



Positive solutions for first-order nonlinear Caputo-Hadamard fractional differential equations

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Abstract

In this paper, we study the existence and uniqueness of positive solutions of the first-order nonlinear Caputo-Hadamard fractional differential equation

$$\begin{cases} \mathfrak{D}_1^\alpha (x(t) - g(t, x(t))) = f(t, x(t)), & 1 < t \leq e, \\ x(1) = x_0 > g(1, x_0) > 0, \end{cases}$$

where $0 < \alpha \leq 1$. In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct appropriate mappings and employ the Krasnoselskii and Banach fixed point theorems and the method of upper and lower solutions to show the existence and uniqueness of a positive solution of this equation. Finally, an example is given to illustrate our results.

Keywords

Fixed points, fractional differential equations, positive solutions, existence, uniqueness.

AMS Subject Classification

34A08, 34A12.

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1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have

received the attention of many authors, see [1]–[14], [16] and the references therein.

Zhang in [16] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), & 0 < t \leq 1, \\ x(0) = 0, \end{cases}$$

where D^α is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$ and $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional differential equation

$$\begin{cases} {}^C D^\alpha x(t) = f(t, x(t)) + {}^C D^{\alpha-1} g(t, x(t)), & 0 < t \leq T, \\ x(0) = \theta_1 > 0, x'(0) = \theta_2 > 0, \end{cases}$$

has been investigated in [5], where ${}^C D^\alpha$ is the standard Caputo's fractional derivative of order $1 < \alpha \leq 2$, $f, g : [0, T] \times [0, \infty) \rightarrow [0, \infty)$ are given continuous functions, g is nondecreasing on x and $\theta_2 \geq g(0, \theta_1)$. By employing the method of

the upper and lower solutions and Schauder and Banach fixed point theorems, the authors obtained positivity results.

In [2], Abdo, Wahash and Panchat discussed the existence and uniqueness of the positive solution of the following non-linear fractional differential equation with integral boundary conditions

$$\begin{cases} {}^C D^\alpha x(t) = f(t, x(t)), & 0 < t \leq 1, \\ x(0) = \lambda \int_0^1 x(s) ds + d, \end{cases}$$

where $0 < \alpha \leq 1, \lambda \geq 0, d > 0$ and $f: [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a given continuous function. By using the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the existence and uniqueness of solutions has been established.

Ahmad and Ntouyas in [4] studied the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_1^\alpha \left(\mathfrak{D}_1^\beta u(t) - g(t, u_t) \right) = f(t, u_t), & t \in [1, b], \\ u(t) = \phi(t), & t \in [1-r, 1], \\ \mathfrak{D}_1^\beta u(1) = \eta \in \mathbb{R}, \end{cases}$$

where \mathfrak{D}_1^α and \mathfrak{D}_1^β are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional differential equations. Inspired and motivated by the works mentioned above and the papers [1]–[14], [16] and the references therein, we concentrate on the positivity of solutions for the first-order nonlinear Caputo-Hadamard fractional differential equation

$$\begin{cases} \mathfrak{D}_1^\alpha (x(t) - g(t, x(t))) = f(t, x(t)), & 1 < t \leq e, \\ x(1) = x_0 > g(1, x_0) > 0, \end{cases} \quad (1.1)$$

where $0 < \alpha \leq 1, g, f: [1, e] \times [0, \infty) \rightarrow [0, \infty)$ are continuous. To show the existence and uniqueness of the positive solution, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use the Krasnoselskii and Banach fixed point theorems.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1.1) and the Banach and Krasnoselskii fixed point theorems. For details on the Banach and Krasnoselskii theorems we refer the reader to [15]. In Sections 3 and 4, we give and prove our main results on positivity and we provide an example to illustrate our results.

2. Preliminaries

Let $X = C([1, e])$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1, e]$, endowed with the maximum norm. Define the cone

$$\mathcal{E} = \{x \in X : x(t) \geq 0, \forall t \in [1, e]\}.$$

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [10, 14].

