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The extended tanh method for certain system of nonlinear ordinary differential equations

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Abstract

We propose a method to obtain Tanh-solution based on leading order analysis of Painlevè test. The crucial aspect is that this point of view gives "exactly truncation of the series expansion applicable to Tanh-method". This approach gives all possible leading orders of solutions. Each branches can be treated separately and obtained closed form solutions.

Keywords: Ordinary differential equations, Tanh-method, Singularity analysis.

2010 MSC: 02; 02.30.Ik; 02.30.Hq.

1 Introduction

For many years, nonlinearity is playing an important role in various fields of mathematics, physics and biology. Finding the exact solutions of the nonlinear ordinary differential equations and partial differential equations are quite difficult. So far, many methods have been proposed by many authors for finding exact solutions of nonlinear differential equations. We mentioned some of them here: tanh−expansion method [1] − [7], the simplest equation method [11], the Jacobi elliptic−function method [12], the modified simplest equation method [13], the exp−function method [14] − [16], the G'/G-expansion method [18] and application of the Hirota method for non integrable nonlinear differential equation [17]. Recently, Willy Malfliet et al. and Abdul−Majid WazWaz [7] have successfully refined the tanh method for solving a lot of systems of autonomous partial differential equations and obtained solutions of them successfully. For the first time, best of our knowledge, we employ this method directly to ordinary differential equations. Here, we implement the leading order analysis or ARS method to determine all leading orders in the expansion of all solutions of differential equations. We remind the readers that we are not going to test the Painlevé property here. Thus, the approach is equally applicable for both integrable and non-integrable differential equations. We truncate the expression looking at the leading term. That is, if the leading term starts with τ^{-p}, $p > 0$ then the expression terminates at τ^{p}. To find the full expression of this expansion, we determine the each coefficients of the expansion by comparing the various powers of ξ and obtain an over-determined system of algebraic equation for the unknowns. Solving them consistently, we can obtain the values of the coefficients uniquely. Thus, tanh solution is determined uniquely for a given equation. If there are more than one leading orders then each order will give the appropriate series solutions separately. Interestingly the present approach gives a concrete way of finding all leading terms. That is if a given equation admits more than one branch of solutions then it could be determined uniquely.

In this paper, we explain the extended tanh-method with all possible leading orders and apply to certain physically important problems.

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2 Review of leading order analysis of Painlevé test [9]

Let us consider the system of ordinary differential equations

$$f_1(x,y,z,\dot{x},\dot{y},\dot{z},....)=0, \tag{2.1}$$

$$f_2(x,y,z,\dot{x},\dot{y},\dot{z},....)=0, \tag{2.2}$$

$$f_3(x,y,z,\dot{x},\dot{y},\dot{z},....)=0, \tag{2.3}$$

where '˙' denotes derivative with respect to t. Assume that the leading order of the solutions are in the form

$$x\sim\tau^p, \tag{2.4}$$

$$y\sim\tau^q, \tag{2.5}$$

$$z\sim\tau^r, \tag{2.6}$$

where $p,\ q$ and r are the integers to be determined and $\tau=t-t_0$. Substituting Eqs.(2.4)-(2.6) into Eqs.(2.1)-(2.3) then equating the all dominant terms then we can get the all possible choices of $p,\ q$ and r. Some times we may get two or more choices of $p,\ q$ and r. We demonstrate these concepts with the following example

Example

Consider the third-order ordinary differential equation [9]

$$\dddot{x}+x\ddot{x}-2x^3+\lambda x^2+\alpha x+\beta=0. \tag{2.7}$$

Substituting Eq.(2.4) in Eq.(2.7) then we get

$$p(p-1)(p-2)\tau^{p-3}+p(p-1)\tau^{2p-2}-3\tau^{3p}\approx 0. \tag{2.8}$$

Equating the various powers of τ and find p as follows

1. $p-3=2p-2$ this implies $p=-1$
2. $2p-2=3p$ this implies $p=-2$.

