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Some results of Morse functions in digital images

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Abstract
In many years authors have adapted some notions of topology and combinatorial topology to the digital topology. In this paper we apply some definition of discrete Morse theory to the digital topology. We define a new definition of adjacency relation to show that digital subcomplexes are digitally homotopy equivalent. We conclude that if there is no digitally critical simplex in the digital interval $[m,n]_{\mathbb{Z}}$, then the digital subcomplexes $K(m)$ and $K(n)$ are digitally homotopy equivalent.

Keywords
digital morse theory, digital topology, digital simplicial complex.

AMS Subject Classification
55N99, 55U10, 68R05, 68U10.

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Contents

1. Introduction

The importance of digital image analysis has increased considerably with technological advances. Image analysis has a lot of applications in many fields such as industry, medicine and environmental sciences.

Since topological invariants are extremely useful in the digital image analysis and computer graphics, digital topology is constructed by benefits from topology and algebraic topology methods.

Digital topology is first introduced by Rosenfeld [17] and the concept of topology has become more useful with Rosenfeld's definition of an adjacency relation.

Boxer [2] has expanded the work of Rosenfeld [18] and defined digital analogs of specific types such as continuous functions, homeomorphisms and homotopy. Also, Boxer [3] has studied the digital homotopy first given by Kong [16] and obtained some properties of digital homotopy groups in [4].

Digital versions of the simplex concept in algebraic topology and many properties about simplicial complexes have been appeared in [1]. By analysing the topology of the simplicial complex, it is thought that a different approach can be brought to many problems in various mathematical fields. Moreover, many theories have been developed for smooth manifolds. One of them is a Morse Theory. Discrete Morse Theory has been given as a combinatorial adaptation of Morse Theory in [12]. Forman has discussed the discrete Morse Theory and applied it to new problems in his works [11] and [12].

In this study, we try to carry some basic definitions and theorems in discrete Morse Theory to digital topology. We shall try to apply some properties and theorems from discrete Morse Theory to digital topology. We give digital topology versions of some important properties and theorems about the theory in [12]. We cannot prove a digital analog of an important theorem and some lemmas about the homotopy equivalence of subcomplexes in discrete Morse theory in [12] by using known adjacency relations. In order to prove the theorem we define a new adjacency relation.

2. Preliminaries

Let $\mathbb{Z}$ denote the set of integers and $\mathbb{Z}^n$ be the set of lattice points in the Euclidean n-dimensional space. The pair (X,κ) is called a *digital image*, where X is a subset of $\mathbb{Z}^n$ and κ is an adjacency relation on X [3]. Several adjacency relations are used in the study of digital images. The following terminology is used in [16]. Two points p and q in $\mathbb{Z}^2$ are 8-*adjacent*, if they are distinct and differ by at most 1 in each coordinate; p and q in $\mathbb{Z}^2$ are 4-*adjacent*, if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathbb{Z}^3$ are 26-*adjacent*, if they are distinct and differ by at most 1 in each coordinate; they are 18-*adjacent*, if they are 26-adjacent and differ in at most two coordinates; they are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate. For an adjacency relation κ, a κ- *neighbor* of a lattice point $p \in \mathbb{Z}^n$ is a point which is κ-adjacent to p [14]. A digital image $X \subset \mathbb{Z}^n$ is κ-*connected*, where κ is an adjacency relation defined on $\mathbb{Z}^n$ if and only if for every pair of different points $x, y \in X$, there exists a set $\{x_0, x_1, \ldots, x_r\} \subset X$ such that $x = x_0$, $x_r = y$, and x_i and x_{i+1} are κ-neighbors, $i \in \{0, 1, \ldots, r-1\}$ [14].

For $a, b \in \mathbb{Z}$ with $a < b$ if the set is of the form

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\},$$

then it is called *a digital interval* in [3] in which 2-adjacency is given.

Definition 2.1. *[3] Let (X,κ) and (Y,λ) be digital images and $f : X \to Y$ be a function. We say f is (κ,λ)-continuous if the image under f of every κ-connected subset of X is λ-connected in Y.*

The following proposition is a simple result of Definition 2.1.

Proposition 2.2. [3] Let (X,κ) and (Y,λ) be digital images. A function $f : X \to Y$ is (κ,λ)-continuous if and only if for every κ-adjacent pair x_0, x_1 in X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are λ-adjacent in Y.

