Fractional integral inequalities for continuous random variables

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Abstract

By introducing new concepts on the probability theory, new integral inequalities are established for the fractional expectation and the fractional variance for continuous random variables. These inequalities generalize some interested results in [N.S. Barnett, P. Cerone, S.S. Dragomir and J. Roumeliotis: *Some inequalities for the dispersion of a random variable whose p.d.f. is defined on a finite interval*, J. Inequal. Pure Appl. Math., Vol. 2 Iss. 1 Art. 1 (2001), 1-18].

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1 Introduction

It is well known that the integral inequalities play a fundamental role in the theory of differential equations and applied sciences. Significant development in this theory has been achieved for the last two decades. For details, we refer to [4, 7, 11, 16, 19, 20, 21, 23] and the references therein. Moreover, the study of fractional type inequalities is also of great importance. We refer the reader to [2, 3, 6, 8, 10] for further information and applications. Let us introduce now the results that have inspired our work. The first one is given in [5]; in their paper, using Korkine identity and Holder inequality for double integrals, N.S. Barnett et al. established several integral inequalities for the expectation $E(X)$ and the variance $\sigma^2(X)$ of a random variable X having a probability density function (p.d.f.) $f : [a, b] \to \mathbb{R}^+$. In [13, 14], P. Kumar presented new inequalities for the moments and for the higher order central moments of a continuous random variable. In [15], Y. Miao and G. Yang gave new upper bounds for the standard deviation $\sigma(X)$, for the quantity $\sigma^2(X) + (t - E(X))^2, t \in [a, b]$ and for the L^p absolute deviation of a random variable X. Recently, G.A. Anastassiou et al. [2] proposed a generalization of the weighted Montgomery identity for fractional integrals with weighted fractional Peano kernel. More recently, M. Niezgoda [18] proposed new generalizations of the results of P. Kumar [14], by applying some Ostrowski-Gruss type inequalities. Other paper deal with these probability inequalities can be found in [1, 17, 22].

In this paper, we introduce new concepts on "fractional random variables". Then, we obtain new integral inequalities for the fractional dispersion and the fractional variance functions of a continuous random variable X having the probability density function $(p.d.f.)f : [a, b] \to \mathbb{R}^+$. We also present new results for the "fractional expectation and the fractional variance". For our results, some classical integral inequalities of Barnet et al. [5] can be deduced as some special cases.

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2 Preliminaries

Definition 2.1. *[12] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function h on $[a, b]$ is defined as*

$$J^\alpha[h(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} h(\tau) d\tau; \quad \alpha > 0, a < t \leq b,$$
$$J^0[h(t)] = h(t), \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

We give the following properties:

$$J^\alpha J^\beta[h(t)] = J^{\alpha+\beta}[h(t)], \alpha \geq 0, \beta \geq 0, \tag{2.2}$$

and

$$J^\alpha J^\beta[h(t)] = J^\beta J^\alpha[h(t)], \alpha \geq 0, \beta \geq 0. \tag{2.3}$$

We introduce also the following new concepts and definitions:

Definition 2.2. *The fractional expectation function of order $\alpha \geq 0$, for a random variable X with a positive p.d.f. f defined on $[a, b]$ is defined as*

$$E_{X,\alpha}(t) := J^\alpha[tf(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \tau f(\tau) d\tau; \quad \alpha \geq 0, a < t \leq b. \tag{2.4}$$

In the same way, we define the fractional expectation function of $X - E(X)$ by:

Definition 2.3. *The fractional expectation function of order $\alpha \geq 0$, for a random variable $X - E(X)$ is defined as*

$$E_{X-E(X),\alpha}(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} (\tau - E(X)) f(\tau) d\tau; \quad \alpha \geq 0, a < t \leq b, \tag{2.5}$$

where $f : [a, b] \to \mathbb{R}^+$ is the p.d.f. of X.

