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Systems of first-order nabla dynamic equations on time scales

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Abstract

In this article, we study the existence of solutions to systems of first-order ∇-dynamic equations on time scales with periodic boundary or terminal value conditions. where the right member of the system is ∇-Carathéodory. We employ the method of solution-tube and Schauder's fixed-point theorem.

Keywords

∇-dynamic equation, systems of first-order equation on time scales, solution-tube, existence theorems, Schauder's fixed-point theorem.

AMS Subject Classification

34K11, 39A10, 39A99, 34C10, 39A11.

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Contents

1. Introduction

The concept of dynamic equations on time scales was initiated by Hilger [16] with the motivation of providing a unified approach to discrete and continuous calculus. The reader interested on the subject of time scales is referred in [3, 4, 7–9, 15].

This paper considers the existence of solutions to systems of first-order ∇-dynamic equations on time scales:

$$\begin{cases} x^{\nabla}(t) = f(t, x(\rho(t))), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x \in (BC). \end{cases} \tag{1.1}$$

$$\begin{cases} x^{\nabla}(t) = f(t, x(t)), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b). \end{cases} \tag{1.2}$$

Here $\mathbb{T}$ is an arbitrary compact time scales, such that $a = \min \mathbb{T}$, $b = \max \mathbb{T}, \mathbb{T}_0 = \mathbb{T} \backslash \{a\}$, $f : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a ∇-Carathéodory function and (BC) denotes the terminal value conditions:

$$x(b) = x_0, \tag{1.3}$$

or periodic boundary value conditions:

$$x(a) = x(b). \tag{1.4}$$

Existence results for system (1.2) were obtained in [5] with f is a continuous function. In the particular case where $n = 1$, existence results for first-order ∇-dynamic equation on time scales were obtained in [20] for the dynamic initial value problem:

$$x^{\nabla}(t) = f(t, x(t)),\ t \in (0, b]_{\mathbb{T}}, \text{ and } x(0) = 0,$$

with f is a left-Hilger-continuous function, their results were established with the method of lower and upper solutions. Existence results were obtained in [2, 11, 12, 14, 19], for systems of Δ-dynamic equations on time scales. In [14] Gilbert introduced the notion of solution-tube to systems of first order Δ-dynamic equations which generalizes the notions of lower and upper solutions. The results in this paper were motivated by results in [5, 14].

This paper is organized as follows. We start with some notations, definitions and results which are used throughout this paper. The third section presents an existence result for the problem (1.1) and the problem (1.2).

2. Preliminaries

In this section, we recall some notions and results which we will use in this article.

2.1 Definitions and basic properties

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators as follows: define $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

We say that t is right-scattered (resp., left-scattered) if $\sigma(t) > t$ (resp., if $\rho(t) < t$); that t is isolated if it is right-scattered and left-scattered. Also, if $t < \sup \mathbb{T}$ and $t = \sigma(t)$, we say that t is right-dense. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, we say that t is left dense. Points that are right dense and left dense are called dense. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T}\backslash\{M\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0,\infty)$ is defined by $\nu(t) := t - \rho(t)$. If $\mathbb{T}$ has a right-scattered minimum M, then $\mathbb{T}_k = \mathbb{T}\backslash\{M\}$, otherwise, $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ is bounded, then $\mathbb{T}_0 \subseteq \mathbb{T}_k$ where $\mathbb{T}_0 = \mathbb{T}\backslash\{\min \mathbb{T}\}$. For $a,b \in \mathbb{T}$ we define the closed interval $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \le t \le b\}$.

Definition 2.1. *The function $f : \mathbb{T} \to \mathbb{R}^n$ is called ld-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and has a right-sided limits exist at right-dense points in $\mathbb{T}$, write $f \in C_{ld}(\mathbb{T},\mathbb{R}^n)$.*

Definition 2.2. *[20](Left-Hilger-continuous functions)* A mapping $f : (a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is called left-Hilger-continuous at a point (t,x) if f is continuous at each (t,x) where t is left-dense and the limits

$$\lim_{(s,y)\to(t^+,x)} f(s,y) \quad \text{and} \quad \lim_{y\to x} f(t,y),$$

both exist and are finite at each (t,x) where t is right-dense.

Definition 2.3. *[18] For $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}_k$, the ∇-derivative of f at t, denoted by $f^\nabla(t)$, is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$\left\| f^\rho(t) - f(s) - f^\nabla(t)(\rho(t) - s) \right\| \le \varepsilon\,|\rho(t) - s|, \quad for\ all\ s \in U.$$

We say that f is ∇-differentiable if $f^\nabla(t)$ exists for every $t \in \mathbb{T}_k$. The function $f^\nabla : \mathbb{T} \to \mathbb{R}^n$ is then called the ∇-derivative of f on $\mathbb{T}_k$. The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ which are ∇-differentiable and whose ∇-derivative is ld-continuous is denoted by $C^1_{ld}(\mathbb{T},\mathbb{R}^n)$.

The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ which are ∇-differentiable and whose ∇-derivative is ld-continuous is denoted by $C^1_{ld}(\mathbb{T},\mathbb{R}^n)$.

Theorem 2.4. *[5] Let W be an open set of $\mathbb{R}^n$ and $t \in \mathbb{T}$ be a left-dense point. If $g : \mathbb{T} \to \mathbb{R}^n$ is ∇-differentiable at t and if $f : w \to \mathbb{R}$ is differentiable at $g(t) \in w$, then $f \circ g$ is ∇-differentiable at t and $(f \circ g)^\nabla(t) = \langle f'(g(t)), g^\nabla(t)\rangle$.*

Example 2.5. *[5] Assume $x : \mathbb{T} \to \mathbb{R}^n$ is $\nabla-$ différentiable at $t \in \mathbb{T}$. We know that $\|.\| : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$ is differentiable. If $t = \rho(t)$, by the previous theorem, we have*

$$\|x(t)\|^\nabla = \frac{< x(t), x^\nabla(t) >}{\|x(t)\|}.$$

Definition 2.6. *[8] The function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive if*

$$1 - \nu(t)p(t) \ne 0,\ for\ all\ t \in \mathbb{T}_k.$$

The set of all ν-regressive and ld-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathscr{R}_\nu = \mathscr{R}_\nu(\mathbb{T},\mathbb{R})$. We define the set $\mathscr{R}^+{}_\nu = \{p \in \mathscr{R}_\nu : 1 - \nu(t)p(t) > 0\}$ for all $t \in \mathbb{T}$.