Definition 2.3. *[2] Let X and Y be digital images. Let $f : X \to Y$ be a digitally continuous function that is one to one and onto. If $f^{-1} : Y \to X$ is a digitally continuous function, then f is called a digital isomorphism, and we say X and Y are digitally isomorphic.*

In Definition 2.3 it is offered to use the terminology digitally isomorphism instead of digitally homeomorphic to avoid misunderstandings. We use the term homeomorphic for two closed interval which are subsets of Euclidean space since they are homeomorphic sets in algebraic topology sense. But their digital models may not be digitally homeomorphic. For example, consider two digital intervals $[0,2]_{\mathbb{Z}}$ and $[0,6]_{\mathbb{Z}}$. These are not digitally homeomorphic since their cardinalities are different.

The notion of the homotopy is modified in a digital setting as follows.

Definition 2.4. *[3] Let (X,κ) and (Y,λ) be digital images. We say two (κ,λ)-continuous functions $f, g : X \to Y$ are (κ,λ)-homotopic if there is a positive integer m and a function*

$$F : X \times [0,m]_{\mathbb{Z}} \to Y$$

such that

- *for all $x \in X$, $F(x,0) = f(x)$ and $F(x,m) = g(x)$;*
- *for all $x \in X$, the induced function $F_x : [0,m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$ is $(2,\lambda)$-continuous; and*
- *for all $t \in [0,m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x,t)$ for all $x \in X$ is (κ,λ)-continuous.*

In that case, F is a said to be a digital (κ,λ)-homotopy between f and g and if such a homotopy function exists between f and g, we write $f \simeq_{(\kappa,\lambda)} g$ for short.

In [3] it is mentioned that for the (κ,λ)-continuous function $f : X \to Y$, if there exists a (λ,κ)-continuous function $g : Y \to X$ such that

$$f \circ g \simeq_{(\lambda,\lambda)} 1_Y \quad \text{and} \quad g \circ f \simeq_{(\kappa,\kappa)} 1_X,$$

then we say digital images X and Y have the same (κ,λ)-homotopy type and these are (κ,λ)-homotopy equivalent.

Definition 2.5. *[1] Let S be a set of nonempty subset of a digital image (X,κ). Then the members of S are called simplices of (X,κ) if the following statements hold:*

1. *If p and q are distinct points of $s \in S$, then p and q are κ-adjacent,*
2. *If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.*

A *digital m-simplex* α^m, is a digital simplex $|\alpha^m| = m+1$. If α' is a nonempty proper subset of α^m, then α' is called a *digital face* of α^m. We write $Vert(\alpha^m)$ to denote the vertex set of α^m, namely, the set of all digital 0-simplexes in α^m.

Throughout this paper the digital p-simplex will be denoted by α^p.

Definition 2.6. *[1] Let (X,κ) be a finite collection of digital m-simplices, $0 \le m \le d$ for some non-negative integer d. Then we call (X,κ) as a finite digital simplicial complex if the following statements hold:*

1. *If P belongs to X, then every face of P also belongs to X,*
2. *If $P, Q \in X$, then $P \cap Q$ is either empty or common face of P and Q.*

Example 2.7. *The digital image (X,κ) on the left has two paths in Figure 1. One of them is between a and d, the other one is between c and f. So (X,κ) has only one 8-connected component. But (Y,λ) has two 8-connected components. Therefore, these digital images are not digitally homotopy equivalent.*

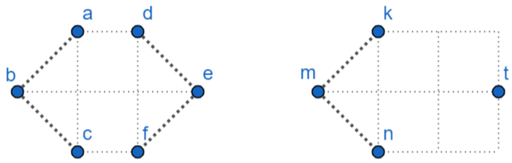

Figure 1

If we consider the usual adjacency relations in digital topology, then we see from Example 2.7 these two images are not digitally homotopy equivalent. However, we want to show that these digital images are homotopy equivalent in the sense of algebraic topology. In order to do it, we propose an alternative definition for the adjacency relation as follows:

Definition 2.8. *Let (X,κ) be a digital image. Consider a digital simplicial complex K whose digital 0-simplices are the elements of X. Then two distinct points $x, x^{'} \in Vert(K)$ are relatively κ-adjacent iff x and $x^{'}$ are two faces of some digital 1-simplex in K. We write $x \propto x^{'}$ if x and $x^{'}$ are relatively κ-adjacent.*

Note that if two points in a digital image are relatively κ-adjacent, they are also κ-adjacent.

Now according to the Definition 2.8 we can say $a, d \in (X,\kappa)$ in Figure 1 are not relatively κ-adjacent so (X,κ) also has two connected components. Hence these digital images are digitally homotopy equivalent.