For $t = b$, we introduce the following concept:

Definition 2.4. *The fractional expectation of order $\alpha \geq 0$, for a random variable X with a positive p.d.f. f defined on $[a, b]$ is defined as*

$$E_{X,\alpha} = E_{X,\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^b (b - \tau)^{\alpha-1} \tau f(\tau) d\tau; \quad \alpha \geq 0. \tag{2.6}$$

For the fractional variance of X, we introduce the two definitions:

Definition 2.5. *The fractional variance function of order $\alpha \geq 0$ for a random variable X having a p.d.f. $f : [a, b] \to \mathbb{R}^+$ is defined as*

$$\sigma_{X,\alpha}^2(t) := J^\alpha[(t - E(X))^2 f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} (\tau - E(X))^2 f(\tau) d\tau;$$
$$\alpha \geq 0, a < t \leq b. \tag{2.7}$$

where $E(X) := \int_a^b \tau f(\tau) d\tau$ is the classical expectation of X.

Definition 2.6. *The fractional variance of order $\alpha \geq 0$, for a random variable X with a p.d.f. $f : [a, b] \to \mathbb{R}^+$ is defined as*

$$\sigma_{X,\alpha}^2 = \frac{1}{\Gamma(\alpha)} \int_a^b (b - \tau)^{\alpha-1} (\tau - E(X))^2 f(\tau) d\tau; \alpha >\geq 0. \tag{2.8}$$

We give the following important properties:

$(P1*)$: If we take $\alpha = 1$ in Definition 2.4, we obtain the classical expectation: $E_{X,1} = E(X)$.
$(P2*)$: If we take $\alpha = 1$ in Definition 2.6, we obtain the classical variance: $\sigma_{X,1}^2 = \sigma^2(X) = \int_a^b (\tau - E(X))^2 f(\tau) d\tau$.
$(P3*)$: For $\alpha > 0$, the p.d.f. f satisfies $J^\alpha[f(b)] = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$.
$(P4*)$: For $\alpha = 1$, we have the well known property $J^\alpha[f(b)] = 1$.

3   Main Results

In this section, we present new results for fractional continuous random variables. The first main result is the following theorem:

Theorem 3.1. *Let X be a continuous random variable having a p.d.f. $f : [a, b] \to \mathbb{R}^+$. Then we have:*
(a) : For all $a < t \le b, \alpha \ge 0$,

$$J^\alpha[f(t)]\sigma^2_{X,\alpha}(t) - (E_{X-E(X),\alpha}(t))^2 \le ||f||^2_\infty \left[\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J^\alpha[t^2] - (J^\alpha[t])^2 \right], \tag{3.9}$$

provided that $f \in L_\infty[a, b]$.
(b) : The inequality

$$J^\alpha[f(t)]\sigma^2_{X,\alpha}(t) - (E_{X-E(X),\alpha}(t))^2 \le \frac{1}{2}(t-a)^2(J^\alpha[f(t)])^2 \tag{3.10}$$

is also valid for all $a < t \le b, \alpha \ge 0$.

Proof. Let us define the quantity

$$H(\tau, \rho) := (g(\tau) - g(\rho))(h(\tau) - h(\rho)); \tau, \rho \in (a, t), a < t \le b. \tag{3.11}$$

Taking a function $p : [a, b] \to \mathbb{R}^+$, multiplying (3.11) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} p(\tau); \tau \in (a, t)$, then integrating the resulting identity with respect to τ from a to t, we can state that

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} p(\tau) H(\tau, \rho) d\tau$$

$$= J^\alpha[pgh(t)] - g(\rho)J^\alpha[ph(t)] - h(\rho)J^\alpha[pg(t)] + g(\rho)h(\rho)J^\alpha[p(t)]. \tag{3.12}$$

Now, multiplying (3.12) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} p(\rho); \rho \in (a, t)$ and integrating the resulting identity with respect to ρ over (a, t), we can write

$$\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} p(\tau)p(\rho) H(\tau, \rho) d\tau d\rho$$

$$= 2J^\alpha[p(t)]J^\alpha[pgh(t)] - 2J^\alpha[pg(t)]J^\alpha[ph(t)]. \tag{3.13}$$