Definition 2.1 ([10]). *The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \rightarrow \mathbb{R}$ is defined as*

$$\mathfrak{J}_1^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}, \quad \alpha > 0.$$

Definition 2.2 ([10]). *The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \rightarrow \mathbb{R}$ is defined as*

$$\mathfrak{D}_1^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n(x)(s) \frac{ds}{s},$$

where $n-1 < \alpha < n, \delta^n = \left(t \frac{d}{dt}\right)^n, n \in \mathbb{N}$.

Lemma 2.3 ([10]). *Let $n-1 < \alpha \leq n, n \in \mathbb{N}$ and $x \in C^n([1, T])$. Then*

$$(\mathfrak{J}_1^\alpha \mathfrak{D}_1^\alpha x)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(1)}{\Gamma(k+1)} (\log t)^k.$$

Lemma 2.4 ([10]). *For all $\mu > 0$ and $\nu > -1$,*

$$\frac{1}{\Gamma(\mu)} \int_1^t \left(\log \frac{t}{s}\right)^{\mu-1} (\log s)^\nu \frac{ds}{s} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\log t)^{\mu+\nu}.$$

Lemma 2.5. *Let $x \in C([1, e]), x'$ and $\frac{\partial g}{\partial t}$ exist, then x is a solution of (1.1) if and only if*

$$\begin{aligned} x(t) &= x_0 - g(1, x_0) + g(t, x(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}, \end{aligned} \quad (2.1)$$

where $1 \leq t \leq e$.

Proof. Suppose x satisfies (1.1), then applying \mathfrak{J}_1^α to both sides of (1.1), we have

$$\mathfrak{J}_1^\alpha \mathfrak{D}_1^\alpha (x(t) - g(t, x(t))) = \mathfrak{J}_1^\alpha f(t, x(t)).$$

In view of Lemma 2.3 and the integral boundary condition, we get

$$\begin{aligned} x(t) &= x_0 - g(1, x_0) + g(t, x(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}. \end{aligned}$$

Conversely, suppose x satisfies (2.1), then applying \mathfrak{D}_1^α to both sides of (2.1), we obtain

$$\begin{aligned} \mathfrak{D}_1^\alpha x(t) &= \mathfrak{D}_1^\alpha (x_0 - g(1, x_0) + g(t, x(t))) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ &= \mathfrak{D}_1^\alpha g(t, x(t)) + \mathfrak{D}_1^\alpha \mathfrak{J}_1^\alpha f(t, x(t)) \\ &= \mathfrak{D}_1^\alpha g(t, x(t)) + f(t, x(t)). \end{aligned}$$



Then

$$\mathfrak{D}_1^\alpha (x(t) - g(t, x(t))) = f(t, x(t)).$$

Moreover, the boundary condition $x(1) = x_0$ holds. \square

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a positive solution of (1.1).

Definition 2.6. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{H} : X \rightarrow X$. The operator \mathcal{H} is a contraction operator if there is an $\lambda \in (0, 1)$ such that $x, y \in X$ imply

$$\|\mathcal{H}x - \mathcal{H}y\| \leq \lambda \|x - y\|.$$

Theorem 2.7 (Banach [15]). Let \mathcal{K} be a nonempty closed convex subset of a Banach space X and $\mathcal{H} : \mathcal{K} \rightarrow \mathcal{K}$ be a contraction operator. Then there is a unique $x \in \mathcal{K}$ with $\mathcal{H}x = x$.

Theorem 2.8 (Krasnoselskii fixed point theorem [15]). If \mathcal{K} is a nonempty bounded, closed and convex subset of a Banach space X , \mathcal{A} and \mathcal{B} two operators defined on \mathcal{K} with values in X such that

- i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{K}$, for all $x, y \in \mathcal{K}$,
- ii) \mathcal{A} is continuous and compact,
- iii) \mathcal{B} is a contraction.

Then there exists $z \in \mathcal{K}$ such that $z = \mathcal{A}z + \mathcal{B}z$.

3. Existence of positive solutions

In this section, we consider the results of existence problem for many cases of (1.1). Moreover, we introduce the following conditions.

(H1) $g, f : [1, e] \times [0, \infty) \rightarrow [0, \infty)$ are continuous functions and g is nondecreasing on x .

