Malaya J. Mat. 2(2)(2014) 91–102

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Meromorphic parabolic starlike functions with a fixed point involving Srivastava-Attiya operator

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Abstract

In the present investigation, we introduce a new class of meromorphic parabolic starlike functions with a fixed point defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ by making use of the Srivastava-Attiya Operator $\mathcal{J}_b^s$. We obtained Coefficient inequalities, growth and distortion inequalities, as well as closure results for functions $f \in \mathcal{M}_b^s(\lambda, \beta, \gamma)$. We further established some results concerning convolution and the partial sums.

Keywords: Meromorphic functions, starlike function, convolution, positive coefficients, coefficient inequalities, integral operator.

2010 MSC: 30C50.

1 Introduction

Let ξ be a fixed point in the unit disc $:= \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{H}()$ the class of functions which are regular and

$$\mathcal{A}(\xi) = \{f \in H() : f(\xi) = f'(\xi) - 1 = 0\}.$$

Also denote by

$$\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi) : f \text{ is univalent in } \},$$

the subclass of $\mathcal{A}(\xi)$ consist of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n, \tag{1.1}$$

that are analytic in the open unit disc . Note that $\mathcal{S}_0 = \mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in . By $\mathcal{S}_\xi^*(\gamma)$ and $\mathcal{K}_\xi(\gamma)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\Re\left(\frac{(z-\xi)f'(z)}{f(z)}\right) > \gamma, \Re\left(1 + \frac{(z-\xi)f''(z)}{f'(z)}\right) > \gamma$$

and $z \in$ for $0 \le \gamma < 1$, introduced and studied by Kanas and Ronning [11]. The class $\mathcal{S}_\xi^*(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}_\xi(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [8] and [9] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathcal{K}_0 = \mathcal{K}(0)$ and $\mathcal{S}_0^* = \mathcal{S}^*(0)$ respectively, are the well-known standard class of convex and starlike functions(see [21]).

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Let Σ denote the class of meromorphic functions f of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, \tag{1.2}$$

defined on the punctured unit disk $\Delta^*:=\{z\in\mathbb{C}:0<|z|<1\}$.

Denote by Σ_ξ be the subclass of $\mathcal{A}(\xi)$ consist of the functions of the form

$$f(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}a_n(z-\xi)^n, a_n\geq 0; z\neq\xi. \tag{1.3}$$

A function f of the form (1.3) is in the class of *meromorphic starlike of order* γ $(0\leq\gamma<1)$ denoted by $\Sigma^*_\xi(\gamma)$, if

$$-\Re\left(\frac{(z-\xi)f'(z)}{f(z)}\right)>\gamma,\quad z-\xi\in\Delta:=\Delta^*\cup\{0\} \tag{1.4}$$

and is in the class of *meromorphic convex of order* γ $(0\leq\gamma<1)$ denoted by $\Sigma^K_\xi(\gamma)$, if

$$-\Re\left(1+\frac{(z-\xi)f''(z)}{f'(z)}\right)>\gamma,\quad z-\xi\in\Delta:=\Delta^*\cup\{0\}.$$

For functions $f(z)$ given by (1.3) and $g(z)=\frac{1}{(z-\xi)}+\sum_{n=1}^{\infty}b_n(z-\xi)^n$, $(b_n\geq 0)$ we define the Hadamard product or convolution of f and g by

$$(f*g)(z):=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}a_n\,b_n(z-\xi)^n.$$

The study of operators plays a vital role in the geometric function theory and its associated fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical investigation and also helps to understand the geometric properties of such operators better.

We recall a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (see [24])

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s} \tag{1.5}$$

$$(a\in\mathbb{C}\setminus\{\mathbb{Z}_0^-\}; s\in\mathbb{C},\Re(s)>1\ \text{ and }\ |z|=1)$$

where, as usual, $\mathbb{Z}_0^-:=\mathbb{Z}\setminus\{\mathbb{N}\}$ $(\mathbb{Z}:=\{0,\pm1,\pm2,\pm3,...\};\mathbb{N}:=\{1,2,3,...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [5], Lin and Srivastava [12], Lin et al. [13], and see the references stated therein.

