb{F}_R^{'})$ *by* $h_R(x) = x$, *then is a R-cubic pre-continuous mapping and a R-cubic* β*-continuous mapping, but not a R-cubic continuous mapping.*

Theorem 4.9. *A mapping* $f_R : X_R \to Y_R$ *is a R-cubic* α*-continuous mapping if and only if it is both a R-cubic semi continuous mapping and R-cubic pre-continuous mapping.*

Proof. Let f_R be both a R-cubic semi continuous mapping and a R-cubic pre-continuous mapping. Let A be a R-cubic open set in Y_R, then by hypothesis $f_R^{-1}(A)$ is a R-cubic semi open set and R-cubic pre-open set. Hence by proposition 4.25[6], f_R^{-1} is a P-cubic α open set and hence it is a R-cubic α-continuous mapping . The converse is immediate. □

Theorem 4.10. *Let* $f_R : X_R \to Y_R$ *be a mapping. Then*

(i) $f_R^{-1}(B^c) = [f_R^{-1}(B)]^c$ *for all cubic sets B in Y.*

(ii) $[f_R(A)]^c \subseteq_R f_R(A^c)$ *for all cubic sets A in X.*

(iii) $B_1 \subset_R B_2$ *implies* $f_R^{-1}(B_1) \subset_R f_R^{-1}(B_2)$, *where* B_1 *and* B_2 *are cubic sets in Y.*

(iv) $A_1 \subset_R A_2$ *implies* $f_R(A_1) \subset_R f_R(A_2)$, *where* A_1 *and* A_2 *are cubic sets in Y.*

(v) $f_R(f_R^{-1}(B)) \subset_R B$ *the equality holds if* f_R *is surjective, for all cubic sets B in Y.*

(vi) $A \subset_R f_R^{-1}(f_R(A))$ *the equality holds if* f_R *is injective, for all cubic sets A in X.*

(vii)

$$f_R^{-1}(\bigcup_{i\in\Lambda} {}_R B_i) = \bigcup_{i\in\Lambda} {}_R f_R^{-1}(B_i)$$

for all cubic sets B_i *in Y.*

(viii)

$$f_R^{-1}(\bigcap_{i\in\Lambda} {}_R B_i) = \bigcap_{i\in\Lambda} {}_R f_R^{-1}(B_i)$$

for all cubic sets B_i *in Y.*

Proof. (i) Let $B = \langle \beta, \eta \rangle$ be a cubic set in Y. Then

$$\begin{aligned} f_R^{-1}(B^c)(x) &= \langle x, f_R^{-1}(\beta^c)(x), f_R^{-1}(\eta^c)(x) \rangle \\ &= \langle x, \beta^c(f_R)(x), \eta^c(f_R)(x) \rangle \\ &= \langle x, 1 - \beta(f_R)(x), 1 - \eta(f_R)(x) \rangle \\ &= [f_R^{-1}(B)]^c \end{aligned}$$

(ii) $A = \langle \mu, \lambda \rangle$ be a cubic set in X and $f_R^{-1}(y) \neq \phi$. Then $A^c = \{\langle x, [1 - \mu^+(x), 1 - \mu^-(x)], 1 - \lambda(x) \rangle\}$, we have

$$\begin{aligned} [f_R(A)]^c(y) &= 1 - f_R(A)(y) \\ &= 1 - \langle y, [\sup\ (\mu^-(x)), \sup\ (\mu^+(x))], \\ &\qquad \sup(\lambda(x)) \rangle \\ &= \langle 1 - y, [1 - \sup(\mu^-(x)), 1 - \sup(\mu^+(x))], \\ &\qquad 1 - \sup(\lambda(x)) \rangle \\ [f_R(A)]^c(y) &= \langle y, [\sup\{1 - \mu^-(x)\}, \sup\{1 - \mu^+(x)\}], \\ &\qquad \sup\{1 - \lambda(x)\} \rangle \end{aligned}$$

Therefore, $[f_R(A)]^c \subseteq_R f_R(A^c)$
(iii) $f_R^{-1}(B_1) = B_1(f_R(x))$ and $f_R^{-1}(B_2) = B_2(f_R(x))$ for all $x \in X$. Since $B_1 \subseteq_R B_2$, $B_1 f_R(x) \leq B_2 f_R(x)$ for all $x \in X$ Therefore, $f_R^{-1}(B_1)(x) \leq f_R^{-1}(B_2)(x)$. Hence $f_R^{-1}(B_1) \subset_R f_R^{-1}(B_2)$
(iv) Let $A_1 = \langle \mu_1, \lambda_1 \rangle$ and $A_2 = \langle \mu_2, \lambda_2 \rangle$ be any two cubic sets in X. Then $f_R(A_1)(x) = \langle x, [\sup\ (\mu_1^-(x)), \sup\ (\mu_1^+(x))], \sup(\lambda_1(x)) \rangle$ and $f_R(A_2)(x) = \langle x, [\sup\ (\mu_2^-(x)), \sup\ (\mu_2^+(x))], \sup(\lambda_2(x)) \rangle$. Since $A_1 \subseteq_R A_2$, $\sup\ (\mu_1^-(x)) \leq \sup\ (\mu_2^-(x))$, $\sup\ (\mu_1^+(x)) \leq \sup\ (\mu_2^+(x))$ and $\sup(\lambda_1(x) \leq \sup(\lambda_2(x)$. Hence $f_R(A_1) \subseteq_R f_R(A_2)$.

