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New oscillation criteria for forced superlinear neutral type differential equations

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Abstract

Some new oscillation criteria are established for the neutral type differential equation

$$(a(t)((x(t)+p(t)x(\tau(t)))')^{\alpha})'+q(t)x^{\beta}(t)=e(t),\ t\geq t_0,$$

which are applicable to equations with nonnegative forcing term. Examples are provided to illustrate the results.

Keywords: Neutral differential equation, second order, oscillation, superlinear.

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1 Introduction

Consider the forced second order neutral type differential equation of the form

$$(a(t)((x(t)+p(t)x(\tau(t)))')^{\alpha})'+q(t)x^{\beta}(t)=e(t),\quad t\geq t_0, \tag{1.1}$$

where $\alpha>0,\ \beta>0$ are the quotient of odd positive integers, $a(t),p(t),q(t),\tau(t),$ $e(t)\in C([t_0,\infty))$ and $a(t)>0,\ \int\limits_{t_0}^{\infty}\frac{1}{a^{\frac{1}{\alpha}}(t)}dt=\infty,\quad 0\leq p(t)\leq p<1,\quad q(t)>0, e(t)\geq 0,\ \tau(t)\leq t,\ \tau'(t)\geq 0$ and $\lim\limits_{t\to\infty}\tau(t)=\infty.$

Set $z(t)=x(t)+px(\tau(t))$. By a solution of equation (1.1) we mean a function $x(t)\in C([T_x,\infty)),\ T_x\geq t_0$, which has the properties $z(t)\in C^1([T_x,\infty)),\ a(t)(z'(t))^{\alpha}\in C^1([T_x,\infty))$, and satisfies equation (1.1) on $[T_x,\infty)$.
We consider only those solutions $x(t)$ of equation (1.1) which satisfy $\sup\{|x(t)|:t\geq T\}>0$ for all $T\geq T_x$. We assume that equation (1.1) possess such a solution. A solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[t_x,\infty)$ and otherwise it is said to be nonoscillatory. Also a solution $x(t)$ is said to be almost oscillatory if either $x(t)$ is oscillatory or $x'(t)$ is oscillatory or $x(t)\to 0$ as $t\to\infty$.

When $p(t)=0$ and $\alpha=1$ then equation (1.1) reduces to the following equation

$$(a(t)x'(t))'+q(t)x^{\beta}(t)=e(t),\quad t\geq t_0. \tag{1.2}$$

The oscillatory behavior of solutions of equation (1.2) has been discussed in many papers, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the references cited therein. In [2, 14], the authors studied oscillatory behavior of equation (1.1) or (1.2) with the assumption that $e(t)$ changes sign and therefore in this paper we establish conditions for the oscillatory behavior of equation (1.1) when $e(t)$ does not changes sign.

In Section 2, we present some oscillation criteria for equation (1.1) and in Section 3, we provide several examples to illustrate our main results.

In the sequel, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

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2 Oscillation Results

We begin with a lemma which can be easily proved using differential calculus.

Lemma 2.1. *Set* $F(x) = ax^{\beta-\alpha} + \frac{b}{x^{\alpha}}$ *for* $x > 0$. *If* $a \geq 0$, $b \geq 0$ *and* $\beta > \alpha \geq 1$ *then* $F(x)$ *attains its minimum with*

$$F_{min} = \frac{\beta a^{\frac{\alpha}{\beta}} b^{1-\frac{\alpha}{\beta}}}{\alpha^{\frac{\alpha}{\beta}} (\beta-\alpha)^{1-\frac{\alpha}{\beta}}}.$$

Theorem 2.1. *Assume that there exists a real valued positive function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)Q^*(s) - \frac{a(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right) ds = \infty, \tag{2.1}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\int_{t_0}^{s} (Mq(u) \pm e(u))du\right) ds = \infty \tag{2.2}$$

where

$$Q(t) = \frac{\beta q^{\frac{\alpha}{\beta}}(t) e^{1-\frac{\alpha}{\beta}}(t)(1-p)^{\alpha}}{\alpha^{\frac{\alpha}{\beta}} (\beta-\alpha)^{1-\frac{\alpha}{\beta}}},$$

$$Q^*(t) = \min\{Q(t), d^{(\beta-\alpha)} q(t)(1-p)^{\beta} - d^{-\alpha} e(t)\},$$