Definition 2.7. *[8] If $p \in \mathscr{R}_\nu$, then we define the nabla exponential function $\hat{e}_p$ by:*

$$\hat{e}_p(t,s) = \exp\Big(\int_s^t \hat{\xi}_{\mu(\tau)}(p(\tau))\nabla\tau\Big),$$

for $t,s \in \mathbb{T}$, where the ν-cylinder transformation is as in :

$$\hat{\xi}_h(z) = \begin{cases} \dfrac{1}{h}\log(1 - zh); & if\ h > 0; \\ z; & if\ \ h = 0, \end{cases}$$

where log *is the principal logarithm function.*

Theorem 2.8. *[8] For $p \in \mathscr{R}_\nu$, the (nabla) exponential function $\hat{e}_p(t,t_0) : \mathbb{T} \to \mathbb{R}$ is defined as the unique solution to the initial value problem*

$$x^\nabla(t) = px(t), \qquad x(t_0) = 1.$$

2.2 Lebesgue ∇-measure and Lebesgue ∇-integral

We recall some notions and results related to the theory of ∇-measure and Lebesgue ∇-integration for an arbitrary bounded time scale $\mathbb{T}$ where $a = \min\mathbb{T} < \max\mathbb{T} = b$ introduced in [3, 8, 15].

Definition 2.9. *Let $\mathfrak{F}$ denote the family of all right closed and left open intervals of $\mathbb{T}$ of the form*

$$(r,s] = \{t \in \mathbb{T} : r \le t < s\},$$

with $r,s \in \mathbb{T}$ and $r \le s$. The interval $(r,r]$ is understood as the empty set. We define an additive measure $m_1 : \mathfrak{F} \to [0,\infty)$ by $m_1((r,s]) = s - r$. Using m_1, the outer measure $m_1^* : \mathscr{P}(\mathbb{T}) \to \mathbb{R}$, defined for each $E \subset \mathbb{T}$ as:

- *If $a \notin E$, then*

$$m_1^*(E) := \inf\left\{ \sum_{k=1}^{k=m} (s_k - r_k) : E \subset \bigcup_{k=1}^{k=m} (r_k, s_k],\ (r_k, s_k] \in \mathfrak{F} \right\}.$$

- *If $a \in E$, then $m_1^*(E) = +\infty$.*

Definition 2.10. *A set $A \subset \mathbb{T}$ is said to be ∇-measurable if, for every set $E \subset \mathbb{T}$*

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \backslash A)).$$

The Lebesgue ∇-measure on

$$\mathscr{M}(m_1^*) = \{A \subset \mathbb{T} : A \text{ is } \nabla\text{-mesurable}\},$$

denoted by μ_∇, is the restriction of m_1^* to $\mathscr{M}(m_1^*)$.
So, $(\mathbb{T}, \mathscr{M}(m_1^*), \mu_\nabla)$ is a complete measurable space.

Lemma 2.11. *[8] For each t_0 in $\mathbb{T}_0$ the ∇-measure of the single-point set $\{t_0\}$ is given by $\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0)$.*

Lemma 2.12. *[8]* 1. *If $r, s \in \mathbb{T}$ and $r \le s$, then*

$$\mu_\nabla((r,s]) = s - r,\ \mu_\nabla((r,s)) = \rho(s) - r.$$

2. *If $r, s \in \mathbb{T}_0$ and $r \le s$, then*

$$\mu_\nabla([r,s)) = \rho(s) - \rho(r),\ \mu_\nabla([r,s]) = s - \rho(r).$$

The following lemma can be proved analogously to Lemma 3.1 in [10].

Lemma 2.13. *The set of all left-scattered points of $\mathbb{T}$ is at most countable, that is, there are $J \subseteq \mathbb{N}$ and $\{t_j\}_{j\in J} \subset \mathbb{T}$ such that $\mathscr{L}_\mathbb{T} := \{t \in \mathbb{T}, \rho(t) < t\} = \{t_j\}_{j\in J}$.*

The following Proposition can be proved analogously to Proposition 3.1 in [10].

Proposition 2.14. *Let $A \subset \mathbb{T}$. Then A is a ∇-measurable if and only if, A is Lebesgue measurable. In this case the following properties hold for every ∇-measurable set A :*
1. If $a \notin A$, then $\mu_\nabla(A) = \mu_L(A) + \sum_{j\in J_A} \nu(t_j)$.
2. $\mu_\nabla(A) = \mu_L(A)$ if and only if $a \notin A$ and A has no left-scattered point.

The notions of ∇-measurable and ∇-integrable functions $f : \mathbb{T} \to \mathbb{R}^n$ can be defined similarly to the theory of Lebesgue integral.

Definition 2.15. *We say that $f : \mathbb{T} \longrightarrow \overline{\mathbb{R}} = [-\infty, +\infty]$ is ∇-measurable if for every $\alpha \in \mathbb{R}$, the set $f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$ is ∇-measurable.*

In order to compare the Lebesgue ∇-integral on $\mathbb{T}$ and the Lebesgue integral on $[a,b]$, given a function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$, we need an auxiliary function which extends $\widetilde{f}$ to the interval $[a,b]$ defined as

$$\widetilde{f}(t) := \begin{cases} f(t), & \text{if } t \in \mathbb{T}, \\ f(t_j), & \text{if } t \in (\rho(t_j), t_j)), \text{ for all } j \in J. \end{cases} \tag{2.1}$$

Let $E \subset \mathbb{T}$, we define $J_E := \{j \in J : t_j \in E \cap \mathscr{L}_\mathbb{T}\}$ and

$$\widetilde{E} := E \cup \bigcup_{j\in J_E} (\rho(t_j), t_j). \tag{2.2}$$

The following Theorem can be proved analogously to Theorem 5.1 in [10].

Theorem 2.16. *Let $E \subset \mathbb{T}$ be a ∇-measurable such that $a \notin E$, let $\widetilde{E}$ be the set defined in (2.2), let $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ be a ∇-measurable function and $\widetilde{f} : [a,b] \longrightarrow \mathbb{R}^n$ be the extension of f to $[a,b]$. Then, f is Lebesgue ∇-integrable on E if and only if $\widetilde{f}$ is Lebesgue integrable on $\widetilde{E}$ and we have*

$$\int_E f(t)\nabla t = \int_{\widetilde{E}} \widetilde{f}(t)\,dt = \int_E f(t)\,dt + \sum_{j\in J_E} \nu(t_j) f(t_j). \tag{2.3}$$

2.3 Sobolev's spaces on time Scales

In this section, we develop the Sobolev's spaces on bounded time scale $\mathbb{T}$ where $a = \min \mathbb{T} < \max \mathbb{T} = b$, $\mathbb{T}_0 = \mathbb{T} \backslash \{a\}$ and their important properties.