For the class of analytic functions denote by $\mathcal{A}$ consisting of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,(z\in\ )$$

Srivastava and Attiya [23] introduced and investigated the linear operator:

$$\mathcal{J}_{s,b}:\mathcal{A}\to\mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_{s,b}f(z)=G_{b,s}*f(z) \tag{1.6}$$

where, for convenience,

$$G_{b,s}(z):=(1+b)^s[\Phi(z,s,b)-b^{-s}] \tag{1.7}$$

$(z\in\ ;b\in\mathbb{C}\setminus\{\mathbb{Z}_0^-\};s\in\mathbb{C};f\in\mathcal{A})$. For $f\in\mathcal{A}$ it is easy to observe from (1.6) and (1.7) that

$$\mathcal{J}_{s,b}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^s a_nz^n,\quad (z\in .) \tag{1.8}$$

It is well known that the Srivastava-Attiya operator $\mathcal{J}_{s,b}$ contains, among its special cases, the integral operators introduced and investigated earlier by (for example) Alexander [1], Libera [14], Bernardi [4], and Jung et al. [10].

Motivated essentially by the above mentioned Srivastava-Attiya operator, in this paper we define a new linear operator

$$\mathcal{J}_b^s : \Sigma_\xi \to \Sigma_\xi$$

in terms of Hadamard product given by

$$\mathcal{J}_b^s f(z) = \mathcal{G}_{b,p}^s * f(z) \tag{1.9}$$

$$(z-\xi \in \Delta := \Delta^* \cup \{0\}; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}; f \in \Sigma_\xi),$$

where, for convenience

$$\mathcal{G}_{b,p}^s(z) := (1+b)^s[\Phi_p(z,s,b) - b^{-s}] \tag{1.10}$$

and

$$\Phi_p(z,s,b) = \frac{1}{b^s} + \frac{(z-\xi)^{-1}}{(1+b)^s} + \frac{(z-\xi)}{(2+b)^s} + \dots .$$

For $f \in \Sigma_\xi$, it is easy to observe from the above equations (1.9) and (1.10) that

$$\mathcal{J}_b^s f(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} C_b^s(n) a_n (z-\xi)^n, \quad (z-\xi \in \Delta := \Delta^* \cup \{0\}) \tag{1.11}$$

where

$$C_b^s(n) = \left| \left(\frac{1+b}{n+1+b} \right)^s \right| \tag{1.12}$$

and (throughout this paper unless otherwise mentioned) the parameters s, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}$.

Motivated by earlier works on meromorphic functions by function theorists(see [2, 3, 7, 15, 16, 17, 18, 19, 20, 25]), we define the following new subclass of functions in Σ_ξ by making use of the generalized operator $\mathcal{J}_b^s$.

For $0 \leq \gamma < 1$ and $0 \leq \lambda < 1/2$, we let $\mathcal{M}_b^s(\lambda, \beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (1.3) satisfying the condition that

$$\begin{aligned} - \quad & \Re\left(\frac{(z-\xi)(\mathcal{J}_b^s f(z))' + \lambda(z-\xi)^2(\mathcal{J}_b^s f(z))''}{(1-\lambda)\mathcal{J}_b^s f(z) + \lambda(z-\xi)(\mathcal{J}_b^s f(z))'} \right) \\ > \quad & \beta \left| \frac{(z-\xi)(\mathcal{J}_b^s f(z))' + \lambda(z-\xi)^2(\mathcal{J}_b^s f(z))''}{(1-\lambda)\mathcal{J}_b^s f(z) + \lambda(z-\xi)(\mathcal{J}_b^s f(z))'} + 1 \right| + \gamma \end{aligned} \tag{1.13}$$

where $\mathcal{J}_b^s f$ is given by (1.11).

Further shortly we can state this condition by

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)} \right) > \beta \left| \frac{(z-\xi)G'(z)}{G(z)} + 1 \right| + \gamma, \tag{1.14}$$

where

$$G(z) = (1-\lambda)\mathcal{J}_b^s f(z) + \lambda(z-\xi)(\mathcal{J}_b^s f(z))' = \frac{1-2\lambda}{z-\xi} + \sum_{n=1}^{\infty} (n\lambda - \lambda + 1) C_b^s(n) a_n (z-\xi)^n, \quad a_n \geq 0. \tag{1.15}$$

It is of interest to note that, on specializing the parameters λ, β and s, b we can define various new subclasses of Σ_ξ. We illustrate two important subclasses in the following examples.

Example 1.1. *For $\lambda = 0$, we let $\mathcal{M}_b^s(0, \beta, \gamma) = \mathcal{M}_b^s(\beta, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (1.3) satisfying the condition that*

$$- \Re\left(\frac{(z-\xi)(\mathcal{J}_b^s f(z))'}{\mathcal{J}_b^s f(z)} \right) > \beta \left| \frac{(z-\xi)(\mathcal{J}_b^s f(z))'}{\mathcal{J}_b^s f(z)} + 1 \right| + \gamma \tag{1.16}$$

where $\mathcal{J}_b^s f(z)$ is given by (1.11).