Hence, there are two set of dominant terms $(\dddot{x},x\ddot{x})$ and $(x\ddot{x},x^3)$ which are balancing each other in Eq.(2.7) [9].

3 Review of extended Tanh-method [1] − [7]

Now we use the extended tanh-method [1] − [7] for finding the exact solutions of system of nonlinear autonomous ordinary differential equations. we introduce a new independent variable

$$\xi = \tanh(\mu t), \tag{3.9}$$

then (3.10)

$$\frac{d}{dt} = \mu(1-\xi^2)\frac{d}{d\xi}, \tag{3.11}$$

$$\frac{d^2}{dt^2} = -2\mu^2\xi(1-\xi^2)\frac{d}{d\xi}+\mu^2(1-\xi^2)^2\frac{d^2}{d\xi^2}, \tag{3.12}$$

$$\frac{d^3}{dt^3} = 2\mu^3(1-\xi^2)(3\xi^2-1)\frac{d}{d\xi}-6\mu^3\xi(1-\xi^2)^2\frac{d^2}{d\xi^2}+\mu^3(1-\xi^2)^3\frac{d^3}{d\xi^3}, \tag{3.13}$$

$$\frac{d^4}{dt^4} = -8\mu^4\xi(1-\xi^2)(3\xi^2-2)\frac{d}{d\xi}+4\mu^4(1-\xi^2)^2(9\xi^2-2)\frac{d^2}{d\xi^2} -12\mu^4\xi(1-\xi^2)^3\frac{d^3}{d\xi^3}+\mu^4(1-\xi^2)^4\frac{d^4}{d\xi^4}. \tag{3.14}$$

holds. Now consider the series expansion

$$\begin{aligned} x[t] &= X[\xi] = \sum_{i=-p}^{p} a_i \xi^i \\ y[t] &= Y[\xi] = \sum_{i=-q}^{q} b_i \xi^i \\ z[t] &= Z[\xi] = \sum_{i=-r}^{r} c_i \xi^i \end{aligned}$$

where p, q and r which were identified from leading order analysis.

4 Applications

4.1 Example

Consider the system of ODE [9]

$$\dddot{x} + x\ddot{x} - 2x^3 + \lambda x^2 + \alpha x + \beta = 0. \tag{4.15}$$

First, one has to change the given Eq.(4.15) in terms of new independent variable ξ by using Eqs.(3.11), (3.12) and (3.13). Thus, we obtain

$$\begin{aligned} &\mu^3\left(1-\xi^2\right)^3 x''' - 6\mu^3\xi\left(1-\xi^2\right)^2 x'' + x\left(-2\mu^2\xi\left(1-\xi^2\right)x' + \mu^2\left(1-\xi^2\right)^2 x''\right) \\ &\quad +2\mu^3\left(1-\xi^2\right)\left(-1+3\xi^2\right)x' - 2x^3 + \lambda x^2 + \alpha x + \beta = 0, \end{aligned} \tag{4.16}$$

where $''$ denote the derivatives with respect to new independent variable ξ.

Since, we have obtained two possible leading orders $p = -1$ and $p = -2$, it is evident that there are two branches of solutions exist for Eq.(4.15). we treat each case separately.

Case (a) p=-1:
We assume that the solution of the form

$$x[t] = X[\xi] = a_{-1}\xi^{-1} + a_0 + a_1\xi. \tag{4.17}$$

On substitution Eq.(4.17) into Eq.(4.16) and collecting the coefficients of various powers of ξ than we obtain a system of over-determined equations for a_i, where $i = -1,\ 0$ and 1.