Definition 2.17. *Let $p \in [1, +\infty)$, $E \subset \mathbb{T}$ be a ∇-measurable set and $f : \mathbb{T} \to \mathbb{R}^n$ be a ∇-measurable function. We say that $f \in L^p_\nabla(E, \mathbb{R}^n)$ (respectively $f \in L^p_\nabla(\mathbb{T}, \mathbb{R}^n)$) provided*

$$\int_E \|f(s)\|^p \nabla s < +\infty \ (\textit{respectively} \int_{\mathbb{T}_0} \|f(s)\|^p \nabla s < +\infty).$$

Proposition 2.18. *Assume $f \in L^1_\nabla(E, \mathbb{R}^n)$. Then,*

$$\left\| \int_E f(s)\nabla s \right\| \le \int_E \|f(s)\| \nabla s.$$

Here is an analog of the Lebesgue dominated convergence theorem.

Theorem 2.19. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of function in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$. If there exists a function $f : \mathbb{T}_0 \to \mathbb{R}^n$ such that $f_k(t) \to f(t)$ ∇-a.e. $t \in \mathbb{T}_0$ and if there exists a function $g \in L^1_\nabla(\mathbb{T}_0)$ such that $\|f_k(t)\| \le g(t)$ ∇-a.e. $t \in \mathbb{T}_0$ and for every $k \in \mathbb{N}$. Then $f_k \to f$ in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$.*

The following Proposition can be proved analogously to Proposition 3.1 in [6].

Proposition 2.20. *Let $p \in [1, +\infty)$, $L^p_\nabla(\mathbb{T}, \mathbb{R}^n)$ is a Banach space equipped with the norm*

$$\|f\|_{L^p_\nabla(\mathbb{T},\mathbb{R}^n)} := \left(\int_{\mathbb{T}_0} \|f(t)\|^p \nabla t \right)^{\frac{1}{p}}.$$

Now we introduce the concept of absolutely continuous function on $\mathbb{T}$.

Definition 2.21. *A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be absolutely continuous on $\mathbb{T}$ if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{(a_k, b_k], k = 1, ..., m\}$, with $a_k, b_k \in \mathbb{T}$, is a finite pairwise disjoint family of subintervals of $\mathbb{T}$ satisfying*

$$\sum_{k=1}^{k=m} (b_k - a_k) < \eta \ \textit{ then } \ \sum_{k=1}^{k=m} \|f(b_k) - f(a_k)\| < \varepsilon.$$

The following Theorem can be proved analogously to Theorem 4.1 in [9].

Theorem 2.22. *A function $f:\mathbb{T}\to\mathbb{R}^n$ is absolutely continuous on $\mathbb{T}$ if and only if f is ∇-différentiable ∇-almost everywhere on $\mathbb{T}_0$, $f^{\nabla}\in L^1_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)$ and*

$$\int_{(t,b]\cap\mathbb{T}} f^{\nabla}(s)\nabla s = f(b)-f(t), \qquad \text{for every } t\in\mathbb{T}.$$

The following two Propositions can be proved analogously to Proposition 2.19 and Proposition 2.20 in [14].

Proposition 2.23. *Let $f\in L^1_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)$, then $F:\mathbb{T}\to\mathbb{R}^n$ defined by*

$$F(t)=\int_{(t,b]\cap\mathbb{T}} f(s)\nabla s \text{ satisfies } F^{\nabla}(t)=f(t),\ \nabla\text{-a.e. on } \mathbb{T}_0.$$

Proposition 2.24. *Let $u:\mathbb{T}\to\mathbb{R}$ be an absolutely continuous function, then the ∇-measure of the set $\{t\in\mathbb{T}_0\backslash\mathscr{L}_{\mathbb{T}_0}: u(t)=0 \text{ and } u^{\nabla}(t)\neq 0\}$ is zero.*

The following Theorem can be proved analogously to Theorem 3.2 in [6].

Theorem 2.25. *Let $p\in[1,\infty)$, then $\boldsymbol{C}(\mathbb{T},\mathbb{R}^n)$ is dense in $L^p_{\nabla}(\mathbb{T},\mathbb{R}^n)$.*

We now define a notion of Sobolev's space.

Definition 2.26. *Let $p\in[1,\infty)$, and $f:\mathbb{T}\to\mathbb{R}^n$. Say that f belongs to $W^{1,p}_{\nabla}(\mathbb{T},\mathbb{R}^n)$ if and only if $f\in L^p_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)$ and there exists $g:\mathbb{T}_k\to\mathbb{R}^n$ such that $g\in L^p_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)$ and*

$$\int_{\mathbb{T}_0}\left(f.\phi^{\nabla}\right)(s)\nabla s=-\int_{\mathbb{T}_0}(g.\phi^{\rho})(s)\nabla s, \text{ for all } \phi\in C^1_{0,ld}(\mathbb{T}), \tag{2.4}$$

with

$$C^1_{0,ld}(\mathbb{T}):=\left\{\phi\in C^1_{ld}(\mathbb{T}):\phi(a)=\phi(b)=0\right\}.$$

The following Theorem can be proved analogously to Theorem 3.4 in [1].

Theorem 2.27. *Suppose that $u\in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n)$ and that (2.4) holds for a function $g\in L^1_{\nabla}(\mathbb{T},\mathbb{R}^n)$. Then, there exists a unique function $x:\mathbb{T}\longrightarrow\mathbb{R}^n$ absolutely continuous such that ∇-almost everywhere on $\mathbb{T}_0$, one has $x=u$ and $x^{\nabla}=g$. Moreover, if g is rd-continuous on $\mathbb{T}_0$, then there exists a unique function $x\in C^1_{ld}(\mathbb{T},\mathbb{R}^n)$ such that $x=u$ ∇-almost everywhere on $\mathbb{T}_0$ and such $x^{\nabla}=g$ on $\mathbb{T}_0$.*

Theorem 2.28. *Let $p\in[1,\infty)$. The set $W^{1,p}_{\nabla}(\mathbb{T},\mathbb{R}^n)$ is a Banach space together with the norm defined for every $f\in W^{1,p}_{\nabla}(\mathbb{T},\mathbb{R}^n)$ as*

$$\|f\|_{W^{1,p}_{\nabla}(\mathbb{T},\mathbb{R}^n)}=\|f\|_{L^p_{\nabla}(\mathbb{T},\mathbb{R}^n)}+\left\|f^{\nabla}\right\|_{L^p_{\nabla}(\mathbb{T}_k,\mathbb{R}^n)}.$$

The proof is analogous to that of Theorem 3.5 in [1].

We now define a notion of ∇-Carathéodory functions on a compact time scale.

