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Recurrence relations of multiparameter K-Mittag-Leffler function

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Abstract

In this paper we evaluate the functional relation between Multiparameter K-Mittag-Leffler function defined by [2] and K-Series defined by [3]. Also we evaluate the recurrence relations and integral representation of Multiparameter K-Mittag-Leffler function. Some particular cases have been discussed.

Keywords: Multiparameter K-Mittag-Leffler function, K-Series, K-Pochhammer symbol, K- Gamma function.

2010 MSC: 26A33, 33B15, 33E12.

1 Introduction

In [8] the author introduce the generalized K-Gamma Function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k > 0, x \in C\backslash kZ^-, \tag{1.1}$$

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k).....(x+(n-1)k), x \in C, k \in R, n \in N^+. \tag{1.2}$$

K-Gamma function is given by,

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, x \in C, k \in R, Re(x) > 0, \tag{1.3}$$

and it follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}). \tag{1.4}$$

$$\Gamma_k(x+k) = x\Gamma_k(x). \tag{1.5}$$

$$(x)_{n,k} = k^n(\frac{x}{k})_n. \tag{1.6}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}. \tag{1.7}$$

$$nk(x)_{n-1,k} = (x)_{n,k} - (x-k)_{n,k}. \tag{1.8}$$

$$(x)_{n+j,k} = (x)_{j,k}(x+jk)_{n,k} \tag{1.9}$$

The Multiparameter K-Mittag-Leffler function defined by [2], as

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Definition 1.1. *Let $k \in R_+ = (0, \infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ $(j = 1, 2, .., p; r = 1, 2, .., q; i = 1, 2, .., m)$. Then the Multiparameter K-Mittag-Leffler function defined as,*

$$ {}_pK_{q,k}^{(\beta,\eta)_m}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m; z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)}, \tag{1.10}$$

where $\Gamma_k(x)$ is the K-Gamma function given by (1.1) and $(\gamma)_{n,k}$ is the K-Pochhammer symbol given by (1.2).

The series (1.10) is defined when none of the parameter $b_r (r = 1, 2, .., q)$ is negative integer or zero. If any parameter $a_j (j = 1, 2, .., p)$ in (1.10) is zero or negative, the series terminates into polynomial in z.
Convergent conditions for the series (1.10) are given by Ratio test,
(i) If $p < q + \sum_{i=1}^m (\frac{\eta_i}{k})$, then the power series on the right of (1.10) is absolutely convergent for all $z \in C$.
(ii) If $p = q + \sum_{i=1}^m (\frac{\eta_i}{k})$, then the power series on the right of (1.10) is absolutely convergent for all $|k^{p-q-\sum_{i=1}^m (\frac{\eta_i}{k})} z| < \prod_{i=1}^m (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$ and $|k^{p-q-\sum_{i=1}^m (\frac{\eta_i}{k})} z| = \prod_{i=1}^m (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$, $Re(\sum_{r=1}^q (\frac{b_r}{k}) + \sum_{i=1}^m (\frac{\beta_i}{k}) - \sum_{j=1}^p (\frac{a_j}{k})) > \frac{2+q+m-p}{2}$.

2 Main Results

In this section we evaluate the functional relation between Multiparameter K-Mittag-Leffler Function and K-Series. Also we evaluate the recurrence relations and integral representation of Multiparameter K-Mittag-Leffler Function. Nine particular cases have been evaluated for different values of parameters.

Theorem 2.1. *The functional relation between Multiparameter K-Mittag-Leffler function and K-Series is given by,*

$$ {}_pK_{q,k}^{(\beta,\eta)_m}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m; z]$$

$$ = k^{\sum_{i=1}^m (1-\frac{\beta_i}{k})}\, {}_pK_q^{(\beta,\eta)_m}[(\frac{a_j}{k})_{j=1}^p; (\frac{b_r}{k})_{r=1}^q, (\frac{\beta_i}{k}, \frac{\eta_i}{k})_{i=1}^m; zk^{p-q-\sum_{i=1}^m \frac{\eta_i}{k}}]. \tag{2.1}$$

And its counter part is given by

$$ {}_pK_q^{(\beta,\eta)_m}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m; z]$$

$$ = k^{\sum_{i=1}^m (\beta_i - 1)}\, {}_pK_{q,k}^{(\beta,\eta)_m}[(ka_j)_{j=1}^p; (kb_r)_{r=1}^q, (k\beta_i, k\eta_i)_{i=1}^m; zk^{\sum_{i=1}^m \eta_i + q - p}]. \tag{2.2}$$