Example 1.2. *For $\lambda = 0, \beta = 0$ we let $\mathcal{M}_b^s(0,0,\gamma) = \mathcal{M}_b^s(\gamma)$ denote a subclass of Σ_ξ consisting functions of the form (1.3) satisfying the condition that*

$$-\Re\left(\frac{(z-\xi)(\mathcal{J}_b^s f(z))'}{\mathcal{J}_b^s f(z)}\right) > \gamma \tag{1.17}$$

where $\mathcal{J}_b^s f(z)$ is given by (1.11).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, as well as closure results for the function class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Properties of certain integral operator and convolution properties of the new class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ are also discussed.

2 Coefficients Inequalities

In order to obtain the necessary and sufficient condition for a function $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, we recall the following lemmas.

Lemma 2.1. *If γ is a real number and w is a complex number, then $\Re(w) \geq \gamma \Leftrightarrow |w+(1-\gamma)| - |w-(1+\gamma)| \geq 0$.*

Lemma 2.2. *If w is a complex number and γ, k are real numbers, then*

$$\Re(w) \geq k|w-1| + \gamma \Leftrightarrow \Re\{w(1+ke^{i\theta}) - ke^{i\theta}\} \geq \gamma, \ \ -\pi \leq \theta \leq \pi.$$

Analogous to the lemma proved by Dziok et.al [7], we state the following lemma without proof.

Lemma 2.3. *Suppose that $\gamma \in [0,1)$, $r \in (0,1]$ and the function $H \in \Sigma_\xi(\gamma)$ is of the form $H(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} b_n(z-\xi)^n$, $\ 0 < |z-\xi| < r$, with $b_n \geq 0$ then*

$$\sum_{n=1}^{\infty}(n+\gamma)b_n r^{n+1} \leq 1-\gamma. \tag{2.1}$$

Theorem 2.1. *Let $f \in \Sigma_\xi$ be given by (1.3). Then $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}[n(1+\beta)+(\gamma+\beta)](n\lambda-\lambda+1)\ C_b^s(n)a_n \leq (1-2\lambda)(1-\gamma). \tag{2.2}$$

Proof. If $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, then by (1.14) we have,

$$-\Re\left(\frac{(z-\xi)G'(z)}{G(z)}\right) > \beta\left|\frac{(z-\xi)G'(z)}{G(z)}+1\right| + \gamma.$$

Making use of Lemma 2.2

$$-\Re\left(\frac{(z-\xi)(1+\beta e^{i\theta})G'(z)+\beta e^{i\theta}G(z)}{G(z)}\right) > \gamma,$$

where $G(z)$ is given by (1.15). Substituting for $G(z), G'(z)$ and letting $|z-\xi| < r \to 1^-$, we have

$$\left\{\frac{(1-2\lambda)(1-\gamma) - \sum_{n=1}^{\infty}[n(1+\beta)+(\gamma+\beta)](n\lambda-\lambda+1)C_b^s(n)a_n}{(1-2\lambda) - \sum_{n=1}^{\infty}(n\lambda-\lambda+1)C_b^s(n)a_n}\right\} > 0.$$

This shows that (2.2) holds.

Conversely, assume that (2.2) holds. Since $-\Re(w) > \gamma$, if and only if $|w+1| < |w-(1-2\gamma)|$, it is sufficient to show that

$$\left|\frac{w+1}{w-(1-2\gamma)}\right| < 1 \ \text{ and } \ |w-(1-2\gamma)| \neq 0 \text{ for } \ |z-\xi| < r \leq 1, \quad (z-\xi) \in \Delta.$$

Using (2.2) and taking $w(z) = \frac{(z-\xi)(1+\beta e^{i\theta})G'(z)+\beta e^{i\theta}G(z)}{G(z)}$ we get

$$\left|\frac{w+1}{w-(1-2\gamma)}\right| \leq \frac{\sum_{n=1}^{\infty}(n\lambda-\lambda+1)[(n+1)(1+\beta)]C_b^s(n)a_n}{2(1-\gamma)(1-2\lambda) - \sum_{n=1}^{\infty}(n\lambda-\lambda+1)[n(1+\beta)+(\beta+2\gamma-1)]C_b^s(n)a_n} \leq 1.$$

Thus we have $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$. $\square$

For the sake of brevity throughout this paper we let

$$d_n(\lambda,\beta,\gamma) := [n(1+\beta)+(\gamma+\beta)](n\lambda-\lambda+1) \tag{2.3}$$
$$d_1(\lambda,\beta,\gamma) = (1+\gamma+2\beta)$$

unless otherwise stated.