(H2) There exists $L_g \in (0, 1)$ such that

$$|g(t, x) - g(t, y)| \leq L_g |x - y|.$$

(H3) There exists $L_f > 0$ such that

$$|f(t, x) - f(t, y)| \leq L_f |x - y|.$$

We note that to apply Theorem 2.8 we need to construct two mappings, one is contraction and the other is completely continuous. Therefore, we express (2.1) as

$$x(t) = (\mathcal{A}x)(t) + (\mathcal{B}x)(t) = (\mathcal{H}x)(t), \tag{3.1}$$

where the operators $\mathcal{A}, \mathcal{B} : \mathcal{E} \rightarrow X$ are defined by

$$(\mathcal{A}x)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s},$$

and

$$(\mathcal{B}x)(t) = x_0 - g(1, x_0) + g(t, x(t)).$$

We need the following lemmas to establish our results.

Lemma 3.1. Assume that (H1) holds. Then, the operator $\mathcal{A} : \mathcal{E} \rightarrow \mathcal{E}$ is completely continuous.

Proof. By taking into account that f is continuous nonnegative function, we get that $\mathcal{A} : \mathcal{E} \rightarrow \mathcal{E}$ is continuous. The function $f : [1, e] \times B_\eta \rightarrow [0, \infty)$ is bounded, then there exists $\rho > 0$ such that $0 \leq f(t, x(t)) \leq \rho$, where $B_\eta = \{x \in \mathcal{E}, \|x\| \leq \eta\}$. We obtain

$$\begin{aligned} |(\mathcal{A}x)(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\ &\leq \frac{\rho}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{\rho (\log t)^\alpha}{\Gamma(\alpha + 1)}. \end{aligned}$$

Thus, $\|\mathcal{A}x\| \leq \frac{\rho}{\Gamma(\alpha + 1)}$. Hence, $\mathcal{A}(B_\eta)$ is uniformly bounded.

Now, we will prove that $\mathcal{A}(B_\eta)$ is equicontinuous. Let $x \in B_\eta$, then for any $t_1, t_2 \in [1, e], t_2 > t_1$, we have

$$\begin{aligned} &|(\mathcal{A}x)(t_1) - (\mathcal{A}x)(t_2)| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left| \left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1} \right| \\ &\quad \times |f(s, x(s))| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\ &\leq \frac{\rho}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1} \right) \frac{ds}{s} \\ &\quad + \frac{\rho}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{\rho}{\Gamma(\alpha + 1)} \left((\log t_1)^\alpha - (\log t_2)^\alpha + 2 \left(\log \frac{t_2}{t_1}\right)^\alpha \right) \\ &\leq \frac{2\rho}{\Gamma(\alpha + 1)} \left(\log \frac{t_2}{t_1}\right)^\alpha, \end{aligned}$$

which is independent of x and tends to zero as $t_2 \rightarrow t_1$. Thus that $\mathcal{A}(B_\eta)$ is equicontinuous. So, the compactness of \mathcal{A} follows by Ascoli Arzela's theorem. \square

Lemma 3.2. Assume that (H1) and (H2) hold. Then, the operator $\mathcal{B} : \mathcal{E} \rightarrow \mathcal{E}$ is a contraction.

Proof. By taking into account that g is continuous nonnegative function and $x_0 > g(0, x_0)$, we get that $\mathcal{B} : \mathcal{E} \rightarrow \mathcal{E}$. For $x, y \in \mathcal{E}$ we have

$$\begin{aligned} |(\mathcal{B}x)(t) - (\mathcal{B}y)(t)| &= |g(t, x(t)) - g(t, y(t))| \\ &\leq L_g \|x - y\|. \end{aligned}$$



Thus

$$\|\mathcal{B}x - \mathcal{B}y\| \leq L_g \|x - y\|.$$

Hence \mathcal{B} is a contraction. \square

Now for any $x \in [a, b] \subset \mathbb{R}^+$, we define respectively the upper and lower control functions as follows

$$H(t, x) = \sup_{a \leq y \leq x} f(t, y), \quad h(t, x) = \inf_{x \leq y \leq b} f(t, y).$$

It is clear that these functions are nondecreasing on $[a, b]$.