Definition 2.29. *A function $f:\mathbb{T}_0\times\mathbb{R}^n\to\mathbb{R}^n$ is called a ∇-Carathéodory function if the three following conditions hold.*

(i) for every $x\in\mathbb{R}^n$, the function $t\mapsto f(t,x)$ is ∇-measurable;

(ii) the function $x\mapsto f(t,x)$ is continuous ∇-almost every $t\in\mathbb{T}_0$;

(iii) for every $r>0$, there exists a function $h_r\in L^1_{\nabla}(\mathbb{T}_0,[0,\infty))$ such that $\|f(t,x)\|\leq h_r(t)$ for ∇-almost every $t\in\mathbb{T}_0$ and for all $x\in\mathbb{R}^n$ such that $\|x\|\leq r$.

3. Main Results

In this section, we establish an existence result for the problem (1.1) (resp.,(1.2)). Let us recall that,$\mathbb{T}$ is compact and $a=\min\mathbb{T}<\max\mathbb{T}=b$. A solution of the problem (1.1) (resp.,(1.2)) will be a function $x\in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n)$ for which (1.1)(resp.,(1.2))is satisfied.

3.1 Existence Theorem for the Problem (1.1)

We introduce the notion of solution tube for the problem (1.1).

Definition 3.1. *Let $(v,M)\in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n)\times W^{1,1}_{\nabla}(\mathbb{T},[0,\infty))$. We say that (v,M) is a solution tube of (1.1) if*

1) *$\langle x-v(\rho(t)),f(t,x)-v^{\nabla}(t)\rangle\geq M(\rho(t))M^{\nabla}(t)$ ∇-a.e. $t\in\mathbb{T}_0$ and for every $x\in\mathbb{R}^n$ such that $\|x-v(\rho(t))\|=M(\rho(t))$,*

2) *$v^{\nabla}(t)=f(t,v(\rho(t)))$ and $M^{\nabla}(t)=0$, ∇-a.e. $t\in\mathbb{T}_0$ such that $M(\rho(t))=0$,*

3) *$M(t)=0$, for every $t\in\mathbb{T}_0$ such that $M(\rho(t))=0$,*

4) *– If (BC) denotes (1.3), then $\|x_0-v(b)\|\leq M(b)$.*
   – If (BC) denotes (1.4), then $\|v(b)-v(a)\|\leq M(b)-M(a)$.

We denote

$$T(v,M)=\{x\in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n):\|x(t)-v(t)\|\leq M(t), \text{ for all } t\in\mathbb{T}\}.$$

We consider the following problem.

$$\begin{cases} x^{\nabla}(t)-x(\rho(t))=f(t,\bar{x}(\rho(t)))-\bar{x}(\rho(t)),\ \nabla\text{-a.e. } t\in\mathbb{T}_0,\\ x\in(BC), \end{cases} \tag{3.1}$$

where

$$\bar{x}(t)=\begin{cases}\frac{M(t)}{\|x-v(t)\|}(x-v(t))+v(t), & \text{if } \|x-v(t)\|>M(t),\\ x(t), & \text{otherwise.}\end{cases} \tag{3.2}$$

Lemma 3.2. *For every $g \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$, the problem*

$$\begin{cases} x^\nabla(t) - x(\rho(t)) = g(t), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b), \end{cases} \tag{3.3}$$

has a unique solution $x \in W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n)$ given by:

$$\begin{aligned} x(t) &= \frac{1}{\hat{e}_{-1}(t,b)} \left(\frac{1}{1-\hat{e}_{-1}(a,b)} \int_{(a,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s \right. \\ &\quad \left. - \int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s \right). \end{aligned}$$

Proof. Let x be a solution to (3.3). By Theorem 3.3 in [8], consider

$$\begin{aligned} [x(t)\hat{e}_{-1}(t,b)]^\nabla &= x^\nabla(t)\hat{e}_{-1}(t,b) - \hat{e}_{-1}(t,b)x(\rho(t)) \\ &= g(t)\hat{e}_{-1}(t,b), \ \ t \in \mathbb{T}_0, \end{aligned}$$

and hence integrating the above on $(t,b]\cap\mathbb{T}$ obtain

$$x(t) = \frac{1}{\hat{e}_{-1}(t,b)} \left(x(b) - \int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s \right). \tag{3.4}$$

If follows from the boundary condition in (3.3) and (3.4) that

$$x(a) = x(b) = \frac{1}{1-\hat{e}_{-1}(a,b)} \int_{(a,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s. \tag{3.5}$$

So by substituting (3.5) into (3.4), the result follows. □

Lemma 3.3. *For every $g \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$, the problem*

$$\begin{cases} x^\nabla(t) - x(\rho(t)) = g(t), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x(b) = x_0, \end{cases} \tag{3.6}$$

has a unique solution $x \in W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n)$ given by:

$$x(t) = \hat{e}_{-1}(b,t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s,b)\nabla s \Big). \tag{3.7}$$

Proof. the result follows in a similar way to the proof of Lemma 3.2. □

We obtain a maximum principle that will be useful to get a priori bounds for solutions of systems considered in this section.

Lemma 3.4. *Let $r \in W^{1,1}_\nabla(\mathbb{T})$ such that $r^\nabla(t) > 0$ ∇-a.e. $t \in \{t \in \mathbb{T}_0 : r(\rho(t)) > 0\}$. If one of the following conditions holds,*

(i) $r(b) \leq 0$;

(ii) $r(b) \leq r(a)$;

then $r(t) \leq 0$, for every $t \in \mathbb{T}$.

Proof. Suppose the conclusion is false. Then, there exists $t_0 \in \mathbb{T}$ such that $r(t_0) = \max_{t\in\mathbb{T}}(r(t) > 0)$, since r is continuous on $\mathbb{T}$. If $\sigma(t_0) > t_0$, then $r^\nabla(\sigma(t_0))$ exists, since $\nu(\sigma(t_0)) = \sigma(t_0) - t_0 > 0$ and because $r \in W^{1,1}_\nabla(\mathbb{T})$. Then,

$$r^\nabla(\sigma(t_0)) = \frac{r(\sigma(t_0)) - r(t_0)}{\sigma(t_0) - t_0} \leq 0,$$

which is a contradiction since $r(t_0) = r(\rho(\sigma(t_0))) > 0$.
If $t_0 = \sigma(t_0) < b$, then there exists an interval $(\sigma(t_0), t_1]$ such that $r(\sigma(t)) > 0$ for all $t \in (\sigma(t_0), t_1] \bigcap \mathbb{T}$. Thus,

$$\begin{aligned} 0 &\leq \int_{(\sigma(t_0),t_1]\bigcap\mathbb{T}} r^\nabla(s)\nabla s = r(t_1) - r(\sigma(t_0)) \\ &= r(t_1) - r(t_0) < 0. \end{aligned}$$

by hypothesis and by Theorem 2.22. Hence, we get a contradiction. The case $t_0 = b$ is impossible if hypothesis (i) holds and if $r(b) \leq r(a)$, we must have $r(a) = r(b)$. If we take $t_0 = a$, by using previous steps of this proof, one can check that $r(a) \leq 0$, and, then, the lemma is proved. □