Our next result gives the coefficient estimates for functions in $\mathcal{M}_b^s(\lambda,\beta,\gamma)$.

Theorem 2.2. *If $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, then*

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}, \qquad n=1,2,3,\ldots.$$

The result is sharp for the functions $f_n(z)$ given by

$$f_n(z) = \frac{1}{z-\xi} + \frac{1-\gamma}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n, \qquad n=1,2,3,\ldots.$$

Proof. If $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, then we have, for each n,

$$d_n(\lambda,\beta,\gamma)C_b^s(n)a_n \leq \sum_{n=1}^{\infty} d_n(\lambda,\beta,\gamma)C_b^s(n)a_n \leq (1-\gamma)(1-2\lambda).$$

Therefore we have

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}.$$

Since

$$f_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n$$

satisfies the conditions of Theorem 2.1, $f_n(z) \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ and the equality is attained for this function. □

Theorem 2.3. *Suppose that there exists a positive number ν*

$$\nu = \inf_{n\in\mathbb{N}} \left\{d_n(\lambda,\beta,\gamma)C_b^s(n)\right\}. \tag{2.4}$$

If $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, then

$$\left|\frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu}r\right| \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r, \quad (|z-\xi|=r)$$

and

$$\left|\frac{1}{r^2} - \frac{(1-\gamma)(1-2\lambda)}{\nu}\right| \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)}{\nu} \quad (|z-\xi|=r).$$

If $\nu = d_1(\lambda,\beta,\gamma)C_b^s(1) = (1+\gamma+2\beta)C_b^s(1)$, then the result is sharp for

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+\gamma+2\beta)C_b^s(1)}(z-\xi). \tag{2.5}$$

Proof. Let the function f given by (1.3) we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n.$$

Since,

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\gamma)(1-2\lambda)}{\nu}.$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r.$$

Similarly

$$|f(z)| \geq \left|\frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu}r\right|.$$

The result is sharp for function (2.5) with $\nu = d_1(\lambda,\beta,\gamma)C_b^s(1) = (1+\gamma+2\beta)C_b^s(1)$.
Similarly we can prove the other inequality $|f'(z)|$. □

3 Radius of starlikeness

In the following theorem we obtain the radius of starlikeness for the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. We say that f given by (1.3) is meromorphically starlike of order ρ, $(0 \le \rho < 1)$, in $|z-\xi| < r$ when it satisfies the condition (1.4) in $|z-\xi| < r$.

Theorem 3.1. *Let the function f given by* (1.3) *be in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Then, if there exists*

$$r_1(\lambda,\gamma,\rho) = \inf_{n\ge 1}\left[\frac{(1-\rho)d_n(\lambda,\beta,\gamma)C_b^s(n)}{(n+\rho)(1-\gamma)(1-2\lambda)}\right]^{\frac{1}{n+1}} \tag{3.1}$$

and it is positive, then f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho)$.

Proof. Let the function $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ be of the form (1.3). If $0 < r \le r_1(\lambda,\gamma,\rho)$, then by (3.1)

$$r^{n+1} \le \frac{(1-\rho)d_n(\lambda,\beta,\gamma)C_b^s(n)}{(n+\rho)(1-\gamma)(1-2\lambda)} \tag{3.2}$$

for all $n \in \mathbb{N}$. From (3.2) we get

$$\frac{n+\rho}{1-\rho}r^{n+1} \le \frac{d_n(\lambda,\beta,\gamma)C_b^s(n)}{(1-\gamma)(1-2\lambda)}$$

for all $n \in \mathbb{N}$, thus

$$\sum_{n=1}^{\infty}\frac{n+\rho}{1-\rho}a_n r^{n+1} \le \sum_{n=1}^{\infty}\frac{d_n(\lambda,\beta,\gamma)C_b^s(n)}{(1-\gamma)(1-2\lambda)}a_n \le 1 \tag{3.3}$$

because of (2.2). Hence, from (3.3) and (2.1), f is meromorphically starlike of order ρ in $|z-\xi| < r \le r_1(\lambda,\gamma,\rho)$. □

Suppose that there exists a number $\widetilde{r}$, $\widetilde{r} > r_1(\lambda,\gamma,\rho)$ such that each $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ is meromorphically starlike of order ρ in $|z-\xi| < \widetilde{r} \le 1$. The function

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n$$

is in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$, thus it should satisfy (2.1) with $\widetilde{r}$:

$$\sum_{n=1}^{\infty}(n+\rho)a_n\widetilde{r}^{n+1} \le 1-\rho, \tag{3.4}$$

while the left–hand side of (3.4) becomes

$$(n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}\widetilde{r}^{n+1} > (n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}\frac{(1-\rho)d_n(\lambda,\beta,\gamma)C_b^s(n)}{(n+\rho)(1-\gamma)(1-2\lambda)} = 1-\rho$$

which contradicts with (3.4). Therefore the number $r_1(\lambda,\gamma,\rho)$ in Theorem 3.1, cannot be replaced with a greater number. This means that $r_1(\lambda,\gamma,\rho)$ is called radius of meromorphically starlikness of order ρ for the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$.