Definition 2.9. *Let (X,κ) and (Y,λ) be digital images and (K,κ) and (L,λ) be two simplical complex whose digital 0-simplices are the elements of X and Y respectively. Then $f : Vert(K) \to Vert(L)$ is relatively (κ,λ)-continuous if for every $\{x_0, x_1\} \subset X$ such that x_0 and x_1 are relatively-κ-adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are relatively-λ-adjacent.*

There are no notions for 1-simplex and 2-simplex in digital topology, because of studying with discrete points. As a result, the concept of a simplicial complex in digital topology is quite different from the concept in algebraic topology.

The homotopy function to be established between two digital simplicial complexes is defined on vertices sets of these simplicial complexes via definition of relatively adjacency relation. We say that two simplicial complexes have the same homotopy type, when their set of vertices have the same homotopy type.

Now the definition of homotopy and homotopy equivalence will be given with respect to the definition of relatively adjacency.

Definition 2.10. *Let (X,κ) and (Y,λ) be digital images and (K,κ) and (L,λ) be two simplical complexes whose digital 0-simplices are the elements of X and Y, respectively. Let $f, g : Vert(K) \to Vert(L)$ be relatively-(κ,λ)-continuous functions and suppose there is a positive integer m and a function*

$$F : Vert(K) \times [0,m]_{\mathbb{Z}} \to Vert(L)$$

such that

- *for all $x \in K$, $F(x,0) = f(x)$ and $F(x,m) = g(x)$;*
- *for all $x \in K$, the induced function $F_x : [0,m]_{\mathbb{Z}} \to L$ defined by $F_x(t) = F(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$ is relatively-$(2,\lambda)$-continuous,*
- *for all $t \in [0,m]_{\mathbb{Z}}$, the induced function $F_t : K \to L$ defined by $F_t(x) = F(x,t)$ for all $x \in K$ is relatively-(κ,λ)-continuous.*

Then F is said to be digital relatively-(κ,λ)-homotopy between f and g and if such a homotopy function exists between f and g, we write $f \asymp_{(\kappa,\lambda)} g$ shortly.

Definition 2.11. *$f : K \to L$ be relatively-(κ,λ)-continuous function and let $g : L \to K$ be a relatively-(λ,κ)-continuous function such that*

$$f \circ g \asymp_{(\lambda,\lambda)} 1_K \quad \text{and} \quad g \circ f \asymp_{(\kappa,\kappa)} 1_L.$$

Then we say K and L have the same relatively-(κ,λ)-homotopy type and that X and Y are relatively-(κ,λ)-homotopy equivalent.

Note that if two digital images have the same relatively-(κ,λ)-homotopy type, they have also the same (κ,λ)-homotopy type.

According to Definition 2.8, a and d are not relatively κ-adjacent, because a and d are not faces of the same digital 1-simplex. Similarly, c and f are not relatively κ-adjacent. Therefore, the digital images (X,κ) and (Y,λ) are not relatively homotopy equivalent.

3. Digital Morse Function

The adaptation of the concept "discrete Morse function" into a digital case is based on [12]. In this section, we start with giving the digital version of the discrete Morse function and critical simplex in [12]. Then we present a few lemmas and propositions related to a critical simplex. Finally, we give the proof of the main result.

Remark 3.1. *For $n < m$, let α^n be a digital n-simplex and γ^m be a digital m-simplex. Note that whenever we state $\alpha^n \subset \gamma^m$ we mean that α^n is a face of γ^m.*

Definition 3.2. *A function*

$$\phi : (K,\kappa) \to \mathbb{R}$$

is a digital Morse function if for every digital p-simplex $\alpha^p \in (K,\kappa)$

$$\#\{\alpha^p \subset \beta^{p+1} \mid \phi(\beta^{p+1}) \leq \phi(\alpha^p)\} \leq 1$$

and

$$\#\{\gamma^{p-1} \subset \alpha^p \mid \phi(\gamma^{p-1}) \geq \phi(\alpha^p)\} \leq 1.$$

Example 3.3. *Consider the minimal simple closed curve* $(MSC_8^{'}, 8)$ *given in Figure 2 and a digital Morse function* $\phi : (K, \kappa) \to \mathbb{R}$ *where* $K = \{\{a\},\{b\},\{c\},\{d\},\{a,b\},\{b,c\},\{c,d\},\{a,d\}\}$ *is a simplicial complex whose 0-simplices are the elements of* $MSC_8^{'}$. *Then the following function*

$$\begin{aligned} \phi : (K,8) &\to \mathbb{R} \\ \{a\} &\mapsto 0 \\ \{b\} &\mapsto 2 \\ \{c\} &\mapsto 4 \\ \{d\} &\mapsto 6 \\ \{a,b\} &\mapsto 1 \\ \{b,c\} &\mapsto 3 \\ \{c,d\} &\mapsto 5 \\ \{a,d\} &\mapsto 7, \end{aligned}$$

is a digital Morse function. But if the function ϕ *maps the digital 0-simplex* $\{d\}$ *to 7 and the digital 1-simplex* $\{a,d\}$ *to 6, then it is not a digital Morse function since* $\{a,d\}$ *and* $\{c,d\}$ *have values less than 7.*