Definition 3.3. Let $\bar{x}, \underline{x} \in \mathcal{E}$, $a \leq \underline{x} \leq \bar{x} \leq b$, satisfying

$$\begin{aligned} \bar{x}(t) &\geq x_0 - g(1, x_0) + g(t, \bar{x}(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} H(s, \bar{x}(s)) \frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} \underline{x}(t) &\leq x_0 - g(1, x_0) + g(t, \underline{x}(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s, \underline{x}(s)) \frac{ds}{s}. \end{aligned}$$

Then the functions \bar{x} and \underline{x} are called a pair of upper and lower solutions for the equation (1.1).

Theorem 3.4. Assume that (H1) and (H2) hold and \bar{x} and \underline{x} are respectively upper and lower solutions of (1.1), then the problem (1.1) has at least one positive solution.

Proof. Let

$$\mathcal{H} = \{x \in \mathcal{E}, \underline{x}(t) \leq x(t) \leq \bar{x}(t), t \in [1, e]\}.$$

As $\mathcal{H} \subset E$ and \mathcal{H} is a nonempty bounded, closed and convex subset. By Lemma 3.1, $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{E}$ is completely continuous. Also, from Lemma 3.2, $\mathcal{B} : \mathcal{H} \rightarrow \mathcal{E}$ is a contraction. Next, we show that if $x, y \in \mathcal{H}$, we have $\mathcal{A}x + \mathcal{B}y \in \mathcal{H}$. For any $x, y \in \mathcal{H}$, we have $\underline{x} \leq x, y \leq \bar{x}$, then

$$\begin{aligned} &(\mathcal{A}x)(t) + (\mathcal{B}y)(t) \\ &= (x_0 - g(1, x_0)) + g(t, y(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ &\leq x_0 - g(1, x_0) + g(t, \bar{x}(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} H(s, \bar{x}(s)) \frac{ds}{s} \\ &\leq \bar{x}(t), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &(\mathcal{A}x)(t) + (\mathcal{B}y)(t) \\ &= x_0 - g(1, x_0) + g(t, y(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ &\geq x_0 - g(1, x_0) + g(t, \underline{x}(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s, \underline{x}(s)) \frac{ds}{s} \\ &\geq \underline{x}(t). \end{aligned} \tag{3.3}$$

Thus, from (3.2) and (3.3), we obtain that $\mathcal{A}x + \mathcal{B}y \in \mathcal{H}$. We now see that all the conditions of the Krasnoselskii's fixed point theorem are satisfied. Thus there exists a fixed point x in \mathcal{H} . Therefore, the problem (1.1) has at least one positive solution x in \mathcal{H} . \square

Corollary 3.5. Assume that (H1) and (H2) hold and there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$ such that

$$\lambda_1 \leq g(t, x) \leq \lambda_2, \quad (t, x) \in [1, e] \times [0, +\infty), \tag{3.4}$$

and

$$\lambda_3 \leq f(t, x) \leq \lambda_4, \quad (t, x) \in [1, e] \times [0, +\infty). \tag{3.5}$$

Then the problem (1.1) has at least one positive solution $x \in X$, moreover

$$x(t) \geq x_0 - g(1, x_0) + \lambda_1 + \lambda_3 \frac{(\log t)^\alpha}{\Gamma(\alpha + 1)}, \tag{3.6}$$

and

$$x(t) \leq x_0 - g(1, x_0) + \lambda_2 + \lambda_4 \frac{(\log t)^\alpha}{\Gamma(\alpha + 1)}. \tag{3.7}$$

Proof. From (3.5) and the definition of control functions, we have

$$\lambda_3 \leq h(t, x) \leq H(t, x) \leq \lambda_4. \tag{3.8}$$

Now, let

$$\begin{aligned} \bar{x}(t) &= x_0 - g(1, x_0) + \lambda_2 + \lambda_4 \frac{(\log t)^\alpha}{\Gamma(\alpha + 1)} \\ &= x_0 - g(1, x_0) + \lambda_2 + \frac{\lambda_4}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}. \end{aligned}$$

Taking into account (3.4) and (3.8), we have

$$\begin{aligned} \bar{x}(t) &= x_0 - g(1, x_0) + \lambda_2 + \frac{\lambda_4}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\geq x_0 - g(1, x_0) + g(t, \bar{x}(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} H(s, \bar{x}(s)) \frac{ds}{s}. \end{aligned}$$

It is clear that \bar{x} is the upper solution of (1.1).