In (3.13), taking $p(t) = f(t), g(t) = h(t) = t - E(X), t \in (a, b)$, we have

$$\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

$$= 2J^\alpha[f(t)]J^\alpha[f(t)(t - E(X))^2] - 2\left(J^\alpha[f(t)(t - E(X))] \right)^2. \tag{3.14}$$

On the other hand, we have

$$\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

$$\le ||f||^2_\infty \frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau-\rho)^2 d\tau d\rho \tag{3.15}$$

$$\le ||f||^2_\infty \left[2\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J^\alpha[t^2] - 2(J^\alpha[t])^2 \right].$$

Thanks to (3.14), (3.15), we obtain the part (a) of Theorem 3.1.
For the part (b), we have

$$\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

$$\le \sup_{\tau,\rho\in[a,t]} |(\tau-\rho)|^2 (J^\alpha[f(t)])^2 = (t-a)^2(J^\alpha[f(t)])^2. \tag{3.16}$$

Then, by (3.14) and (3.16), we get the desired inequality (3.10).   □

We give also the following corollary:

Corollary 3.1. *Let X be a continuous random variable with a p.d.f. f defined on* $[a, b]$*. Then:*
(i) : *If* $f \in L_\infty[a, b]$*, then for any* $\alpha \geq 0$*, we have*

$$\frac{(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\sigma_{X,\alpha}^2 - E_{X,\alpha}^2 \leq ||f||_\infty^2 \left[\frac{2(b-a)^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)} - \left(\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \right)^2 \right]. \tag{3.17}$$

(ii) : *The inequality*

$$\frac{(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\sigma_{X,\alpha}^2 - E_{X,\alpha}^2 \leq \frac{1}{2}\frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha)} \tag{3.18}$$

is also valid for any $\alpha \geq 0$*.*

Remark 3.1. $(r1)$: *Taking* $\alpha = 1$ *in* (i) *of Corollary 3.1, we obtain the first part of Theorem 1 in [5].*
$(r2)$: *Taking* $\alpha = 1$ *in* (ii) *of Corollary 3.1, we obtain the last part of Theorem 1 in [5].*

We shall further generalize Theorem 3.1 by considering two fractional positive parameters:

Theorem 3.2. *Let X be a continuous random variable having a p.d.f.* $f : [a, b] \rightarrow \mathbb{R}^+$*. Then we have:*
$(a*)$: *For all* $a < t \leq b, \alpha \geq 0, \beta \geq 0$*,*

$$J^\alpha[f(t)]\sigma_{X,\beta}^2(t) + J^\beta[f(t)]\sigma_{X,\alpha}^2(t) - 2(E_{X-E(X),\alpha}(t))(E_{X-E(X),\beta}(t))$$

$$\leq ||f||_\infty^2 \left[\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\beta[t^2] + \frac{(t-a)^\beta}{\Gamma(\beta+1)}J^\alpha[t^2] - 2(J^\alpha[t])(J^\beta[t]) \right], \tag{3.19}$$

where $f \in L_\infty[a, b]$*.*
$(b*)$: *The inequality*

$$J^\alpha[f(t)]\sigma_{X,\beta}^2(t) + J^\beta[f(t)]\sigma_{X,\alpha}^2(t) - 2(E_{X-E(X),\alpha}(t))(E_{X-E(X),\beta}(t))$$

$$\leq (t-a)^2 J^\alpha[f(t)]J^\beta[f(t)] \tag{3.20}$$

is also valid for any $a < t \leq b, \alpha \geq 0, \beta \geq 0$*.*

Proof. Using (3.11), we can write

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho$$

$$= J^\alpha[p(t)]J^\beta[pgh(t)] + J^\beta[p(t)]J^\alpha[pgh(t)]$$

$$- J^\alpha[ph(t)]J^\beta[pg(t)] - J^\beta[ph(t)]J^\alpha[pg(t)]. \tag{3.21}$$

Taking $p(t) = f(t), g(t) = h(t) = t - E(X), t \in (a, b)$ in the above identity, yields

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

$$= J^\alpha[f(t)]J^\beta[f(t)(t-E(X))^2] + J^\beta[f(t)]J^\alpha[f(t)(t-E(X))^2]$$

$$- 2J^\alpha[f(t)(t-E(X))]J^\beta[f(t)(t-E(X))]. \tag{3.22}$$

We have also

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

$$\leq ||f||_\infty^2 \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \int_a^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)^2 d\tau d\rho \tag{3.23}$$

$$\leq ||f||_\infty^2 \left[\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J^\beta[t^2] + \frac{(t-a)^\beta}{\Gamma(\beta+1)}J^\alpha[t^2] - 2J^\alpha[t]J^\beta[t] \right].$$

Thanks to (3.22) and (3.23), we obtain $(a*)$.