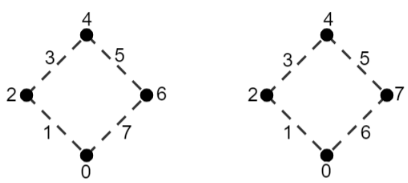

Figure 2. The figure on the left is an example of a digital Morse function while the figure on the right is not.

Example 3.4. *Consider the minimal simple closed curve* $(MSC_8, 8)$ *in Figure 3 and a digital Morse function* $\phi : (K, \kappa) \to \mathbb{R}$ *where* $K = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{a,c\},\{a,b\},\{b,d\},\{c,e\},\{d,f\},\{e,f\}\}$ *is a simplicial complex whose 0-simplices are the elements of* MSC_8. *If we define the function such that*

$$\begin{aligned} \phi : (K,8) &\to \mathbb{R} \\ \{a\} &\mapsto 0 \\ \{b\} &\mapsto 8 \\ \{c\} &\mapsto 3 \\ \{d\} &\mapsto 2 \\ \{e\} &\mapsto 9 \\ \{f\} &\mapsto 1 \\ \{a,c\} &\mapsto 5 \\ \{a,b\} &\mapsto 6 \\ \{b,d\} &\mapsto 11 \\ \{c,e\} &\mapsto 10 \\ \{d,f\} &\mapsto 4 \\ \{e,f\} &\mapsto 7, \end{aligned}$$

then it is a digital Morse function.

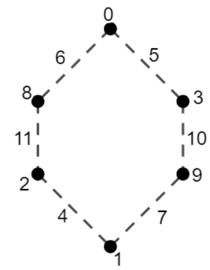

Figure 3. $(\mathrm{MSC}_8, 8)$

Example 3.5. *Consider the minimal simple closed curve* $(MSC_4, 8)$ *in Figure 3 and a digital Morse function* $\phi : (K, \kappa) \to \mathbb{R}$ *where* $K = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\},\{h\},\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,f\},\{f,g\},\{g,h\},\{a,h\}\}$ *is a simplicial complex whose 0-simplices are the elements of* MSC_4. *If we define the function such that*

$$\begin{aligned} \phi : (K,8) &\to \mathbb{R} \\ \{a\},\{c\},\{e\},\{g\} &\mapsto 4 \\ \{b\},\{f\} &\mapsto 6 \\ \{d\},\{h\} &\mapsto 1 \\ \{a,b\},\{e,f\} &\mapsto 5 \\ \{b,c\},\{f,g\}, &\mapsto 7 \\ \{a,h\},\{d,e\} &\mapsto 3 \\ \{g,h\},\{c,d\} &\mapsto 2, \end{aligned}$$

then it is a digital Morse function.

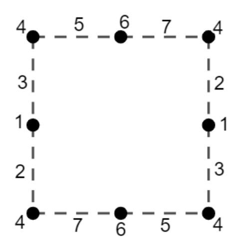

Figure 4. $(\mathrm{MSC}_4, 8)$

The following definition gives us the digital version of the critical simplex in [12].

Definition 3.6. *A digital p-simplex* α^p *is digitally critical if*

$$\#\{\alpha^p \subset \beta^{p+1} \mid \phi(\beta^{p+1}) \leq \phi(\alpha^p)\} = 0,$$

or

$$\#\{\gamma^{p-1} \subset \alpha^p \mid \phi(\gamma^{p-1}) \geq \phi(\alpha^p)\} = 0.$$

Note that in Figure 2 the digital 0-simplex $\{a\}$ is digitally critical since $\{a,b\}$ and $\{a,d\}$ have values greater than 0 in the digital image MSC_8 with the digital Morse function ϕ, in Figure 3 digital 0-simplices $\{a\}$, $\{c\}$, $\{d\}$,

$\{f\}$ and digital 1-simplices $\{d,f\}$, $\{a,c\}$, $\{c,e\}$, $\{b,d\}$ are digitally critical and in Figure 4 digital 0-simplices $\{b\}$, $\{c\}$, $\{d\}$, $\{h\}$ are digitally critical.