$$\begin{aligned} -6\mu^3 a_{-1} + 2\mu^2 a_{-1}^2 &= 0, \\ -2a_{-1}^3 + 2\mu^2 a_{-1} a_0 &= 0, \\ 8\mu^3 a_{-1} + \lambda a_{-1}^2 - 2\mu^2 a_{-1}^2 - 6a_{-1}^2 a_0 + 2\mu^2 a_{-1} a_1 &= 0, \\ \alpha a_{-1} + 2\lambda a_{-1} a_0 - 2\mu^2 a_{-1} a_0 - 6a_{-1} a_0^2 - 6a_{-1}^2 a_1 &= 0, \\ \beta - 2\mu^3 a_{-1} + \alpha a_0 + \lambda a_0^2 - 2a_0^3 - 2\mu^3 a_1 + 2\lambda a_{-1} a_1 & \\ -4\mu^2 a_{-1} a_1 - 12 a_{-1} a_0 a_1 &= 0, \\ \alpha a_1 + 2\lambda a_0 a_1 - 2\mu^2 a_0 a_1 - 6a_0^2 a_1 - 6a_{-1} a_1^2 &= 0, \\ 8\mu^3 a_1 + 2\mu^2 a_{-1} a_1 + \lambda a_1^2 - 2\mu^2 a_1^2 - 6a_0 a_1^2 &= 0, \\ 2\mu^2 a_0 a_1 - 2a_1^3 &= 0, \\ -6\mu^3 a_1 + 2\mu^2 a_1^2 &= 0. \end{aligned} \tag{4.18}$$

Solving them consistently, we arrive at solutions of a_i where $i = -1,\ 0$ and 1. We tabulate the results in table(1).

Table 1: Case (a): p=-1

Cases	Values	Conditions	Solutions
i	$a_{-1} = 0, a_0 = 9,$ $a_1 = \pm\sqrt{\frac{3(\alpha+486)}{10}},$	$\beta = \frac{(-69984 - 108\alpha + \alpha^2)}{150},$ $\lambda = \frac{1944-\alpha}{45},$ $\mu = \pm\sqrt{\frac{\alpha+486}{30}}$	$x[t] = 9 + \sqrt{\frac{3(\alpha+486)}{10}} \tan\left[\sqrt{\frac{\alpha+486}{30}}t\right],$
ii	$a_{-1} = \pm\frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}},$ $a_0 = 9,$ $a_1 = \pm\frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}}$	$\beta = \frac{(-69984 - 108\alpha + \alpha^2)}{150},$ $\lambda = \frac{1944-\alpha}{45},$ $\mu = \pm\frac{1}{2}\sqrt{\frac{\alpha+486}{30}}$	$x[t] = 9 + \frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}} \cot\left[\frac{1}{2}\sqrt{\frac{\alpha+486}{30}}t\right]$ $+\frac{1}{2}\sqrt{\frac{3(\alpha+486)}{10}} \tan\left[\frac{1}{2}\sqrt{\frac{\alpha+486}{30}}t\right],$
iii	$a_{-1} = \pm\sqrt{\frac{3(\alpha+486)}{10}},$ $a_0 = 9,$ $a_1 = 0$	$\beta = \frac{(-69984 - 108\alpha + \alpha^2)}{150},$ $\lambda = \frac{1944-\alpha}{45},$ $\mu = \pm\sqrt{\frac{\alpha+486}{30}}$	$x[t] = 9 + \sqrt{\frac{3(\alpha+486)}{10}} \cot\left[\sqrt{\frac{\alpha+486}{30}}t\right],$

Case (b): p=-2

Assume the solution in the form

$$X[\xi] = a_{-2}\xi^{-2} + a_{-1}\xi^{-1} + a_0 + a_1\xi + a_2\xi^2, \tag{4.19}$$

On substitution Eq.(4.19) into Eq.(4.16) and collecting the coefficients of various powers of ξ than we obtain a system of over-determined equations for a_i where $i = -2, -1, 0, 1$ and 2. The solutions are given in the table(2).