To prove $(b*)$, we use the fact that $\sup_{\tau, \rho \in [a,t]} |(\tau - \rho)|^2 = (t - a)^2$. We obtain

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t - \tau)^{\alpha-1} (t - \rho)^{\beta-1} f(\tau) f(\rho) (\tau - \rho)^2 d\tau d\rho \tag{3.24}$$

$$\leq (t - a)^2 J^\alpha [f(t)] J^\beta [f(t)].$$

And, by (3.22) and (3.24), we get (3.20). $\square$

Remark 3.2. $(r1)$: *Applying Theorem 3.2 for $\alpha = \beta$, we obtain Theorem 3.1.*

$(r2)$: *Taking $\alpha = \beta = 1$ in $(a*)$ of Theorem 3.4, we obtain the first inequality of Theorem 1 in [5].*

$(r3)$: *Taking $\alpha = \beta = 1$ in $(b*)$ of Theorem 3.2, we obtain the last part of Theorem 1 in [5].*

We give also the following fractional integral result:

Theorem 3.3. *Let f be the p.d.f. of X on $[a, b]$. Then for all $a < t \leq b, \alpha \geq 0$, we have:*

$$J^\alpha [f(t)] \sigma^2_{X,\alpha}(t) - (E_{X-E(X),\alpha}(t))^2 \leq \frac{1}{4}(b - a)^2 (J^\alpha [f(t)])^2. \tag{3.25}$$

Proof. Using Theorem 3.1 of [9], we can write

$$\left| J^\alpha [p(t)] J^\alpha [pg^2(t)] - (J^\alpha [pg(t)])^2 \right| \tag{3.26}$$

$$\leq \tfrac{1}{4} \left(J^\alpha [p(t)] \right)^2 (M - m)^2.$$

Taking $p(t) = f(t), g(t) = t - E(X), t \in [a, b]$, then $M = b - E(X), m = a - E(X)$. Hence, (3.25) allows us to obtain

$$0 \leq J^\alpha [f(t)] J^\alpha [f(t)(t - E(X))^2] - \left(J^\alpha [f(t)(t - E(X))] \right)^2 \tag{3.27}$$

$$\leq \tfrac{1}{4} (J^\alpha [f(t)])^2 (b - a)^2.$$

This implies that

$$J^\alpha [f(t)] \sigma^2_{X,\alpha}(t) - (E_{X-E(X),\alpha}(t))^2 \leq \frac{1}{4}(J^\alpha [f(t)])^2 (b - a)^2. \tag{3.28}$$

Theorem 3.3 is thus proved. $\square$

For $t = b$, we propose the following interesting inequality:

Corollary 3.2. *Let f be the p.d.f. of X on $[a, b]$. Then for any $\alpha \geq 0$, we have:*

$$\frac{(b - a)^{(\alpha-1)}}{\Gamma(\alpha)} \sigma^2_{X,\alpha} - (E_{X-E(X),\alpha})^2 \leq \frac{1}{4\Gamma^2(\alpha)} (b - a)^{2\alpha}. \tag{3.29}$$

Remark 3.3. *Taking $\alpha = 1$ in Corollary 3.2, we obtain Theorem 2 of [5].*

We also present the following result for the fractional variance function with two parameters:

Theorem 3.4. *Let f be the p.d.f. of the random variable X on $[a, b]$. Then for all $a < t \leq b, \alpha \geq 0, \beta \geq 0$, we have:*

$$J^\alpha [f(t)] \sigma^2_{X,\beta}(t) + J^\beta [f(t)] \sigma^2_{X,\alpha}(t)$$

$$+ 2(a - E(X))(b - E(X)) J^\alpha [f(t)] J^\beta [f(t)] \tag{3.30}$$

$$\leq (a + b - 2E(X)) \left(J^\alpha [f(t)] (E_{X-E(X),\beta}(t)) + J^\beta [f(t)] (E_{X-E(X),\alpha}(t)) \right).$$

Proof. Thanks to Theorem 3.4 of [9], we can state that:

$$\left[J^\alpha[p(t)]J^\beta[pg^2(t)] + J^\beta[p(t)]J^\alpha(pg^2(t)) - 2J^\alpha[pg(t)]J^\beta[pg(t)] \right]^2$$

$$\leq \left[\left(MJ^\alpha[p(t)] - J^\alpha[pg(t)] \right)\left(J^\beta[pg(t)] - mJ^\beta[p(t)] \right) + \right.$$ (3.31)

$$\left. \left(J^\alpha[pg(t)] - mJ^\alpha[p(t)] \right)\left(MJ^\beta[p(t)] - J^\beta[pg(t)] \right) \right]^2.$$

In (3.31), we take $p(t) = f(t), g(t) = t - E(X), t \in [a, b]$. We obtain

$$\left[J^\alpha[f(t)]J^\beta[f(t)(t - E(X))^2] + J^\beta[f(t)]J^\alpha[f(t)(t - E(X))^2] \right.$$

$$\left. -2J^\alpha[f(t)(t - E(X))]J^\beta[f(t)(t - E(X))] \right]^2$$ (3.32)

$$\leq \left[\left(MJ^\alpha[f(t)] - J^\alpha[f(t)(t - E(X))] \right)\left(J^\beta[f(t)(t - E(X))] - mJ^\beta[f(t)] \right) + \right.$$

$$\left. \left(J^\alpha[f(t)(t - E(X))] - mJ^\alpha[f(t)] \right)\left(MJ^\beta[f(t)] - J^\beta[f(t)(t - E(X))] \right) \right]^2.$$

Combining (3.22) and (3.32) and taking into account the fact that the left hand side of (3.22) is positive, we get:

$$J^\alpha[f(t)]J^\beta[f(t)(t - E(X))^2] + J^\beta[f(t)]J^\alpha[f(t)(t - E(X))^2]$$

$$-2J^\alpha[f(t)(t - E(X))]J^\beta[f(t)(t - E(X))]$$ (3.33)

$$\leq \left(MJ^\alpha[f(t)] - J^\alpha[f(t)(t - (EX))] \right)\left(J^\beta[f(t)(t - E(X))] - mJ^\beta[f(t)] \right) +$$

$$\left(J^\alpha[f(t)(t - E(X))] - mJ^\alpha[f(t)] \right)\left(MJ^\beta[f(t)] - J^\beta[f(t)(t - E(X))] \right).$$

Therefore,

$$J^\alpha[f(t)]J^\beta[f(t)(t - E(X))^2] + J^\beta[f(t)]J^\alpha[f(t)(t - E(X))^2]$$

$$\leq M\left(J^\alpha[f(t)](E_{X-E(X),\beta}(t)) + J^\beta[f(t)](E_{X-E(X),\alpha}(t)) \right)$$ (3.34)

$$+m\left(J^\alpha[f(t)](E_{X-E(X),\beta}(t)) + J^\beta[f(t)](E_{X-E(X),\alpha}(t)) \right)$$

$$-2mMJ^\alpha[f(t)]J^\beta[f(t)].$$

Substituting the values of m and M in (3.28) , then a simple calculation allows us to obtain (3.30). Theorem 3.4 is thus proved. $\square$

To finish, we present to the reader the following corollary:

Corollary 3.3. *Let f be the p.d.f. of X on $[a, b]$. Then for all $a < t \leq b, \alpha \geq 0$, the inequality*

$$\sigma^2_{X,\alpha}(t) + (a - E(X))(b - E(X))J^\alpha[f(t)]$$ (3.35)

$$\leq (a + b - 2E(X))E_{X-E(X),\alpha}(t)$$

is valid.

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