It follows from Definition 3.6 that a digital p-simplex α^p is not critical if and only if either of the following conditions holds [11]:

(i) $\exists \alpha^p \subset \beta^{p+1}$ such that $\phi(\beta^{p+1}) \leq \phi(\alpha^p)$

(ii) $\exists \gamma^{p-1} \subset \alpha^p$ such that $\phi(\gamma^{p-1}) \geq \phi(\alpha^p)$

Lemma 3.7. *If α is not a digitally critical p-simplex, then the conditions (i) and (ii) in the Definition 3.6 cannot be verified at the same time.*

Proof. Assume the both conditions holds. Let α^p is not a digitally critical p-simplex. From condition (i), there is a digital $(p+1)$-simplex β^{p+1} such that $\phi(\beta^{p+1}) \leq \phi(\alpha^p)$. Since ϕ is a digital Morse function, for another p-simplex α_1 which is a face of β^{p+1}, $\phi(\beta^{p+1}) > \phi(\alpha_1)$.

From the condition (ii), there is a digital $(p-1)$-simplex γ^{p-1} such that $\phi(\gamma^{p-1}) \geq \phi(\alpha^p)$. Since ϕ is a digital Morse function, for another p-simplex α_2 that one of faces is γ, $\phi(\gamma^{p-1}) < \phi(\alpha_2)$. γ^{p-1} is a face of α^p and α^p is a face of β^{p+1}, so there is a digital p-simplex which is face of β^{p+1} and has γ^{p-1} as a face. Let's denote this simplex by τ. Then we have,

$$\phi(\alpha) \leq \phi(\gamma) < \phi(\tau) < \phi(\beta) \leq \phi(\alpha)$$

which is a contradiction. □

Lemma 3.7 is a digital version of Lemma 2.5 [11] and it is useful for the proof of Theorem 3.10.

Lemma 3.8. *Let $\phi : (K,\kappa) \to \mathbb{R}$ be a digital Morse function and α be any digital simplex in X. If α attains a minimum value, then α is singleton.*

Proof. Assume that α^p is a digital p-simplex, where $p > 0$ and $\phi(\alpha^p)$ is a minimum value. From the definition of a digital Morse function, at most one of the $(p-1)$-faces of α^p has greater value than $\phi(\alpha^p)$. Since α^p is a digital p-simplex, the number of $(p-1)$-faces of α^p is $p+1$. So the rest p $(p-1)$-faces of α^p have values smaller than $\phi(\alpha^p)$. It contradicts to a minimum value of α^p, because α^p should have a minimum value. Thus, α must be a singleton. □

Proposition 3.9. *Let $\phi : (X,\kappa) \to \mathbb{R}$ be a digital Morse function and α be a digital simplex. If α is a face of a digital 1-simplex γ^1 and α attains a minimum value, then $\phi(\gamma^1)$ cannot be equal to $\phi(\alpha)$.*

Proof. Take α^0 as a face of a digital 1-simplex γ^1, and assume that $\phi(\gamma^1) = \phi(\alpha^0)$. Since γ^1 is a digital 1-simplex, it has exactly two 0-faces, α^0 and β^0.

By the definition of a digital Morse function, a value of only one of the faces of γ^1 can be greater than or equal to $\phi(\gamma^1)$. Since $\phi(\gamma^1) = \phi(\alpha^0)$, the condition holds and hence $\phi(\beta^0)$ must be smaller than $\phi(\gamma^1)$. From our assumption that $\phi(\gamma^1) = \phi(\alpha^0)$, $\phi(\beta^0) < \phi(\gamma^1) = \phi(\alpha^0)$ is obtained. This is a contradiction since α^0 attains a minimum value. So we have $\phi(\gamma^1) \neq \phi(\alpha^0)$. □

Lemma 3.8 and Proposition 3.9 play a crucial role in the proof of Theorem 3.10.

Theorem 3.10. *If a digital 0-simplex α^0 has a minimum value, then α^0 is critical.*

Proof. Assume that α^p is a digitally noncritical p-simplex. Then α^p satisfies exactly one of the followings by Lemma 3.7:

(i) $\exists \tau^{p+1} > \alpha^p$ with $\phi(\tau^{p+1}) \leq \phi(\alpha^p)$,

(ii) $\exists \upsilon^{p-1} < \alpha^p$ with $\phi(\upsilon^{p-1}) \geq \phi(\alpha^p)$.