Table 2: Case (b): p=-2

Cases	Values	Conditions	Solutions
i	$a_0 = \frac{88}{25}, a_{-1} = a_{-2} = 0,$ $a_2 = \frac{12}{25}, a_1 = \pm\frac{24}{25}$	$\beta = \frac{75392}{625}, \alpha = -\frac{58848}{625},$ $\lambda = 24, \mu = \mp\frac{2}{5}$	$x[t] = \frac{88}{25} - \frac{24}{25}\tan\left[\frac{2t}{5}\right] + \frac{12}{25}\tan^2\left[\frac{2t}{5}\right]$
ii	$a_0 = \frac{88}{25}, a_{-2} = \frac{12}{25},$ $, a_1 = a_2 = 0, a_{-1} = \pm\frac{24}{25}$	$\beta = \frac{75392}{625}, \alpha = -\frac{58848}{625},$ $\lambda = 24, \mu = \mp\frac{2}{5}$	$x[t] = \frac{88}{25} - \frac{24}{25}\cot\left[\frac{2t}{5}\right] + \frac{12}{25}\cot^2\left[\frac{2t}{5}\right]$
iii	$a_0 = \frac{94}{25}, a_{-2} = a_2 = \frac{3}{25},$ $a_{-1} = a_1 = \pm\frac{12}{25}$	$\beta = \frac{75392}{625}, \alpha = -\frac{58848}{625},$ $\lambda = 24, \mu = \mp\frac{1}{5}$	$x[t] = \frac{94}{25} - \frac{12}{25}\left(\cot\left[\frac{t}{5}\right] + \tan\left[\frac{t}{5}\right]\right)$ $+\frac{3}{25}\left(\cot^2\left[\frac{t}{5}\right] + \tan^2\left[\frac{t}{5}\right]\right)$
iv	$a_0 = \frac{468}{25}, a_{-2} = a_2 = \frac{-162}{25},$ $a_{-1} = a_1 = \pm\frac{36i\sqrt{6}}{25}$	$\beta = \frac{2239488}{625}, \alpha = -\frac{82944}{125}$ $\lambda = \frac{1656}{25}, \mu = \mp\frac{3i\sqrt{6}}{5}$	$x[t] = -\frac{36}{25}\sqrt{6}\left(\coth\left[\frac{3\sqrt{6}t}{5}\right] + \tanh\left[\frac{3\sqrt{6}t}{5}\right]\right)$ $+\frac{468}{25} + \frac{162}{25}\left(\coth^2\left[\frac{3\sqrt{6}t}{5}\right] + \tanh^2\left[\frac{3\sqrt{6}t}{5}\right]\right)$
v	$a_0 = \frac{792}{25}, a_{-2} = a_{-1} = 0,$ $a_2 = -\frac{648}{25}, a_1 = \pm\frac{72i\sqrt{6}}{25}$	$\beta = \frac{2239488}{625}, \alpha = -\frac{82944}{125}$ $\lambda = \frac{1656}{25}, \mu = \mp\frac{6i\sqrt{6}}{5}$	$x[t] = \frac{792}{25} + \frac{72}{25}\sqrt{6}\tanh\left[\frac{6\sqrt{6}t}{5}\right]$ $+\frac{648}{25}\tanh^2\left[\frac{6\sqrt{6}t}{5}\right]$
vi	$a_0 = \frac{792}{25}, a_{-1} = \frac{\pm 72i\sqrt{6}}{25}$ $a_{-2} = -\frac{648}{25}, a_1 = a_2 = 0$	$\beta = \frac{2239488}{625}, \alpha = -\frac{82944}{125}$ $\lambda = \frac{1656}{25}, \mu = \mp\frac{6i\sqrt{6}}{5}$	$x[t] = \frac{792}{25} - \frac{72}{25}\sqrt{6}\coth\left[\frac{6\sqrt{6}t}{5}\right]$ $+\frac{648}{25}\coth^2\left[\frac{6\sqrt{6}t}{5}\right]$