$M > 0$ *and* $d > 0$. *Then every solution of equation* (1.1) *is almost oscillatory.*

Proof. Suppose that $x(t)$ is not almost oscillatory.Then there is a positive solution of equation (1.1) such that $x(\tau(t)) > 0$ and $x(t) > 0$ for all $t \geq t_1 \geq t_0$. Then by the definition of not almost oscillatory there are two possibilities to consider: (I) $x'(t) > 0$ for all $t \geq t_1$ and (II) $x'(t) < 0$ for all $t \geq t_1$.

Case (I). Assume that $x'(t) > 0$ for all $t \geq t_1$. Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{2.3}$$

then $z'(t) > 0$ for all$t \geq t_1$, and $x(t) \geq (1-p)z(t)$. Then from equation (1.1), we have

$$(a(t)(z'(t))^{\alpha})' + q(t)(1-p)^{\beta} z^{\beta}(t) \leq e(t). \tag{2.4}$$

Define

$$w(t) = \frac{\rho(t)a(t)(z'(t))^{\alpha}}{z^{\alpha}(t)}, \quad t \geq t_1. \tag{2.5}$$

Then inview of (2.4), we obtain

$$w'(t) \leq -\rho(t)\left(q(t)(1-p)^{\beta} z^{\beta-\alpha}(t) - \frac{e(t)}{z^{\alpha}(t)}\right) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha}{(a(t)\rho(t))^{\frac{1}{\alpha}}} w^{1+\frac{1}{\alpha}}(t). \tag{2.6}$$

Set $F(u) = q(t)(1-p)^{\beta} u^{(\beta-\alpha)} - \frac{e(t)}{u^{\alpha}}$. Then, since u is increasing , there is a constant $d > 0$ such that $u \geq d > 0$ and

$$F(u) \geq d^{\beta-\alpha}(1-p)^{\beta} q(t) - d^{-\alpha} e(t). \tag{2.7}$$

Using the inequality

$$Bu - Au^{1+\frac{1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}, \ A > 0, \tag{2.8}$$

we have

$$\frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha}{(a(t)\rho(t))^{\frac{1}{\alpha}}} w^{1+\frac{1}{\alpha}}(t) \leq \frac{a(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}. \tag{2.9}$$

From (2.6),(2.7) and (2.9), we have

$$w'(t) \leq -\left[\rho(t)Q^*(t) - \frac{a(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}\right]. \tag{2.10}$$

Integrating (2.10) from t_1 *to* t , we obtain

$$\int_{t_1}^{t}\left(\rho(s)Q^*(s)-\frac{a(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right)ds \le w(t_1)-w(t)\le w(t_1)$$

for all large t, and this contradicts (2.1). Next, assume $x(t)<0$ for all $t\ge t_1$, and we use the transformation $y(t)=-x(t)$, then we have $y(t)$ is an eventually positive solution of the equation

$$(a(t)((y(t)+p(t)y(\tau(t)))')^{\alpha})'+q(t)y^{\beta}(t)=-e(t).$$

Define

$$w(t)=\rho(t)\frac{a(t)(z'(t))^{\alpha}}{z^{\alpha}(t)},t\ge t_1, \tag{2.11}$$

where $z(t)=y(t)+p(t)y(\tau(t))$. Then $w(t)>0$ and satisfies

$$w'(t)\le-\rho(t)\left(q(t)(1-p)^{\beta}z^{\beta-\alpha}(t)+\frac{e(t)}{z^{\alpha}(t)}\right)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\alpha w^{1+\frac{1}{\alpha}}(t)}{(a(t)\rho(t))^{\frac{1}{\alpha}}}. \tag{2.12}$$

Set $F(u)=q(t)(1-p)^{\beta}u^{\beta-\alpha}+\frac{e(t)}{u^{\alpha}}$. Using Lemma 2.1, we see that

$$F(u)\ge\frac{\beta q^{\frac{\alpha}{\beta}}(t)e^{1-\frac{\alpha}{\beta}}(t)}{\alpha^{\frac{\alpha}{\beta}}(\beta-\alpha)^{1-\frac{\alpha}{\beta}}}(1-p)^{\alpha}$$

and also (2.8) holds. Then the rest of the proof is similar to that of the above and hence is omitted.