Since α^0 is a digital 0-simplex, the condition (ii) cannot be hold. So we have only condition (i) that is there exists a digital 1-simplex τ^1 whose face is α^0 with $\phi(\tau^1) \leq \phi(\alpha^0)$. It can be considered in two separate cases: $\phi(\tau^1) = \phi(\alpha^0)$ or $\phi(\tau^1) < \phi(\alpha^0)$.

First, if $\phi(\tau^1) = \phi(\alpha^0)$, then we get a contradiction by Proposition 3.9. Also we know that $\phi(\alpha^0)$ has minimum value by hypothesis. So $\phi(\tau^1)$ cannot be less than $\phi(\alpha^0)$. As a result, α^0 should be critical. □

Definition 3.11. *For any digital simplicial complex (K,κ) with a digital Morse function $\phi : (K,\kappa) \to \mathbb{R}$, and any real number m, a digital subcomplex $(K(m),\kappa)$ is a digital subcomplex consisting of all digital simplices α of (K,κ) such that $\phi(\alpha) \leq m$ along with all their faces likewise in [12].*

Also in [12], a free face is defined as a face which is not the face of any other simplex in digital subcomplex $(K(m),\kappa)$.

Lemma 3.12. *Let $\phi : (K,\kappa) \to \mathbb{R}$ be a digital Morse function, $[m,n]_{\mathbb{Z}}$ be a digital interval such that $n = m+1$ and $K(m) \neq K(n)$ be subcomplexes. If there is no critical simplex in $[m,n]_{\mathbb{Z}}$, then $\gamma \in K(n) \setminus K(m)$ is a face of $\sigma \in \phi^{-1}(n)$ or $\gamma \in \phi^{-1}(n)$ is a digital t-simplex for $t \geq 1$.*

Proof. Let $K(m) \neq K(n)$ and let γ be a digital simplex such that $\gamma \in K(n) \setminus K(m)$. From the definition of digital subcomplex and the assumption $\gamma \in K(n) \setminus K(m)$, we obtain

$$\gamma \in K(n) = \bigcup_{\substack{\sigma \in K \\ \phi(\sigma) \leq n}} \bigcup_{\tau \leq \sigma} \tau \text{ and } \gamma \notin K(m) = \bigcup_{\substack{\sigma \in K \\ f(\sigma) \leq m}} \bigcup_{\tau \leq \sigma} \tau.$$

Since $\gamma \notin K(m)$, we have $\phi(\gamma) > m$ and there is no $\sigma \in K(m)$ such that $\phi(\sigma) \leq m$ and γ is not the face of σ. Since $\gamma \in K(n)$ we get $\phi(\gamma) \leq n$ or there is a $\sigma \in K(n)$ such that $\phi(\sigma) \leq n$ and γ is the face of σ. But we know that $\phi(\gamma) > m$ and there is no $\sigma \in K(m)$ such that $\phi(\sigma) \leq m$ and γ is not the face of σ, so $\phi(\gamma) = n$ or there is a $\sigma \in K(m)$ such that $\phi(\sigma) = n$. Therefore, $\gamma \in \phi^{-1}(n)$ or there is $\sigma \in \phi^{-1}(n)$ such that γ is a face of σ.

Assume that $\gamma \in \phi^{-1}(n)$ and let γ be a digital 0-simplex. Then γ is a separated point from $K(m)$. In this case, γ is a digitally critical simplex which gives a contradiction. So $\gamma \in \phi^{-1}(n)$ is a digital t-simplex for $t \geq 1$. $\square$

Lemma 3.13. *Let $\phi : (K,\kappa) \to \mathbb{R}$ be a digital Morse function, $[m,n]_{\mathbb{Z}}$ be a digital interval such that $n = m+1$, $K(m) \neq K(n)$ be digital subcomplexes and assume that there is no critical point in $[m,n]_{\mathbb{Z}}$. Then the number of relatively κ-connected subsets of $K(m)$ is equal to the number of relatively κ-connected subsets of $K(n)$.*

Proof. Let $K(m) = \bigcup_{i \in I} K_i$, $I = \{0,\ldots,k\}$, where K_i is a κ-connected subsets of $K(m)$. Assume that we add a new connected component to $K(m)$ and so construct $K(n)$. Lets call this component by K_{k+1}. Hence, $\alpha \in f^{-1}(n)$ is a t-simplex for $t \geq 1$ from Lemma 3.12. Since $\alpha \in f^{-1}(n)$ is not critical from hypothesis, it holds only one of the following conditions:

(i) $\exists \tau^{t+1} > \alpha \in f^{-1}(n)$ such that $f(\tau) \leq f(\alpha) = n$,

(ii) $\upsilon^{t-1} < \alpha \in f^{-1}(n)$ such that $f(\upsilon) \geq f(\alpha) = n$.