Let us define the operator $T_1 : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ by

$$\begin{aligned} T_1(x)(t) &= \hat{e}_{-1}(b,t)\,(x_0 - \\ &\quad \left. \int_{(t,b]\cap\mathbb{T}} \left(f(s,\bar{x}(\rho(s))) - \bar{x}(\rho(s))\right)\hat{e}_{-1}(s,b)\nabla s \right). \end{aligned}$$

Proposition 3.5. *If $(v,M) \in W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n) \times W^{1,1}_\nabla(\mathbb{T}, [0,\infty))$ is a solution tube of (1.1) then the operator $T_1 : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ is compact.*

Proof. We first observe that from Definitions 2.29 and 3.1, there exists a function $h \in L^1_\nabla(\mathbb{T}_0, [0,\infty))$ such that $\|f(t,\bar{x}(\rho(t))) - \bar{x}(\rho(t))\| \leq h(t)$, ∇-a.e. $t \in \mathbb{T}_0$ for every $x \in C(\mathbb{T}, \mathbb{R}^n)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(\mathbb{T}, \mathbb{R}^n)$ converging to $x \in C(\mathbb{T}, \mathbb{R}^n)$. By Proposition 2.18,

$$\begin{aligned} &\|T_1(x_n(t)) - T_1(x(t))\| \\ &= \Big\| \int_{(t,b]\cap\mathbb{T}} |\hat{e}_{-1}(s,t)| \Big[\Big(f(s,\bar{x}_n(\rho(s))) - \bar{x}_n(\rho(s)) \Big) \\ &- \Big(f(s,\bar{x}(\rho(s))) - \bar{x}((\rho(s))) \Big) \Big] \nabla s \Big\| \\ &\leq K \int_{(a,b]\cap\mathbb{T}} \Big\| \Big(f(s,\bar{x}_n(\rho(s))) - \bar{x}_n(\rho(s)) \Big) \\ &- \Big(f(s,\bar{x}(\rho(s))) - \bar{x}((\rho(s))) \Big) \Big\| \nabla s, \end{aligned}$$

where $K := \max_{s,t\in\mathbb{T}} \{|\hat{e}_{-1}(s,t)|\}$. Then, we must show that the sequence $\{g_n\}_{n\in\mathbb{N}}$ defined by $g_n(s) = f(s,\bar{x}_n(\rho(s))) - \bar{x}_n(\rho(s))$ converges to the function $g(s) \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ where $g(s) = f(s,\bar{x}(\rho(s))) - \bar{x}(\rho(s))$.

We can easily check that $\bar{x}_n(t) \to \bar{x}(t)$ for every $t \in \mathbb{T}_0$ and, then, by (ii) of Definition 2.29, $g_n(s) \to g(s)$, ∇-a.e. $s \in \mathbb{T}_0$. Using also the fact that $\|g_n(s)\| \leq h(s)$, ∇-a.e. $s \in \mathbb{T}_0$, we deduce that $g_n(s) \to g(s)$ in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ by Theorem 2.19. This prove the continuity of T_1. For the second part of the proof, we have to show that the set $T_1(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact. Let $y=T_1(x)\in T_1(C(\mathbb{T},\mathbb{R}^n))$. Therefore,

$$\begin{aligned}\|T_1(x(t))\| &\leq K\Big(\|x_0\| \\ &\qquad +K\int_{(a,b]\cap\mathbb{T}}\|f(s,\bar{x}(\rho(s)))-\bar{x}(\rho(s))\|\nabla s\Big). \\ &\leq K\Big(\|x_0\|+K\|h(s)\|_{L^1_\nabla(\mathbb{T}_0,\mathbb{R}^n)}\Big).\end{aligned}$$

So, $T_1(C(\mathbb{T},\mathbb{R}^n))$ is uniformly bounded. This set is also equicontinuous since for every $t_1<t_2\in\mathbb{T}$,

$$\begin{aligned}&\|T_1(x)(t_2)-T_1(x)(t_1)\| \\ &\leq|\hat{e}_{-1}(b,t_2)-\hat{e}_{-1}(b,t_1)|\Big(\|x_0\| \\ &+\int_{(t_2,b]\cap\mathbb{T}}|\hat{e}_{-1}(s,b)|\|f(s,\bar{x}(\rho(s)))-\bar{x}(\rho(s))\|\nabla s\Big) \\ &+\int_{[t_1,t_2]\cap\mathbb{T}}|\hat{e}_{-1}(s,t_1)|\,\|f(s,\bar{x}(\rho(s)))-\bar{x}(\rho(s))\|\nabla s \\ &\leq|\hat{e}_{-1}(b,t_2)-\hat{e}_{-1}(b,t_1)|\Big(\|x_0\| \\ &+K\int_{(a,b]\cap\mathbb{T}}h(s)\nabla s\Big)+K\int_{(t_1,t_2]\cap\mathbb{T}}h(s)\nabla s.\end{aligned}$$

By an analogous version of the Arzelà-Ascoli Theorem adapted to our context, $T_1(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact in $C(\mathbb{T},\mathbb{R}^n)$. Hence, T_1 is compact. □

We now define the operator $T_p:C(\mathbb{T},\mathbb{R}^n)\to C(\mathbb{T},\mathbb{R}^n)$ by:

$$\begin{aligned}&T_p(x)(t) \\ &=\frac{1}{\hat{e}_{-1}(t,b)}\left[\frac{1}{1-\hat{e}_{-1}(a,b)}\int_{(a,b]\cap\mathbb{T}}f(t,\bar{x}(\rho(t)))\nabla s\right. \\ &-\int_{(a,b]\cap\mathbb{T}}\bar{x}(\rho(t))\hat{e}_{-1}(s,b)\nabla s- \\ &\left.\int_{(t,b]\cap\mathbb{T}}(f(t,\bar{x}(\rho(t)))-\bar{x}(\rho(t)))\,\hat{e}_{-1}(s,b))\nabla s\right].\end{aligned}$$

The following result can be proved as the previous one.

Proposition 3.6. *If $(v,M)\in W^{1,1}_\nabla(\mathbb{T},\mathbb{R}^n)\times W^{1,1}_\nabla(\mathbb{T},[0,\infty))$ is a solution tube of (1.1) then the operator $T_p:C(\mathbb{T},\mathbb{R}^n)\to C(\mathbb{T},\mathbb{R}^n)$ is compact.*

Now, we can obtain the main theorem of this section.