Case (II). Assume that $x'(t)$ is negative for all $t\ge t_1$. From the definition of $z(t)$ we obtain $z'(t)=x'(t)+px'(\tau(t))\tau'(t)$. Since $p\ge 0$ and $\tau'(t)>0$ we have $z'(t)<0$ for all $t\ge t_1$. From $x'(t)<0$ we obtain $\lim_{t\to\infty}x(t)=b$. We assert that $b=0$. If not then $x^{\beta}(t)\to b^{\beta}>0$ as $t\to\infty$, and hence there exists a $t_2\ge t_1$ such that $x^{\beta}(t)\ge b^{\beta}$ for $t\ge t_2$. Therefore, we have

$$(a(t)(z'(t))^{\alpha})'\le -q(t)b^{\beta}+e(t).$$

Integrating the last inequality from t_2 *to* t, we obtain

$$a(t)(z'(t))^{\alpha}<a(t)(z'(t))^{\alpha}-a(t_2)(z'(t_2))^{\alpha}\le-\int_{t_2}^{t}(b^{\beta}q(s)-e(s))ds$$

and then

$$z'(t)\le-\left(\frac{1}{a(t)}\int_{t_2}^{t}(b^{\beta}q(s)-e(s))ds\right)^{\frac{1}{\alpha}},\quad t\ge t_2.$$

Again integrating the above inequality from t_2 to t, we obtain

$$z(t)\le z(t_2)-\int_{t_2}^{t}\left(\frac{1}{a(s)}\int_{t_2}^{s}(b^{\beta}q(u)-e(u))du\right)^{\frac{1}{\alpha}}ds.$$

Condition (2.2) implies that $z(t)$ is negative for all $t\ge t_2$, a contradiction. Finally, for $x(t)<0$ for all $t\ge t_1$, we use the transformation $y(t)=-x(t)$ then we have $y(t)$ is an eventually positive solution of the equation

$$(a(t)(z'(t))^{\alpha})'+q(t)y^{\beta}(t)=-e(t)$$

where $z(t)=y(t)+p(t)y(\tau(t))>0$. The rest of the proof is similar to the above and hence omitted. The proof is now complete. □

Corollary 2.1. *Assume that all the conditions of Theorem 2.2 hold, except the condition* (2.1) *is replaced by*

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\rho(s)Q^*(s)ds=\infty,$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{a(s)(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}ds<\infty.$$

Then every solution of equation (1.1) *is almost oscillatory.*

In the following theorem, we provide another sufficient condition for almost oscillation of equation (1.1).

Definition 2.1. *Consider the sets* $D_0=\{(t,s):t>s\geq t_0\}$ *and* $D=\{(t,s):t\geq s\geq t_0\}$. *Assume that* $H\in C(D,R)$ *satisfies the following assumptions:*
(A_1) $H(t,t)=0,\quad t\geq t_0;\quad H(t,s)>0,\quad (t,s)\in D_0;$
(A_2) H *has a nonpositive continuous partial derivative with respect to the second variable in* D_0.
Then the function H *has the property* P.

Theorem 2.2. *Assume that condition* (2.2) *holds. Further assume that* $H\in C(D,R)$ *has the property* P *and there exists a function* $\rho\in C'([t_0,\infty),(0,\infty))$ *such that for all sufficiently large* $t_1\geq t_0$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\rho(s)Q^*(s)-\frac{a(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'(s)}{\rho(s)}H^{\frac{1}{\alpha+1}}(t,s)-h(t,s)\right)^{\alpha+1}\right]ds=\infty, \tag{2.13}$$

where $h(t,s)=\frac{1}{H^{\frac{\alpha}{\alpha+1}}(t,s)}\frac{\partial}{\partial s}H(t,s),\quad (t,s)\in D_0$. *Then every solution of equation* (1.1) *is almost oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1 we have two cases to consider. First assume that $x'(t)>0$ for all $t\geq t_1$. Define $w(t)$ by (2.5) , then $w(t)>0$ and satisfies

$$w'(t)\leq-\rho(t)Q^*(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\alpha}{(a(t)\rho(t))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(t). \tag{2.14}$$