Assume that condition (i) holds. Then there exists a simplex $\tau^{t+1} > \alpha \in \phi^{-1}(n)$ such that $\phi(\tau) \leq \phi(\alpha) = n$. Since $\phi(\tau) \leq n$, the value of $\phi(\tau)$ is either n or less than or equal to m. If $\phi(\tau) \leq m$, the simplex τ would be in the subcomplex $K(m)$ by the definition of the $K(m)$. Since K_{i+1} is a disconnected component, this is a contradiction. If $\phi(\tau) = n$, then only one of the followings holds from Lemma 3.7:

$(i)^*$ $\exists \gamma^{t+2} > \tau$ such that $\phi(\gamma) \leq \phi(\tau) = n$,

$(ii)^{**}$ $\delta^t < \tau$ such that $\phi(\delta) \geq \phi(\tau) = n$.

The condition $(ii)^{**}$ already holds because the simplex $\alpha \in f^{-1}(n)$ was a t-simplex with the value n. Since the digital simplex τ is not critical, other t-faces of τ must have smaller value than n. Let δ be one of the these faces. Since $f(\delta) < n$, $\delta \in K(m)$. This is a contradiction.

Assume now that the condition (ii) holds. Then there exists a $\upsilon^{t-1} < \alpha = \phi^{-1}(n)$ such that $\phi(\upsilon) \geq \phi(\alpha) = n$. Since α is not critical, $t-1$-faces of α must have values smaller than n. Consequently, these faces are in $K(m)$. This is a contradiction. Then the number of the κ-connected component of $K(m)$ and $K(n)$ must be equal. $\square$

We can now express the digital version of the main theorem of discrete Morse theory in [12].

Theorem 3.14. *Let (K,κ) be a digital complex, $(K(m),\kappa)$ and $K((n),\kappa)$ be digital subcomplexes with digital interval $[m,n]$ such that $m < n$, $m,n \in \mathbb{Z}$. Consider a digital Morse function $\phi : (K,\kappa) \to \mathbb{R}$. If there is no digitally critical simplex in $[m,n]$, then the digital subcomplex $K(m)$ is digitally homotopy equivalent to the digital subcomplex $K(n)$.*

Proof. Take $n = m+1$. In this case to construct $K(n)$ from $K(m)$ we add points which are relatively κ-adjacent to some points in $K(m)$ from Lemma 3.13. Thus, the number of κ-connected components of two digital subcomplexes $K(m)$ and $K(n)$ will be same.

These added points must be relatively κ-adjacent to some points in $K(m)$. Hence, we can construct a (κ,κ)-homotopy equivalence between $K(m)$ and $K(n)$ as follows:

Let $X(m) = \mathrm{Ver}(K(m)) = \{q_j\}_{j \in J}$, $J = \{0,...,k\}$, and $X(n) = \mathrm{Ver}(K(n)) = X(m) \cup \{q_j\}_{j \in I}$, $I = \{k+1,...,l\}$. For the point q_j, if $j \in I$ there is an index $j' \in J$ such that q_j and q_j' are relatively κ-adjacent. This means $q_j \propto q_j'$. The function f is defined by

$$f : X(n) \to X(m)$$
$$q_j \mapsto \begin{cases} q_j & j \in J \\ q_j' & j \in I. \end{cases}$$

Consider points $q_{j_0}, q_{j_1} \in X(n)$ such that $q_{j_0} \propto q_{j_1}$. We have the following three cases for these points.

<u>Case 1</u>: Let $q_{j_0}, q_{j_1} \in X(m)$ be points such that $q_{j_0} \propto q_{j_1}$. From the definition of f, $f(q_{j_0}) = q_{j_0}$ and $f(q_{j_1}) = q_{j_1}$. So we have $f(q_{j_0}) \propto f(q_{j_1})$.

<u>Case 2</u>: Let $q_{j_0} \in X(m)$, $q_{j_1} \in X(n) \setminus X(m)$ be points such that $q_{j_0} \propto q_{j_1}$. Since $q_{j_0} \in X(m)$, we have $f(q_{j_0}) = q_{j_0}$ and since $q_{j_1} \in X(n) \setminus X(m)$, we get $f(q_{j_1}) = q_j'$. Hence, we obtain $f(q_{j_0}) \propto f(q_{j_1})$.