Proof. From equation (1.10), we have

$$ A \equiv {}_pK_{q,k}^{(\beta,\eta)_m}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)},$$

using equations (1.4) and (1.6), we obtain

$$ A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p k^{pn} (\frac{a_j}{k})_n\, z^n}{\prod_{r=1}^q k^{qn} (\frac{b_r}{k})_n \prod_{i=1}^m k^{\frac{\eta_i n + \beta_i}{k} - 1} \Gamma(\frac{\eta_i}{k} n + \frac{\beta_i}{k})},$$

$$ A \equiv k^{\sum_{i=1}^m (1-\frac{\beta_i}{k})} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\frac{a_j}{k})_n\, (zk^{p-q-\sum_{i=1}^m \frac{\eta_i}{k}})^n}{\prod_{r=1}^q (\frac{b_r}{k})_n \prod_{i=1}^m \Gamma(\frac{\eta_i}{k} n + \frac{\beta_i}{k})},$$

$$ A \equiv k^{\sum_{i=1}^m (1-\frac{\beta_i}{k})}\, {}_pK_q^{(\beta,\eta)_m}[(\frac{a_j}{k})_{j=1}^p; (\frac{b_r}{k})_{r=1}^q, (\frac{\beta_i}{k}, \frac{\eta_i}{k})_{i=1}^m; zk^{p-q-\sum_{i=1}^m \frac{\eta_i}{k}}].$$

□

Theorem 2.2. *Let $b \in C, \beta \in R$ and the convergent conditions of Multiparameter K-Mittag-Leffler function are satisfies, then*

$$ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (b, \beta); z]$$

$$ = b\, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (b+k, \beta); z]$$

$$ + \beta z \frac{d}{dz}\, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (b+k, \beta); z]. \tag{2.3}$$

Proof. Consider the right hand side of equation (2.3) and using equation (1.10), we have

$$A \equiv b\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(b+k,\beta);z]$$

$$+\beta z\frac{d}{dz}\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(b+k,\beta);z],$$

$$A \equiv b\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\beta n+b+k)}$$

$$+\beta z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\beta n+b+k)},$$

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}(\beta n+b)\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\beta n+b+k)},$$

using equation (1.5), we obtain

$$A \equiv\ {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(b,\beta);z].$$

□

Theorem 2.3. *Let $a\in C$ and the convergent conditions of Multiparameter K-Mittag-Leffler Function are satisfies, then*

$${}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p},a+k;(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(k,1);z]$$

$$-\ {}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p},a;(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(k,1);z]$$

$$=\frac{kz\prod_{j=1}^{p}(a_j)}{\prod_{r=1}^{q}(b_r)}\ {}_pK_{q,k}^{(\beta+\eta,\eta)_{m+1}}[(a_j+k)_{j=1}^{p};(b_r+k)_{r=1}^{q},(\beta_i+\eta_i,\eta_i)_{i=1}^{m},(1,1);z]. \tag{2.4}$$

Proof. Consider the left hand side of equation (2.4) and using equation (1.10), we have

$$A \equiv\ {}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p},a+k;(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(k,1);z]$$

$$-\ {}_{p+1}K_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p},a;(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(k,1);z],$$

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(n+k)}[(a+k)_{n,k}-(a)_{n,k}],$$

using equation (1.8), we obtain

$$A \equiv \sum_{n=1}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(n+k)}[nk(a+k)_{n-1,k}],$$

replacing n by $n+1$, we obtain

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n+1,k}\,z^{n+1}}{\prod_{r=1}^{q}(b_r)_{n+1,k}\prod_{i=1}^{m}\Gamma_k(\eta_i(n+1)+\beta_i)\Gamma_k(n+1+k)}[(n+1)k(a+k)_{n,k}],$$

using equations (1.5) and (1.9), we obtain

$$A \equiv \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{1,k}(a_j+k)_{n,k}\,z^{n+1}[(n+1)k(a+k)_{n,k}]}{\prod_{r=1}^{q}(b_r)_{1,k}(b_r+k)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i+\eta_i)(n+1)\Gamma_k(n+1)},$$

$$A \equiv \frac{kz\prod_{j=1}^{p}(a_j)}{\prod_{r=1}^{q}(b_r)}\ {}_pK_{q,k}^{(\beta+\eta,\eta)_{m+1}}[(a_j+k)_{j=1}^{p};(b_r+k)_{r=1}^{q},(\beta_i+\eta_i,\eta_i)_{i=1}^{m},(1,1);z].$$

□

Theorem 2.4. *Let $\beta \in C, Re(\beta) > 0, \alpha \in R$ and the convergent conditions of Multiparameter K-Mittag-Leffler Function are satisfies, then*

$$
\begin{aligned}
&{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);z] \\
&-k\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);z] \\
&= z^2\alpha^2\,{}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z] \\
&+z\{\alpha^2+2\alpha(\beta+k)\}\,{}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z] \\
&+\beta(\beta+2k)\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]. \qquad (2.5)
\end{aligned}
$$

Proof. From equations (1.10) and (1.5), we have

$$
{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);z]
$$

$$
= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)(\alpha n+\beta)\Gamma_k(\alpha n+\beta)}. \qquad (2.6)
$$