4.2 Fourth order equation

Consider the fourth order ODE [19]

$$x^{(4)} + x(\ddot{x} + \beta) - \frac{3}{4}\dot{x}^2 - 3(\alpha + 1) = 0,. \qquad (4.20)$$

In [19] expensive studies have been made from geometrical and numerical point of view. However, no exact analytical solutions been presented for Eq.(4.20). In this paper, we present a class of new exact closed form solutions for Eq.(4.20). Due to the importance of this equation from geometric point of view, we believe that the solutions presented here are significant in many ways. Painlené leading order analysis gives $p = -2$ for Eq.(4.20). On substitution this value into $X[\xi]$ and follow the tanh procedure then we tabulate the results below

Cases	Values	Conditions	Solutions
i	$a_0 = 5\sqrt{\frac{\beta}{21}}, a_{-1} = a_1 = 0,$ $a_{-2} = a_2 = -\frac{5}{2}\sqrt{\frac{3\beta}{7}}$	$\alpha = \frac{\left(-63 - 10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu = \pm\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t] = -\frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\coth^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$ $+\frac{5\sqrt{\beta}}{\sqrt{21}} - \frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\tanh^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$
ii	$a_{-1} = a_1 = 0, a_0 = 20\sqrt{\frac{\beta}{21}},$ $a_2 = -10\sqrt{\frac{3\beta}{7}}, a_{-2} = 0$	$\alpha = \frac{\left(-63 - 10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu = \pm\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t] = \frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\tanh^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
iii	$a_{-1} = a_1 = 0, a_0 = 20\sqrt{\frac{\beta}{21}},$ $a_{-2} = -10\sqrt{\frac{3\beta}{7}}, a_2 = 0$	$\alpha = \frac{\left(-63 - 10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu = \pm\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t] = \frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\coth^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
iv	$a_0 = -20\sqrt{\frac{\beta}{21}}, a_{-1} = 0,$ $a_2 = 10\sqrt{\frac{3\beta}{7}}, a_{-2} = a_1 = 0,$	$\alpha = \frac{\left(-63 + 10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu = \pm\frac{1}{2}i\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t] = -\frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\tan^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
v	$a_0 = -20\sqrt{\frac{\beta}{21}}, a_1 = a_2 = 0$ $a_{-2} = 10\sqrt{\frac{3\beta}{7}}, a_{-1} = 0,$	$\alpha = \frac{\left(-63 + 10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu = \pm\frac{1}{2}i\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t] = -\frac{20\sqrt{\beta}}{\sqrt{21}}$ $-10\sqrt{\frac{3}{7}}\sqrt{\beta}\cot^2\left[\frac{1}{2}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right].$
vi	$a_0 = -5\sqrt{\frac{\beta}{21}}, a_{-1} = a_1 = 0,$ $a_{-2} = a_2 = \frac{5}{2}\sqrt{\frac{3\beta}{7}}$	$\alpha = \frac{\left(-63 - 10\sqrt{21}\beta^{3/2}\right)}{63},$ $\mu = \pm\frac{1}{4}i\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}$	$x[t] = -\frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\cot^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$ $+\frac{5\sqrt{\beta}}{\sqrt{21}} - \frac{5}{2}\sqrt{\frac{3}{7}}\sqrt{\beta}\tan^2\left[\frac{1}{4}\left(\frac{3}{7}\right)^{1/4}\beta^{1/4}t\right]$

5 Conclusions

In this paper, we have successfully employed extended tanh-method by using leading order analysis of Painlevé test. Thus we could able to find all possible branches of solutions for the given differential equations. Also the choice of the leading term and truncation is indeed not arbitrary uniquely determined by the leading order analysis. Our method is successful to find large class of solutions of certain well-known systems. Finally, we remark that this approach can equally applied to nonintegrable systems as well including systems from Biology [20].

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