4 Results Involving Modified Hadamard Products

For functions

$$f_j(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,j}(z-\xi)^n, a_{n,j} \ge 0 \tag{4.5}$$

we define the Hadamard product or convolution of f_1 and f_2 by

$$(f_1 * f_2)(z) := \frac{1}{z-\xi} + \sum_{n=1}^{\infty}a_{n,1}a_{n,2}(z-\xi)^n.$$

Let

$$\Psi(n,\lambda) = \frac{(n\lambda-\lambda+1)}{(1-2\lambda)}C_b^s(n). \tag{4.6}$$

Theorem 4.2. *For functions $f_j (j = 1,2)$ defined by (4.5), let $f_1 \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ and $f_2 \in \mathcal{M}_b^s(\lambda,\beta,\delta)$. Then $f_1 * f_2 \in \mathcal{M}_b^s(\lambda,\beta,\eta)$ where*

$$\eta = 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda) - 2(1-\gamma)(1-\delta)} \tag{4.7}$$

and $\Psi(1,\lambda) = \frac{C_b^s(1)}{1-2\lambda}$. The results is the best possible for

$$\begin{aligned} f_1(z) &= \frac{1}{z-\xi} + \frac{1-\gamma}{(1+\gamma+2\beta)\Psi(1,\lambda)}(z-\xi), \\ f_2(z) &= \frac{1}{z-\xi} + \frac{1-\delta}{(1+\delta+2\beta)\Psi(1,\lambda)}(z-\xi) \end{aligned}$$

where $\Psi(1,\lambda) = \frac{C_b^s(1)}{1-2\lambda}$.

Proof. In the view of Theorem 2.1, it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)}\Psi(n,\lambda)a_{n,1}a_{n,2} \le 1$$

where η is defined by (4.7) under the hypothesis. It follows from (2.2) and the Cauchy's-Schwarz inequality that

$$\sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}\Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}} \le 1. \tag{4.8}$$

Thus we need to find largest η such that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\eta+\beta)]}{(1-\eta)}\Psi(n,\lambda)a_{n,1}a_{n,2} \\ \le\ &\sum_{n=1}^{\infty} \frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}}\Psi(n,\lambda)\sqrt{a_{n,1}a_{n,2}} \\ \le\ &1. \end{aligned}$$

By virtue of (4.8) it is sufficient to find the largest η, such that

$$\begin{aligned} &\frac{\sqrt{(1-\gamma)(1-\delta)}}{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}\Psi(n,\lambda)} \\ \le\ &\frac{[n(1+\beta)+(\gamma+\beta)]^{1/2}[n(1+\beta)+(\delta+\beta)]^{1/2}}{\sqrt{(1-\gamma)(1-\delta)}} \frac{1-\eta}{[n(1+\beta)+(\eta+\beta)]}, \end{aligned}$$

which yields

$$\eta \le 1 - \frac{(1-\gamma)(1-\delta)(2n+1+\beta)}{[n(1+\beta)+(\gamma+\beta)][n(1+\beta)+(\delta+\beta)]\Psi(n,\lambda) - (1-\gamma)(1-\delta)(n+1)}$$

for $n \ge 1$ where $\Psi(n,\lambda)$ is given by (4.6) and since $\Psi(n,\lambda)$ is a decreasing function of n $(n \ge 1)$, we have

$$\eta = 1 - \frac{(1-\gamma)(1-\delta)(3+\beta)}{(1+\gamma+2\beta)(1+\delta+2\beta)\Psi(1,\lambda) - 2(1-\gamma)(1-\delta)}$$

and $\Psi(1,\lambda) = \frac{C_b^s(1)}{1-2\lambda}$, which completes the proof. □

Theorem 4.3. *Let the functions $f_j, (j = 1,2)$ defined by (4.5) be in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Then $(f_1 * f_2)(z) \in \mathcal{M}_b^s(\lambda,\beta,\eta)$ where*

$$\eta = 1 - \frac{(1-\gamma)^2(3+\beta)}{(1+\gamma+2\beta)^2\Psi(1,\lambda) - 2(1-\gamma)^2}$$

with $\Psi(1,\lambda) = \frac{C_b^s(1)}{1-2\lambda}$.