Now, let

$$\begin{aligned} \underline{x}(t) &= x_0 - g(1, x_0) + \lambda_1 + \lambda_3 \frac{(\log t)^\alpha}{\Gamma(\alpha + 1)} \\ &= x_0 - g(1, x_0) + \lambda_1 + \frac{\lambda_3}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}. \end{aligned}$$

By (3.4) and (3.8), we obtain

$$\begin{aligned} \underline{x}(t) &= x_0 - g(1, x_0) + \lambda_1 + \frac{\lambda_3}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq x_0 - g(1, x_0) + g(t, \underline{x}(t)) \\ &\quad + \frac{\lambda_3}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s, \underline{x}(s)) \frac{ds}{s} \end{aligned}$$



Then, \underline{x} is the lower solution of (1.1). Therefore, from Theorem 3.4, we conclude that the problem (1.1) has at least one positive solution $x \in \mathcal{E}$ which verifies the inequalities (3.6) and (3.7). \square

4. Uniqueness of positive solution

In this section, we shall prove the uniqueness of the positive solution using the contraction mapping principle.

Theorem 4.1. Assume that (H1) – (H3) hold and

$$L_g + \frac{L_f}{\Gamma(\alpha + 1)} < 1, \tag{4.1}$$

then the problem (1.1) has a unique positive solution $x \in \mathcal{X}$.

Proof. From Theorem 3.4, it follows that (1.1) has at least one positive solution in \mathcal{X} . Hence, we need only to prove that the operator \mathcal{H} defined in (3.1) is a contraction on X . In fact, since for any $x_1, x_2 \in \mathcal{X}$, (H2) and (H3) are verified, then we have

$$\begin{aligned} & |(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)| \\ & \leq |g(t, x_1(t)) - g(t, x_2(t))| \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, x_1(s)) - f(s, x_2(s))| \frac{ds}{s} \\ & \leq L_g \|x_1 - x_2\| + \frac{(\log t)^\alpha}{\Gamma(\alpha + 1)} L_f \|x_1 - x_2\| \\ & \leq \left(L_g + \frac{L_f}{\Gamma(\alpha + 1)}\right) \|x_1 - x_2\|. \end{aligned}$$

Thus,

$$\|\mathcal{H}x_1 - \mathcal{H}x_2\| \leq \left(L_g + \frac{L_f}{\Gamma(\alpha + 1)}\right) \|x_1 - x_2\|.$$

Hence, the operator \mathcal{H} is a contraction mapping by (4.1). Therefore, by the contraction mapping principle, we conclude that the problem (1.1) has a unique positive solution $x \in \mathcal{X}$. \square

Finally, we give an example to illustrate our results.

Example 4.2. We consider the following nonlinear fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_1^{\frac{1}{3}} \left(x(t) - \frac{x(t)+2}{x(t)+3}\right) = \frac{1}{e+t} \left(\frac{tx(t)}{x(t)+1} + e\right), & 1 < t \leq e, \\ x(1) = 1, \end{cases} \tag{4.2}$$

where $x_0 = 1$, $g(t, x) = \frac{x+2}{x+3}$, $f(t, x) = \frac{1}{e+t} \left(\frac{tx}{x+1} + e\right)$ and $g(1, x_0) = \frac{3}{4}$. Since g is nondecreasing on x and

$$\frac{2}{3} \leq g(t, x) \leq 1, \quad \frac{1}{2} \leq f(t, x) \leq 1,$$

for $(t, x) \in [1, e] \times [0, \infty)$, hence by Corollary 3.5, (4.2) has a positive solution which verifies $\underline{x}(t) \leq x(t) \leq \bar{x}(t)$ where

$$\bar{x}(t) = \frac{5}{4} + \frac{(\log t)^\alpha}{\Gamma(4/3)} \text{ and } \underline{x}(t) = \frac{11}{12} + \frac{(\log t)^\alpha}{2\Gamma(4/3)},$$

are respectively the upper and lower solutions of (4.2). Also, we have

$$L_g + \frac{L_f}{\Gamma(\alpha + 1)} \simeq 0.671 < 1,$$

then by Theorem 4.1, (4.2) has a unique positive solution which is bounded by \underline{x} and \bar{x} .

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