Theorem 3.7. *If $(v,M)\in W^{1,1}_\nabla(\mathbb{T},\mathbb{R}^n)\times W^{1,1}_\nabla(\mathbb{T},[0,\infty))$ is a solution tube of (1.1) then the problem (1.1) has a solution $x\in W^{1,1}_\nabla(\mathbb{T},\mathbb{R}^n)\cap T(v,M)$.*

Proof. By Proposition 3.5 (resp., Proposition 3.6), T_1(resp., T_p) is compact. It has a fixed point by the Schauder fixed-point Theorem. Lemma 3.2 (resp., Lemma 3.3) implies that this fixed point is a solution for the problem (3.1). Then, it suffices to show that for every solution x of (3.1), $x\in T(v,M)$. Consider the set $A=\{t\in\mathbb{T}_0:\|x(\rho(t))-v(\rho(t))\|>M(\rho(t))\}$. By Example 2.5, ∇-a.e. on the set $\tilde{A}=\{t\in A:t=\rho(t)\}$, we have

$$\begin{aligned}&\|x(t)-v(t)\|-M(t))^\nabla \\ &=\frac{<x(\rho(t))-v(\rho(t)),x^\nabla(t)-v^\nabla(t)>}{\|x(\rho(t))-v(\rho(t))\|}-M^\nabla(t).\end{aligned}\tag{3.8}$$

If $t\in A$ is left scattered, then $\nu(t)=t-\rho(t)>0$ and

$$\begin{aligned}&(\|x(t)-v(t)\|-M(t))^\nabla \\ &=\frac{\|x(\rho(t))-v(\rho(t))\|\|x(t)-v(t)\|-\|x(\rho(t))-v(\rho(t))\|^2}{\nu(t)\|x(\rho(t))-v(\rho(t))\|} \\ &-M^\nabla(t) \\ &\geq\frac{\langle x(\rho(t))-v(\rho(t)),(x(t)-v(t))-(x(\rho(t))-v(\rho(t)))\rangle}{\nu(t)\|x(\rho(t))-v(\rho(t))\|} \\ &-M^\nabla(t) \\ &=\frac{\langle x(\rho(t))-v(\rho(t)),x^\nabla(t)-v^\nabla(t)\rangle}{\|x(\rho(t))-v(\rho(t))\|}-M^\nabla(t).\end{aligned}$$

Since (v,M) is a solution tube of (1.1), we have ∇-a.e. on $\{t\in A:M(\rho(t))>0\}$,

$$\begin{aligned}&(\|x(t)-v(t)\|-M(t))^\nabla \\ \geq\;&\frac{\langle x(\rho(t))-v(\rho(t)),f(t,\bar{x}(\rho(t)))-\bar{x}(\rho(t))}{\|x(\rho(t))-v(\rho(t))\|} \\ &\frac{+x(\rho(t))-v^\nabla(t)\rangle}{\|x(\rho(t))-v(\rho(t))\|}-M^\nabla(t) \\ =\;&\frac{\langle\bar{x}(\rho(t))-v(\rho(t)),f(t,\bar{x}(\rho(t)))-v^\nabla(t))\rangle}{M(\rho(t))} \\ &+\frac{1}{\|x(\rho(t))-v(\rho(t))\|}\frac{\langle x(\rho(t))-v(\rho(t)),}{\|x(\rho(t))-v(\rho(t))\|} \\ &+\frac{(\|x(\rho(t))-v(\rho(t))\|-M(\rho(t)))\,(x(\rho(t))-v(\rho(t)))\rangle}{\|x(\rho(t))-v(\rho(t))\|} \\ &-M^\nabla(t) \\ =\;&\frac{M(\rho(t))M^\nabla(t)}{M(\rho(t))}+\|x(\rho(t))-v(\rho(t))\| \\ &-M(\rho(t))-M^\nabla(t).\end{aligned}$$

On the other hand, we have ∇-a.e. on $\{t\in A:M(\rho(t))=0\}$ that

$$\begin{aligned}&(\|x(t)-v(t)\|-M(t))^\nabla \\ &\geq\frac{\langle x(\rho(t))-v(\rho(t)),f(t,\bar{x}(\rho(t)))-\bar{x}(\rho(t))+x(\rho(t))-v^\nabla(t)\rangle}{\|x(\rho(t))-v(\rho(t))\|} \\ &-M^\nabla(t) \\ &=\frac{\langle x(\rho(t))-v(\rho(t)),f(t,v(\rho(t)))-v^\nabla(t))\rangle}{\|x(\rho(t))-v(\rho(t))\|} \\ &+\|x(\rho(t))-v(\rho(t))\|-M^\nabla(t)>0.\end{aligned}$$

This last equality follows from Definition 3.1(3) and Proposition 2.24.

If we set $r(t)=\|x(t)-v(t)\|-M(t)$, then $r^\nabla(t)>0$ ∇-a.e. $t\in\{t\in\mathbb{T}_0:r(\rho(t))>0\}$. Moreover, since (v,M) is a solution tube of (1.1) and x satisfies (1.3)(resp., x satisfies (1.4)), then $r(a)<0$ (resp.,$r(a)-r(b)\leq\|v(a)-v(b)\|-(M(a)-M(b))\leq 0$. Lemma 3.4 implies that $A=\emptyset$. So, $x\in T(v,M)$ and the theorem is proved. □

3.2 Existence Theorem for the Problem (1.2)

We introduce the notion of solution tube for the problem (1.2).

Definition 3.8. *Let $(v,M) \in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T},[0,\infty))$. We say that (v,M) is a solution tube of (1.2) if*

1. $\langle x-v(t),\ f(t,x)-v^{\nabla}(t)\rangle \leq M(t)M^{\nabla}(t)$ *∇-a.e. $t \in \mathbb{T}_0$ and for every $x \in \mathbb{R}^n$ such that $\|x-v(t)\| = M(t)$,*
2. $v^{\nabla}(t) = f(t,v(t))$ *∇-a.e. $t \in \mathbb{T}_0$ such that $M(t)=0$,*
3. $\|v(b)-v(a)\| \leq M(a)-M(b)$.

We consider the following problem.

$$\begin{cases} x^{\nabla}(t)+x(t) = f(t,\bar{x}(t))+\bar{x}(t), & \nabla\text{-.a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b). \end{cases} \tag{3.9}$$

where $\bar{x}(t)$ is defined in (3.2).