Proof. By taking $\delta = \gamma$ in the above theorem, the results follows. □

For functions in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ we can prove the following inclusion property.

Theorem 4.4. *Let the functions $f_j (j=1,2)$ defined by (4.5) be in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Then the function h defined by*

$$h(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}(a_{n,1}^2+a_{n,2}^2)(z-\xi)^n$$

is in the class $\mathcal{M}_b^s(\lambda,\beta,\delta)$ where

$$\delta \le 1-\frac{4(1-\gamma)^2(1+\beta)}{[1+\gamma+2\beta]^2\Psi(1,\lambda)+2(1-\gamma)^2}, \tag{4.9}$$

and $\Psi(1,\lambda)=\frac{C_b^s(1)}{1-2\lambda}$.

Proof. In view of Theorem 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{(1-\delta)}(a_{n,1}^2+a_{n,2}^2)\le 1 \tag{4.10}$$

where $f_j\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$ $(j=1,2)$, we find from (4.5) and Theorem 2.1, that

$$\sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2 a_{n,j}^2\le \sum_{n=1}^{\infty}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}a_{n,j}\right]^2\le 1, \tag{4.11}$$

which would yields

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2(a_{n,1}^2+a_{n,2}^2)\le 1. \tag{4.12}$$

On comparing (4.10) and (4.12) it can be seen that inequality (4.9) will be satisfied if

$$\Psi(n,\lambda)\frac{[n(1+\beta)+(\delta+\beta)]}{1-\delta}(a_{n,1}^2+a_{n,2}^2)\le\frac{1}{2}\left[\Psi(n,\lambda)\frac{[n(1+\beta)+(\gamma+\beta)]}{1-\gamma}\right]^2(a_{n,1}^2+a_{n,2}^2).$$

That is, if

$$\delta\le 1-\frac{2(1-\gamma)^2[(n+1)(1+\beta)]}{[n(1+\beta)+(\gamma+\beta)]^2\Psi(n,\lambda)+2(1-\gamma)^2} \tag{4.13}$$

where $\Psi(n,\lambda)$ is given by (4.6) and $\Psi(n,\lambda)$ is a decreasing function of n $(n\ge 1)$, we get (4.9), which completes the proof. □

5 Closure Theorems

We state the following closure theorems for $f\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$ without proof (see [7, 16, 18]).

Theorem 5.5. *Let the function $f_k(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}a_{n,k}(z-\xi)^n$ be in the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ for every $k=1,2,...,m$. Then the function f defined by*

$$f(z)=\frac{1}{z-\xi}+\sum_{n=1}^{\infty}a_{n,k}(z-\xi)^n,\ (a_{n,k}\ge 0)$$

belongs to the class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$, where $a_{n,k}=\frac{1}{m}\sum_{k=1}^{m}a_{n,k}$, $(n=1,2,..)$.

Theorem 5.6. *Let $f_0(z)=\frac{1}{z-\xi}$ and $f_n(z)=\frac{1}{z-\xi}+\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\beta,\gamma)C_b^s(n)}(z-\xi)^n$ for $n=1,2,\ldots$. Then $f\in\mathcal{M}_b^s(\lambda,\beta,\gamma)$ if and only if f can be expressed in the form $f(z)=\sum_{n=0}^{\infty}\eta_n f_n(z)$ where $\eta_n\ge 0$ and $\sum_{n=0}^{\infty}\eta_n=1$.*

Theorem 5.7. *The class $\mathcal{M}_b^s(\lambda,\beta,\gamma)$ is closed under convex linear combination.*

Now, we prove that the class is$\mathcal{M}_b^s(\lambda,\beta,\gamma)$ closed under integral transforms .

Theorem 5.8. *Let the function $f(z)$ given by (1.3) be in $\mathcal{M}_b^s(\lambda,\beta,\gamma)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $\mathcal{M}_b^s(\lambda,\beta,\delta)$, where

$$\delta \leq \frac{n^2(1+\beta) + n[(\gamma+\beta) + (1+\beta)(1+c\gamma)] + (c+1)(\gamma+\beta) + c\beta(1-\gamma)}{n^2(1+\beta) + n[(\gamma+\beta) + (1+c)(1+\beta)] + (1+c)(\gamma+\beta) + c(1-\gamma)}.$$

The result is sharp for the function $f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+\gamma+2\beta)C_b^s(1)}(z-\xi)$.