Again,

$$
{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);z]
$$

$$
= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)(\alpha n+\beta+k)(\alpha n+\beta)\Gamma_k(\alpha n+\beta)}. \qquad (2.7)
$$

$$
= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)} \frac{1}{k}\left[\frac{1}{(\alpha n+\beta)} - \frac{1}{(\alpha n+\beta+k)}\right]
$$

using equation (2.6), we obtain

$$
\begin{aligned}
S = {}&{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);z] \\
&-k\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);z]. \qquad (2.8)
\end{aligned}
$$

Where

$$
S = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)}. \qquad (2.9)
$$

Applying a simple identity $\frac{1}{u} = \frac{k}{u(u+k)} + \frac{1}{(u+k)}$, for $u = \alpha n+\beta+k$ to equation (2.9), we obtain

$$
S = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)}
\times\left[\frac{k}{(\alpha n+\beta+k)(\alpha n+\beta+2k)} + \frac{1}{(\alpha n+\beta+2k)}\right],
$$

$$
S = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)}
$$

$$
\times\left[\frac{k(\alpha n+\beta)}{(\alpha n+\beta)(\alpha n+\beta+k)(\alpha n+\beta+2k)} + \frac{(\alpha n+\beta)(\alpha n+\beta+k)}{(\alpha n+\beta)(\alpha n+\beta+k)(\alpha n+\beta+2k)}\right],
$$

using equation (1.5), we have

$$
S = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}\, z^n[n^2\alpha^2+2n\alpha(\beta+k)+\beta(\beta+2k)]}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)}. \qquad (2.10)
$$

We express each summation in right side of (2.5) as follows;

$$\frac{d}{dz}\{z\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]\}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}(n+1)\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

$$z\,{}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$+\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}(n+1)\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

$$z\,{}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{n\prod_{j=1}^{p}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)}. \tag{2.11}$$

Again

$$\frac{d^2}{dz^2}\{z^2\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]\}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}(n+2)(n+1)\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

$$z^2\,{}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$+4z\,{}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$+2\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{(n^2+3n+1)\prod_{j=1}^{p}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)},$$

using equation (2.11)

$$z^2\,{}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$+z\,{}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$=\sum_{n=0}^{\infty}\frac{n^2\prod_{j=1}^{p}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta+3k)}. \tag{2.12}$$

using equations (2.11), (2.12) in equation (2.10), we obtain

$$S=z^2\alpha^2\,{}_p\ddot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$+z\{\alpha^2+2\alpha(\beta+k)\}\,{}_p\dot{K}_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$

$$+\beta(\beta+2k)\,{}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z].$$

□

Theorem 2.5. *Let $\beta \in C, Re(\beta) > 0, \alpha \in R$ and the convergent conditions of Multiparameter K-Mittag-Leffler function are satisfies, then*

$$\int_0^1 t^{\beta+k-1} \, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta,\alpha);t^\alpha]dt$$
$$= {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);1]$$
$$-k \, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);1]. \quad (2.13)$$

Proof. Put $z = 1$ in equations (2.8) and (2.9), we have

$$S = {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);1]$$
$$-k \, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);1]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)}. \quad (2.14)$$

Consider the left hand side integral,

$$A \equiv \int_0^1 t^{\beta+k-1} \, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta,\alpha);t^\alpha]dt,$$

using equation (1.10), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} \, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)} \int_0^1 t^{\alpha n+\beta+k-1}dt,$$
$$A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} \, z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n+\beta_i)\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)},$$

from equation (2.14), we obtain

$$A \equiv {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+k,\alpha);1]$$
$$-k \, {}_pK_{q,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+2k,\alpha);1].$$

□

2.1 Particular Cases

The particular cases of this paper are given by particularzing the values of parameters, we obtain the result for different known Mittag-Leffler Functions, given as:

(a) If we set $k = 1$, then we obtain the results for K-Series definded by [3].

(b) If we set $k = 1, p = q = m$ and $b_1 = b_2 = ... = b_m = 1$, we obtain the results for the 3M-Parameter Multi-Index Mittag-Leffler function definded by [4].

(c) If we set $k = 1, p = q = 1, a_1 = \rho, b_1 = 1$, then we obtain the results for the Generalized Mittag-Leffler function studied by [5].

(d) If we set $k = 1, p = q = 1, a_1 = b_1 = 1$ and $\eta_i = \frac{1}{\alpha_i}$, then we obtain the results for the Multi-Index Mittag-Leffler function studied by [10].

(e) If we set $k = 1, m = 1$, then we obtain the results for Generalized M-Series definded by [9].

(f) If we set $p = q = m = 1, a_1 = \delta, b_1 = k$, then we obtain the results for the K- Mittag-Leffler function studied by [1].

(g) If we set $k = 1, p = q = m = 1, a_1 = \delta, b_1 = 1$, then we obtain the results for the Generalized Mittag-Leffler function studied by [7].

(h) If we set $k = 1, p = q = m = 1, a_1 = b_1 = 1$, then we obtain the results for the Mittag-Leffler function studied by [11].

(i) If we set $k = 1, p = q = m = 1, a_1 = b_1 = 1$ and $\beta = 1$, then we obtain the results for the Mittag-Leffler function studied by [6].

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