Lemma 3.9. *For every $g \in L^1_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)$, the problem*

$$\begin{cases} x^{\nabla}(t)+x(t) = g(t), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x(a) = x(b), \end{cases} \tag{3.10}$$

has a unique solution $x \in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n)$ given by:

$$\begin{aligned} x(t) \quad &= \quad \hat{e}_{-1}(t,b)\left(\frac{\hat{e}_{-1}(a,b)}{\hat{e}_{-1}(a,b)-1}\int_{(a,b]\cap\mathbb{T}}\frac{g(s)}{\hat{e}_{-1}(\rho(s),b)}\nabla s \right. \\ &\quad \left. -\int_{(t,b]\cap\mathbb{T}}\frac{g(s)}{\hat{e}_{-1}(\rho(s),b)}\nabla s\right). \end{aligned}$$

Proof. Let x be a solution to (3.10). By Theorem 3.3 in [8], consider

$$\left[\frac{x(t)}{\hat{e}_{-1}(t,b)}\right]^{\nabla} = \frac{x^{\nabla}(t)\hat{e}_{-1}(t,b)+\hat{e}_{-1}(t,b)x(t)}{\hat{e}_{-1}(t,b)\hat{e}_{-1}(\rho(t),b)} = \frac{g(t)}{\hat{e}_{-1}(\rho(t),b)}.$$

Integrating the above on $(t,b]\cap\mathbb{T}$ and using boundary condition in (3.10), the result follows in a similar way to the proof of Lemma 3.2. □

The following lemma is similar to lemma 2.11 in [5].

Lemma 3.10. *Let $r \in W^{1,1}_{\nabla}(\mathbb{T})$ such that $r^{\nabla}(t) < 0$ ∇-a.e. $t \in \{t \in \mathbb{T}_0 : r(t) > 0\}$. If $r(a) \leq r(b)$, then $r(t) \leq 0$, for every $t \in \mathbb{T}$.*

Let us define the operator $T_2 : C(\mathbb{T},\mathbb{R}^n) \to C(\mathbb{T},\mathbb{R}^n)$ by:

$$\begin{aligned} T_2(x)(t) = \hat{e}_{-1}(t,b)&\left(\frac{\hat{e}_{-1}(a,b)}{\hat{e}_{-1}(a,b)-1}\times\right. \\ &\int_{(a,b]\cap\mathbb{T}}\frac{(f(s,\bar{x}(s))+\bar{x}(s))}{\hat{e}_{-1}(\rho(s),b)}\nabla s \\ &\left.-\int_{(t,b]\cap\mathbb{T}}\frac{(f(s,\bar{x}(s))+\bar{x}(s))}{\hat{e}_{-1}(\rho(s),b)}\nabla s\right). \end{aligned}$$

Proposition 3.11. *If $(v,M) \in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T},[0,\infty))$ is a solution tube of (1.2) then the operator $T_2 : C(\mathbb{T},\mathbb{R}^n) \to C(\mathbb{T},\mathbb{R}^n)$ is compact.*

Proof. We first observe that from Definitions 2.29 and 3.8, there exists a function $h \in L^1_{\nabla}(\mathbb{T}_0,[0,\infty))$ such that $\|f(t,\bar{x}(t))+\bar{x}(t)\| \leq h(t)$, ∇-a.e. $t \in \mathbb{T}_0$ for every $x \in C(\mathbb{T},\mathbb{R}^n)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(\mathbb{T},\mathbb{R}^n)$ converging to $x \in C(\mathbb{T},\mathbb{R}^n)$. By Proposition 2.18,

$$\begin{aligned} &\|T_2(x_n(t))-T_2(x(t))\| \\ &\leq \frac{K(C+1)}{m}\int_{(a,b]\cap\mathbb{T}}\|(f(s,\bar{x}_n(s))+\bar{x}_n(s))-(f(s,\bar{x}(s))+\bar{x}(s))\|\nabla s, \end{aligned}$$

where $K := \max_{t\in\mathbb{T}}|\hat{e}_{-1}(t,b)|$, $C = \left|\frac{\hat{e}_{-1}(a,b)}{\hat{e}_{-1}(a,b)-1}\right|$ and $m := \min_{t\in\mathbb{T}}|\hat{e}_{-1}(t,b)|$. Then, we must show that the sequence $\{g_n\}_{n\in\mathbb{N}}$ defined by $g_n(s) = f(s,\bar{x_n}(s))+\bar{x_n}(s)$ converges to the function $g(s) \in L^1_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)$ where $g(s) = f(s,\bar{x}(s))+\bar{x}(s)$. We can easily check that $\bar{x}_n(t) \to \bar{x}(t)$ for every $t \in \mathbb{T}_0$ and, then, by (ii) of Definition 2.29, $g_n(s) \to g(s)$ ∇-a.e. $s \in \mathbb{T}_0$. Using also the fact that $\|g_n(s)\| \leq h(s)$, ∇-a.e. $s \in \mathbb{T}_0$, we deduce that $g_n(s) \to g(s)$ in $L^1_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)$ by Theorem 2.19. This prove the continuity of T_2. For the second part of the proof, we have to show that the set $T_2(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact. Let $y = T_2(x) \in T_2(C(\mathbb{T},\mathbb{R}^n))$. Therefore,

$$\|T_2(x(t))\| \quad \leq \quad \frac{K(C+1)}{m}\left(\|h(s)\|_{L^1_{\nabla}(\mathbb{T}_0,\mathbb{R}^n)}\right).$$

So, $T_2(C(\mathbb{T},\mathbb{R}^n))$ is uniformly bounded. This set is also equicontinuous since for every $t_1 < t_2 \in \mathbb{T}$,

$$\begin{aligned} &\|T_2(x)(t_2)-T_2(x)(t_1)\| \\ &\leq \left|\hat{e}_{-1}(b,t_2)-\hat{e}_{-1}(b,t_1)\right|\frac{(C+1)}{m}\int_{(a,b]\cap\mathbb{T}}h(s)\nabla s \\ &+K\int_{(t_1,t_2]\cap\mathbb{T}}h(s)\nabla s. \end{aligned}$$

By an analogous version of the Arzelà-Ascoli Theorem adapted to our context, $T_2(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact in $C(\mathbb{T},\mathbb{R}^n)$. Hence, T_2 is compact. □

Here is the main existence theorem for problem (1.2).