Proof. Let $f(z) \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$. Then

$$F(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z-w} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n (z-\xi)^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\, d_n(\lambda,\beta,\delta) C_b^s(n)}{(c+n+1)(1-\delta)} a_n \leq 1. \tag{5.14}$$

Since $f \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$, we have

$$\sum_{n=1}^{\infty} \frac{d_n(\lambda,\beta,\gamma) C_b^s(n)}{(1-\gamma)(1-2\lambda)} a_n \leq 1.$$

Note that (5.14) is satisfied if

$$\frac{c\, d_n(\lambda,\beta,\delta) C_b^s(n)}{(c+n+1)(1-\delta)} \leq \frac{d_n(\lambda,\beta,\gamma) C_b^s(n)}{(1-\gamma)(1-2\lambda)}.$$

Solving for δ, we have

$$\delta \leq \frac{n^2(1+\beta) + n[(\gamma+\beta) + (1+\beta)(1+c\gamma)] + (c+1)(\gamma+\beta) + c\beta(1-\gamma)}{n^2(1+\beta) + n[(\gamma+\beta) + (1+c)(1+\beta)] + (1+c)(\gamma+\beta) + c(1-\gamma)} = \Phi(n).$$

A simple computation will show that $\Phi(n)$ is increasing and $\Phi(n) \geq \Phi(1)$. Using this, the results follows. □

6 Partial Sums

Silverman [22] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [22] and Cho and Owa [6] we will investigate the ratio of a function of the form (1.3) to its sequence of partial sums

$$f_k(z) = \frac{1}{z-\xi} + \sum_{n=1}^{k} a_n (z-\xi)^n \tag{6.15}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} d_n(\lambda,\beta,\gamma) C_b^s(n)\, a_n \leq (1-\gamma)(1-2\lambda).$$

More precisely we will determine sharp lower bounds for $\Re\left(\frac{f(z)}{f_k(z}\right)$ and $\Re\left(\frac{f_k(z)}{f(z}\right)$. In this connection we make use of the well known results that $\Re\left(\frac{1+w(z)}{1-w(z)}\right) > 0, \quad (z-\xi \in \Delta)$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n (z-\xi)^n$ satisfies the inequality $|w(z)| \leq |z-\xi|$.

Unless otherwise stated, we will assume that f is of the form (1.3) and its sequence of partial sums is denoted by (6.15).

Theorem 6.9. *Let $f(z) \in \mathcal{M}_b^s(\lambda,\beta,\gamma)$ be given by (1.3) satisfies condition, (2.2) and suppose that all of its partial sums (6.15) don't vanish in Δ. Moreover, suppose that*

$$2-2\sum_{n=1}^{k}|a_n| - \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum_{n=k+1}^{\infty}|a_n| > 0, for \;\; all \;\; k \in \mathbb{N}. \tag{6.16}$$

Then,

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)} \qquad (z-\xi \in \Delta) \tag{6.17}$$

where

$$d_n(\lambda,\beta,\gamma) \geq \begin{cases} (1-\gamma)(1-2\lambda), & if \quad n = 1,2,3,\ldots,k \\ d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1), & if \quad n = k+1,k+2,\ldots. \end{cases} \tag{6.18}$$

The result (6.17) is sharp with the function given by

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}(z-\xi)^{k+1}. \tag{6.19}$$

Proof. Define the function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\left[\frac{f(z)}{f_k(z)} - \left(1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}\right)\right] \\ &= 1 + \frac{\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum\limits_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}}{1+\sum\limits_{n=1}^{k} a_n(z-\xi)^{n+1}}. \end{aligned} \tag{6.20}$$

It suffices to show that $\Re(w(z)) > 0$, hence we find that

$$\left|\frac{1+w(z)}{1-w(z)}\right| \leq \frac{\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n| - \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\sum\limits_{n=k+1}^{\infty}|a_n|} \leq 1$$

From the condition (2.2),it readily yields the assertion (6.17) of Theorem 6.9.

To see that the function given by (6.19) gives the sharp result, we observe that for $z = re^{i\pi/(k+2)}$

$$\frac{f(z)}{f_k(z)} = 1 + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}(z-\xi)^n \to 1 - \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}$$

when $r \to 1^-$ which shows the bound (6.17) is the best possible for each $k \in \mathbb{N}$. □

We next determine bounds for $f_k(z)/f(z)$.

Theorem 6.10. *Under the assumptions of Theorem 6.9, we have*

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)+(1-\gamma)(1-2\lambda)} \qquad (z-w \in \Delta), \tag{6.21}$$

The result (6.21) is sharp with the function given by (6.19).