In (2.14), replace t by s and then multiply both sides by $H(t,s)$, and integrate with respect to s from t_1 to t, we have

$$\int_{t_1}^{t}H(t,s)\rho(s)Q^*(s)ds\leq-\int_{t_1}^{t}H(t,s)w'(s)ds+\int_{t_1}^{t}H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds-\alpha\int_{t_1}^{t}\frac{H(t,s)}{(a(s)\rho(s))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(s)ds.$$

Thus we obtain

$$\begin{aligned}\int_{t_1}^{t}H(t,s)\rho(s)Q^*(s)ds\leq H(t,t_1)w(t_1)&-\int_{t_1}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-\frac{\rho'(s)}{\rho(s)}H(t,s)\right]w(s)ds\\&-\alpha\int_{t_1}^{t}\frac{H(t,s)}{(a(s)\rho(s))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(s)ds.\end{aligned} \tag{2.15}$$

From the last inequality and (2.8), we obtain

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\rho(s)Q^*(s)-\frac{a(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'(s)}{\rho(s)}H^{\frac{1}{\alpha+1}}(t,s)-h(t,s)\right)^{\alpha+1}\right]ds\leq w(t_1)$$

which contradicts (2.13). Next we consider the case when $x(t)<0$ for all $t\geq t_1$ and we use the transformation $y(t)=-x(t)$ then $y(t)$ is a positive solution of the equation

$$(a(t)(z'(t))^{\alpha})'+q(t)y^{\beta}(t)=-e(t)$$

where $z(t)=y(t)+p(t)y(\tau(t))$. Define $w(t)$ by (2.11), then (2.12) holds. The remainder of the proof is similar to that of first case and hence omitted. The proof for the case (II) is similar to that of Theorem 2.2. The proof is now complete. □

Corollary 2.2. *Assume that all the conditions of Theorem 2.2 hold except the condition* (2.13) *is replaced by*

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_1)}\int_{t_1}^{t} H(t,s)\rho(s)Q^*(s)ds = \infty, \tag{2.16}$$

and

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_1)}\int_{t_1}^{t} a(s)\rho(s)\left(\frac{\rho'(s)}{\rho(s)}H^{\frac{1}{\alpha+1}}(t,s) - h(t,s)\right)^{\alpha+1} ds < \infty. \tag{2.17}$$

Then the conclusion of Theorem 2.2 holds.

Remark 2.1. *By choosing the function $H(t,s)$ in appropriate manners, we can derive several oscillation criteria for equation* (1.1). *For example, set*

$$H(t,s) = (t-s)^m, \quad m \geq 1, \quad (t,s) \in D_0$$

we have the following result.

Corollary 2.3. *Assume that all the conditions of Corollary 2.2 are satisfied except the conditions* (2.16) *and* (2.17) *replaced by*

$$\lim_{t\to\infty}\sup \frac{1}{(t-t_1)^m}\int_{t_1}^{t} (t-s)^m\rho(s)Q^*(s)ds = \infty$$

and

$$\lim_{t\to\infty}\sup \frac{1}{(t-t_1)^m}\int_{t_1}^{t} a(s)\rho(s)\left(\frac{\rho'(s)}{\rho(s)}(t-s)^{\frac{m}{\alpha+1}} + m(t-s)^{\frac{m}{\alpha+1}-11}\right)^{\alpha+1} ds < \infty.$$

Then the conclusion of Theorem 2.1 holds.

3 Examples

In this section we present some examples to illustrate the main results.