Theorem 3.12. *If $(v,M) \in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T},[0,\infty))$ is a solution tube of (1.2) then the problem (1.2) has a solution $x \in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n) \cap T(v,M)$.*

Proof. By Proposition 3.11, T_2 is compact. It has a fixed point by the Schauder fixed-point Theorem. Lemma 3.3 implies that this fixed point is a solution for the problem (3.9). Then, it suffices to show that for every solution x of (3.9), $x \in T(v,M)$. Consider the set $A = \{t \in \mathbb{T}_0 : \|x(t)-v(t)\| > M(t)\}$. By example 2.5, ∇-a.e. on the set $\tilde{A} = \{t \in A : t = \rho(t)\}$, we have

$$(\|x(t)-v(t)\|-M(t))^{\nabla} = \frac{\langle x(t)-v(t), x^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t) \tag{3.11}$$

If $t \in A$ is left scattered, then $\nu(t) = t - \rho(t) > 0$ and

$$
\begin{aligned}
&(\|x(t)-v(t)\| - M(t))^{\nabla} \\
&= \frac{\|x(t)-v(t)\|^2 - \|x(\rho(t)) - v(\rho(t))\|\|x(t)-v(t)\|}{\nu(t)\|x(t)-v(t)\|} \\
&- M^{\nabla}(t) \\
&\leq \frac{\langle x(t)-v(t), (x(t)-v(t)) - (x(\rho(t)) - v(\rho(t)))\rangle}{\nu(t)\|x(t)-v(t)\|} \\
&- M^{\nabla}(t) \\
&= \frac{\langle x(t)-v(t),\, x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t).
\end{aligned}
$$

Since (v, M) is a solution tube of (1.2), we have ∇-a.e. on $\{t \in A : M(t) > 0\}$,

$$
\begin{aligned}
&(\|x(t)-v(t)\| - M(t))^{\nabla} \\
&\leq \frac{\langle x(t)-v(t), f(t,\bar{x}(t)) + \bar{x}(t) - x(t) - v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t) \\
&= \frac{\langle \bar{x}(t)-v(t), f(t,\bar{x}(t)) - v^{\nabla}(t)\rangle}{M(t)} \\
&+ \frac{\langle x(t)-v(t), \dfrac{(M(t) - \|x(t)-v(t)\|)}{\|x(t)-v(t)\|}(x(t)-v(t))\rangle}{\|x(t)-v(t)\|} \\
&- M^{\nabla}(t) = \frac{M(t)M^{\nabla}(t)}{M(t)} - \\
&(\|x(t)-v(t)\| - M(t)) - M^{\nabla}(t) < 0.
\end{aligned}
$$

On the other hand, we have ∇-a.e. on $\{t \in A : M(t) = 0\}$, that

$$
\begin{aligned}
&(\|x(t)-v(t)\| - M(t))^{\nabla} \\
&\leq \frac{\langle x(t)-v(t), f(t,\bar{x}(t)) + \bar{x}(t) - x(t) - v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t) \\
&= \frac{\langle x(t)-v(t), f(t,v(t)) - v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} \\
&- \|x(t)-v(t)\| - M^{\nabla}(t) < 0.
\end{aligned}
$$

If we set $r(t) = \|x(t)-v(t)\| - M(t)$, then $r^{\nabla}(t) < 0$ ∇-a.e. $t \in \{t \in \mathbb{T}_0 : r(t) > 0\}$. Moreover, since (v, M) is a solution tube of (1.2) and x satisfies (1.4), then $r(a) - r(b) \leq \|v(a) - v(b)\| - (M(a) - M(b)) \leq 0$. Lemma 3.10 implies that $A = \emptyset$. So, $x \in T(v, M)$ and the theorem is proved. □

Example 3.13. *The following is a modified version, considering a periodic condition, of Example 4.1 in [14]:*

$$
\begin{cases}
x^{\nabla}(t) = a_1\|x(t)\|^2 x(t) - a_2 x(t) + a_3\varphi(t), & t \in \mathbb{T}_0, \\
x(a) = x(b).
\end{cases}
\tag{3.12}
$$

where $a_1, a_2, a_3 \in \mathbb{R}_+$ such that $a_2 \geq a_1 + a_3$ and $\varphi : \mathbb{T}_0 \to \mathbb{R}^n$ is a continuous function satisfying $\|\varphi(t)\| = 1$ for every $t \in \mathbb{T}_k$.

It is easy to check that $v = 0$ and $M = 1$, is a tube solution. By Theorem 3.12, problem (3.12) has a solution $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ such that $\|x(t)\| \leq 1$ for every $t \in \mathbb{T}_0$.

Remark 3.14. *Definition 3.8 generalizes the notions of lower and upper solutions α and β introduced in [20] in the particular case where the problem (1.2) is considered with $n = 1$, and the periodic boundary condition replaced by $x(0) = 0$ and f is left-Hilger continuous on $(0, b]_{\mathbb{T}} \times \mathbb{R}$. We recall these definitions.*
Consider the problem:

$$
\begin{cases}
x^{\nabla}(t) = f(t, x(t)), & \textit{for all } t \in (0,\, b]_{\mathbb{T}}, \\
x(0) = x(b).
\end{cases}
\tag{3.13}
$$

Definition 3.15. Let α, β be nabla differentiable functions on $(0, b]_{\mathbb{T}}$. We call α a lower solution to (3.13) on $[0, b]_{\mathbb{T}}$ if

(*i*) $\alpha^{\nabla}(t) \leq f(t, \alpha(t))$, for all $t \in (0, b]_{\mathbb{T}}$;

(*ii*) $\alpha(0) = \alpha(b)$.

Similarly, we call β an upper solution to (3.13) on $[0, b]_{\mathbb{T}}$ if

(*i*) $\beta^{\nabla}(t) \geq f(t, \beta(t))$, *for all* $t \in (0, b]_{\mathbb{T}}$;

(*ii*) $\beta(0) = \beta(b)$.

Remark 3.16. *If α, $\beta \in \mathbb{R}$, are, respectively, lower and upper solutions of (3.13) such that $\alpha(t) \leq \beta(t)$ for every $t \in (0, b]_{\mathbb{T}}$, then $v = (\alpha + \beta)/2$ and $M = (\beta - \alpha)/2$ is a solution tube for this problem. Conversely, if (v, M) is a solution tube of (3.13) with v and M of class C^1, $v(0) = v(b)$, and $M(0) = M(b)$, then $\alpha = v - M$ and $\beta = v + M$ are, respectively, lower and upper solutions of (3.13).*

Example 3.17. Consider the problem:

$$
\begin{cases}
x^{\nabla}(t) = x^3(t) - t,\ t \in (0,1]_{\mathbb{T}}, \\
x(0) = x(1).
\end{cases}
\tag{3.14}
$$

Verify that with $v = 0$ and $M = 1$, (v, M) is a solution-tube of (3.14). By Theorem 3.12, the problem (3.14) has a solution x such that $|x(t)| \leq 1$ for every $t \in (0,1]_{\mathbb{T}}$. Observe that $\alpha = v - M$ and $\beta = v + M$ are, respectively, lower and upper solutions of (3.14) and $-1 \leq x \leq 1$.

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