(v) Let $B=\langle\beta,\eta\rangle$ be a cubic set in Y and $y\in Y$.
Case I: $f_R^{-1}(y)\neq\phi$

$$\begin{aligned}&[f_R(f_R^{-1}(\beta))(y)]^-\\&=\sup_{y=f_R(x)}[f_R^{-1}(\beta)(x)]^-\\&=\sup_{y=f_R(x)}[\beta f_R(x)]^-=[\beta(y)]^-\end{aligned}$$

Similarly,
$$[f_R(f_R^{-1}(\beta))(y)]^+=[\beta(y)]^+$$
and
$$[f_R(f_R^{-1}(\eta))(y)]=[\eta(y)]$$
Case I:$f_R^{-1}(y)=0$
$$[f_R(f_R^{-1}(\beta))(y)]^-=0$$
$$[f_R(f_R^{-1}(\beta))(y)]^+=0$$
and
$$[f_R(f_R^{-1}(\eta))(y)]=0$$
Therefore, by cases I and II we get $f_R(f_R^{-1}(B)\subseteq_R B$ when f_R is surjective, for all $y\in Y$. So by case I the equality holds. (i) Let $A=\langle\mu,\lambda\rangle$ be a cubic set in X. Then

$$\begin{aligned}&f_R^{-1}(f_R(A))(x)\\&=\langle x,[f_R(\mu(f_R(x)))^-,f_R(\mu(f_R(x)))^+],\\&\qquad f_R(\lambda(f_R(x))\rangle\\&=\langle z,[\sup_{z=f_R^{-1}(f_R(x))}\mu^-(z),\sup_{z=f_R^{-1}(f_R(x))}\mu^+(z)],\\&\qquad\sup_{z=f_R^{-1}(f_R(x))}\lambda(z)\rangle\\&\geq\langle x,[\mu^-(x),\mu^+(x)],\lambda(x)\rangle\forall x\in X\end{aligned}$$

Therefore $A\subset_R f_R^{-1}(f_R(A))$
(ii) Let $B_i=\langle\beta_i,\eta_i\rangle$ be a cubic set in Y and $y\in Y$. Then

$$\begin{aligned}f_R^{-1}(\bigcup_{i\in\Lambda}{}_RB_i)(y)&=\left(\bigcup_{i\in\Lambda}{}_RB_i\right)f_R(y)\\&=\bigcup_{i\in\Lambda}{}_RB_i(f_R(y))\\&=\bigcup_{i\in\Lambda}{}_Rf_R^{-1}(B_i(y))\end{aligned}$$

(iii) Let $B_i=\langle\beta_i,\eta_i\rangle$ be a cubic set in Y and $y\in Y$. Then

$$\begin{aligned}f_R^{-1}(\bigcap_{i\in\Lambda}{}_RB_i)(y)&=\left(\bigcap_{i\in\Lambda}{}_RB_i\right)f_R(y)\\&=\bigcap_{i\in\Lambda}{}_RB_i(f_R(y))\\&=\bigcap_{i\in\Lambda}{}_Rf_R^{-1}(B_i(y))\end{aligned}$$

□

Theorem 4.11. *If X_R and Y_R are any two R-cubic topological spaces and f_R is a mapping from X_R to Y_R, then the following statements are equivalent:*

(i) The mapping f_R is continuous

(ii) The inverse image of every R-cubic closed set is R-cubic closed

(iii) For each cubic point R_x [3] in X the inverse image of every neighbourhood of $f_R(R_x)$ under f_R is a neighbourhood of R_x.

(iv) For each cubic point R_x in X and each neighbourhood V of $f_R(R_x)$, there is a neighbourhood W of R_x such that $f_R(W)\subset_R V$.

Proof. (i) ⇔ (ii) : The result is obvious as $f_R^{-1}(B^c)=[f_R^{-1}(B)]^c$ for any cubic set B.
(i) ⇔ (iii) Assume that the mapping f_R is continuous and let B be a neighbourhood of $f_R(R_x)$. Then there exists a R-cubic open set U such that $f_R(R_x)\in U\subseteq_R B$. Now $R_x\in f_R^{-1}(f_R(R_x))\in\subseteq R$, where $f_R^{-1}(B)$ is a R-cubic open set in X implying that the inverse of every neighbourhood of $f_R(R_x)$ under f_R is a neighbourhood of R_x.
(iii) ⇔ (i): Let $f_R(R_x)$ be an arbitrary R-cubic point of a R-cubic open set B of Y_R. Then B is a neighbourhood of $f_R(R_x)$. By hypothesis, $f_R^{-1}(B)$ is a neighbourhood of R_x, then there is a R-cubic open set U_x such that $R_x\in U_x\subseteq_R f_R^{-1}(B)$. Then $B=\bigcup_{R_x\in B}U_x$ is a R-union of R-cubic open set of X_R which implies $f_R^{-1}(B)$ is a R-cubic open set of X_R. (iii) ⇔ (iv): Let R_x be a cubic point in X and V be a neighbourhood of $f_R(R_x)$, then by (iii) $f_R^{-1}(V)$ is a neighbourhood of R_x, we have $f_R(W)=f_R[f_R^{-1}(V)]\subseteq_R V$ where $W=f_R^{-1}(V)$
(iv) ⇔ (iii): Let V be a neighbourhood of $f_R(R_x)$. Then there is a neighbourhood W of R_x such that $f_R(W)\subseteq_R V$. Hence $f_R^{-1}[f_R(W)]\subseteq_R f_R^{-1}(V)$. Futhermore, since $W\subset_R f_R[f_R^{-1}(W)],f_R^{-1}(V)$ is a neighbourhood of R_x.

□

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