Example 3.1 Consider the differential equation

$$(((x(t) + 2x(t-2))')^3)' + tx^5(t) = \frac{1}{t^2}, \quad t \geq 1. \tag{3.1}$$

Here $p = 2,\ \alpha = 3,\ \beta = 5,\ \tau(t) = t-2,\ q(t) = t$ and $e(t) = \frac{1}{t^2}$. By taking $\rho(t) = 1$, we see that all conditions of Theorem 2.1 are satisfied. Hence every solution of equation (3.1) is almost oscillatory.

Example 3.2 Consider the differential equation

$$(t(x(t) + \frac{1}{2}x(\frac{t}{2}))')' + t^3(t+1)x^3(t) = t + 1 + \frac{2}{t^2}, \quad t \geq 1. \tag{3.2}$$

Here $p = \frac{1}{2},\ \alpha = 1,\ \beta = 3,\ \tau(t) = \frac{t}{2},\ q(t) = t^3(t+1)$ and $e(t) = t+1+\frac{2}{t^2}$. By taking $\rho(t) = 1$, we see that all conditions of Theorem 2.1 are satisfied and hence every solution of equation (3.2) is almost oscillatory. Infact $x(t) = \frac{1}{t}$ is one such solution of equation (3.2) since it satisfies the equation.

Example 3.3 Consider the differential equation

$$(x(t) + 2x(\frac{t}{2}))'' + t^2x^3(t) = t, \quad t \geq 1. \tag{3.3}$$

Here $p = 2,\ \alpha = 1,\ \beta = 3,\ \tau(t) = \frac{t}{2},\ q(t) = t^2$ and $e(t) = t$. By taking $\rho(t) = 1$ and $H(t,s) = (t-s)^2$ we see that all conditions of Corollary 2.3 are satisfied, and hence every solution of equation (3.3) is almost oscillatory.

Remark 3.1. *Since the forcing terms $e(t)$ in the above examples are positive, the results obtained in [2-14] cannot be applied to these examples. So our results are new and applicable to neutral differential equations with positive forcing terms.*

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References

[1] R.P. Agarwal, S.R. Grace and D.O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic, Dordrecht, 2000.

[2] L.H. Erbe, A. Peterson, T.S. Hasan and S.H. Saker, Interval oscillation criteria for forced second order nonlinear delay dynamic equations with oscillatory potential, *Dyn. Cont. Disc. Syst. A,* 17(2010), 533-542.

[3] L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory For Functional Differential Equations*, Marcel Dekker, New York, 1995.

[4] L.H. Erbe, A. Peterson and S.H. Saker, Oscillation criteria for a forced second order nonlinear dynamic equations, *J. Diff. Eqns. Appl.,* 14(2008), 997-1009.

[5] A.G. Kartsatos, On the maintenance of oscillations of n th order equations under the effect of a small forcing term, *J. Differ. Equ.,* 10(1971), 355-363.

[6] A.G. Kartsatos, Maintenance of oscillations under the effect of a periodic forcing term, *Proc. Amer. Math. Soc.,* 33(1972), 377-383.

[7] Q. Kong and J.S.W. Wong, Oscillation of a forced second order differential equations with a deviating argument, *Funct. Diff. Equ.,* 17(2010), 141-155.

[8] Q. Kong and B.G. Zhang, Oscillation of a forced second order nonlinear equation, *Chin. Ann. Math.,* 15B:1(1994), 59-68.

[9] G.S. Ladde,V. Lakshmikantham and B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.

[10] A.H. Nazar, Sufficient conditions for the oscillation of forced superlinear second order differential equations with oscillatory potential, *Proc. Amer. Math. Soc.,* 126(1998), 123-125.

[11] Y.G. Sun and J.S.W. Wong, Forced oscillation of second order superlinear differential equations, *Math. Nachr.,* 278(2005), 1621 -1628.

[12] Y.G. Sun and J.S.W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, *J. Math. Anal. Appl.,* 334(2007), 549-560.

[13] J.S.W.Wong, Second order nonlinear forced oscillations, *SIAM J. Math. Anal.,* 19(1998), 667-675.

[14] Q.G. Yang, Interval oscillation criteria for a forced second order nonlinear ordinary differential equation with oscillatory potential, *Appl. Math.,* 136(2003), 49-64.

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