Proof. By setting

$$w(z) = \left(1 + \frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}\right)\left[\frac{f_k(z)}{f(z)} - \frac{\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}}{1+\frac{d_{k+1}(\lambda,\beta,\gamma)C_b^s(k+1)}{(1-\gamma)(1-2\lambda)}}\right]$$

proceeding as in Theorem 6.9, we get the desired result and so we omit the details. □

Remark 6.1. *We observe that, if we specialize the parameters λ and β as mentioned in Examples 1 and 2 , we obtain the analogous results for the classes $\mathcal{M}_b^s(\beta,\gamma)$ and $\mathcal{M}_b^s(\gamma)$. Further specializing the parameters s,b various other interesting results (as in Theorems 2.1 to 6.10) can be derived easily for the function class based on interesting integral operators. Further by taking $|\xi| = d$ and $|z-\xi| = r+d < 1$, one can easily prove analogous results as in Theorems 2.1 to 6.10. The details involved may be left as an exercise for the interested reader.*

7 Acknowledgment

The authors thank the referee for their valuable suggestions.

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, Ann. Math. Ser.2, 17(1915), 12–22.

[2] M. K. Aouf, On a certain class of meromorphic univalent functions with positive coefficients, Rend. Mat. Appl. 7(11)(2)(1991), 209-219.

[3] M.K.Aouf and G.Murugusundaramoorthy, A subclass of uniformly convex functions defined by the Dziok-Srivastava operator, Aust.J. Math. Anal. Appln., 5(1)(2008), 1–17.

[4] S.D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135(1969), 429–446.

[5] J.Choi and H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta function, Appl. Math. Comput.,170(2005), 399-409.

[6] N.E.Cho and S.Owa, Partial sums of certain meromorphic functions , JIPAM, 5(2)(2004), ???-???.

[7] J.Dziok, G.Murugusundaramoorthy, and J. Sokół, On certain class of meromorphic functions with positive coefficients, Acta Mathematica Scientia, 32B(4)(2012), 1–16.

[8] A.W.Goodman,On uniformly starlike functions, J. Math. Anal. Appl., 155(1991),364–370.

[9] A.W.Goodman,On uniformly convex functions, Ann. Polon. Math., 56(1)(1991), 87–92.

[10] I.B.Jung,Y.C.Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176(1993), 138-147.

[11] S. Kanas and F. Ronning, Uniformaly starlike and convex function and other related classes of univalent functions, Ann. Univ. Marie Curie-Sklodowaska, A53(1991), 95–105.

[12] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, Appl. Math. Comput., 154(2004), 725-733.

[13] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, Integral Transform. Spec. Funct., 17(2006), 817-827.

[14] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16(1969), 755–758.

[15] J. -L. Liu, Properties of some families of meromorphic p-valent functions, Math. Japon. 52(3)(2000), 425-434.

[16] J-L. Liu,Some inclusion properties for certain subclass of meromorphically multivalent functions involving the Srivastava-Attiya operator,Tamsui Oxford Journal of Information and Mathematical Sciences, 28(3)(2012) 267–279.

[17] J.-L. Liu and H.M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, Math. Comput. Modelling, 39(2004), 21-34.

[18] J.-L. Liu and H.M. Srivastava, Subclasses of meromorphically multivalent functions associated with a certain linear operator, Math. Comput.Modelling, 39 (2004) 35-44.

[19] M. L. Mogra, T. R. Reddy and O. P. Juneja,Meromorphic univalent functions with positive coefficients, Bull.Austral. Math. Soc., 32(1985), 161–176.

[20] S. Owa and N. Pascu, Coefficient inequalities for certain classes of meromorphically starlike and meromorphically convex functions, J. Inequal. Pure Appl. Math., 4(1)(2003), 1-6.

[21] Ch. Pommerenke, On meromorphic starlike functions, Pacific J. Math., 13(1963), 221-235.

[22] H. Silverman, Partial sums of starlike and convex functions, *J.Math. Anal. Appl.*, 209(1997), 221 - 227.

[23] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, Integral Transform. Spec. Funct., 18(2007), 207–216.

[24] H.M. Srivastava and J.Choi, Series associated with the Zeta and related functions (Kluwer Academic Publishers, Dordrecht, Boston, London, 2001).

[25] B. A.Uralegaddi and C. Somanatha, Certain differential operators for meromorphic functions, Houston J. Math., 17(2)(1991), 279-284.

Received: January 11, 2014; *Accepted*: January 22, 2014

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