<u>Case 3</u>: For the points $q_{j_0}, q_{j_1} \in X(n) \setminus X(m)$ such that $q_{j_0} \propto q_{j_1}$, both of the points $f(q_{j_0})$ and $f(q_{j_1})$ are equal to a point q_j' by the definition of the function f so that $f(q_{j_0}) \propto f(q_{j_1})$. Thus, the function f is digitally continuous. The function g defined by

$$g : X(m) \to X(n)$$
$$q_j \mapsto q_j$$

is also digitally continuous. As $f \circ g = 1_{X(m)}$, it is enough to show that $g \circ f \simeq_{(\kappa,\kappa)} 1_{X_n}$. Define a function

$g \circ f : X(n) \to X(n)$ by $\quad q_j \mapsto \begin{cases} q_j & j \in J \\ q_j' & j \in I. \end{cases}$

A digital (κ,κ)-homotopy function

$$H : X(n) \times [0,1]_{\mathbb{Z}} \to X(n)$$

is defined by $H(q_j,0) = q_j = 1_{X(n)}(q_j)$ and $H(q_j,1) = g \circ f(q_j)$. For the point $q_j \in X(n)$ restriction of the digital (κ,κ)-homotopy function to t, $H_t(q_j)$ is (κ,κ)-continuous. The identity map and the maps f and g are digitally continuous, so $H_0(q_j) = H(q_j,0) = 1_{X(n)}(q_j)$ and $H_1(q_j) = g \circ f(q_j)$ are digitally continuous. Now, we shall show that restriction of the digital (κ,κ)-homotopy function to the point q_j, $H_{q_j}(t)$ for the point $q_j \in X(n)$, $t \in [0,1]_{\mathbb{Z}}$ is $(2,\kappa)$-continuous. Assume that the points 0 and 1 are 2-adjacent. Then we have two cases with respect to the point q_j:

- For $j \in J$, $H_{q_j}(t) = \{q_j\}$, $t \in [0,1]_{\mathbb{Z}}$ is digitally continuous,
- For $j \in I$, $H_{q_j}(t)$ is digitally continuous, because $H_{q_j}(0) = \{q_j\}$ and $H_{q_j}(1) = \{q_j^{'}\}$.

So we have $g \circ f \simeq_{(\kappa,\kappa)} 1_{X(n)}$. Therefore, the digital subcomplex $K(m)$ is digitally homotopy equivalent to the digital subcomplex $K(n)$ as desired. □

Example 3.15. *In this example we handle the digital minimal simple closed curve $MSC_8^{'}$ and the digital Morse function defined in Example 3.3. Since there is no*

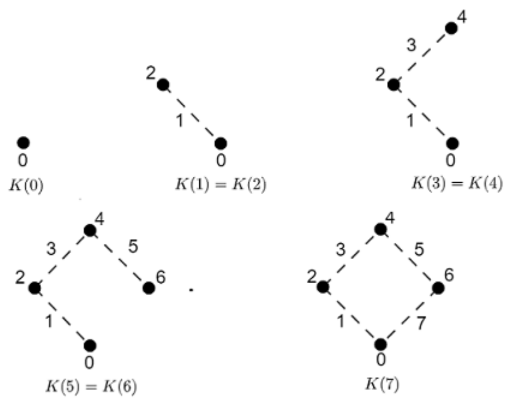

Figure 5. Subcomplexes of the digital image $\mathrm{MSC}_8^{'}$

digitally critical simplex in the digital interval $[2,5]_{\mathbb{Z}}$, we obtain $K(2) \simeq_{(\kappa,\kappa)} K(5)$ from Theorem 3.14. Also $K(a) \simeq_{(\kappa,\kappa)} K(b)$ for $a,b \in [1,6]_{\mathbb{Z}}$ from Theorem 3.14.

Example 3.16. *Consider the digital image X in Figure 6 and a digital Morse function $\phi : (K,\kappa) \to \mathbb{R}$ where K is a digital simplicial complex whose digital 0-simplices are the elements of this digital image. Let define the Digital Morse function as shown in Figure 6. So the digitally critical simplices are $\phi^{-1}(0)$, $\phi^{-1}(1)$, $\phi^{-1}(7)$, $\phi^{-1}(11)$, $\phi^{-1}(12)$ and the digital subcomplexes $K(a) \simeq_{(\kappa,\kappa)} K(b)$ for $a,b \in [2,6]_{\mathbb{Z}}$ or $a,b \in [8,10]_{\mathbb{Z}}$ from Theorem 3.14.*

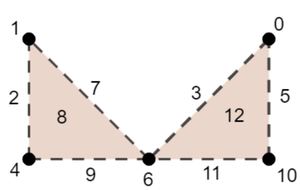

Figure 6. Digital image X

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