On almost contra $\delta_{gp}$-continuous functions in topological spaces

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Abstract
The aim of this paper is to introduce a new class of almost contra continuity. The notion of almost contra $\delta_{gp}$-continuous functions is introduced and studied.

Keywords
$\delta_{gp}$-open set, $\delta_{gp}$-closed set, almost contra pre-continuous function, almost contra $\delta_{gp}$-continuous function.

AMS Subject Classification
54C08, 54C10.

1 Introduction

Recently, Baker (resp, Ekici, Balasubramaniam and Laxmi) introduced and investigated the notions of almost contra continuity [3] (resp, almost contra pre-continuity [10] and almost contra $gp$-continuity [4] as a continuation of research done by Dontchev (resp, S. Jafari and T. Noiri and P. Jeyalakshmi) on the notion of contra continuity [9] (resp, contra pre-continuity [16] and contra $gp$-continuity [18]).

In this paper, we offer a stronger form of almost contra $gp$-continuity called almost contra $\delta_{gp}$-continuity. Also, some properties and characterizations of the said type of functions are investigated.

Throughout this paper, $(U, \tau), (V, \sigma)$ and $(W, \eta)$ (or simply $U, V$ and $W$) represent topological spaces on which no separation axioms are assumed unless explicitly stated and $f: (U, \tau) \rightarrow (V, \sigma)$ or simply $f: U \rightarrow V$ denotes a function $f$ of a topological space $U$ into a topological space $V$. Let $M \subseteq U$, then $cl(M) = \cap \{F: M \subseteq F$ and $F \in \tau \}$ is the closure of $M$. Also, $int(M) = \cup \{O: O \subseteq M$ and $O \in \tau \}$ is the interior of $M$. The class of $\delta_{gp}$-open (resp, $\delta_{gp}$-closed, open, closed, regular open, regular closed, $\delta$-preopen, $\delta$-semiopen, $e^*-open$, pre-open, semiopen, $\beta$-open and clopen) sets of $(U, \tau)$ is denoted by $\delta GPO(U)$ (resp, $\delta GPC(U)$, $O(U)$, $C(U)$, $RO(U)$, $RC(U)$, $\delta PO(U)$, $\delta SO(U)$, $e^*O(U)$, $PO(U)$, $SO(U)$, $\beta O(U)$ and $CO(U)$).

2 Preliminaries

Definition 2.1. A set $M \subseteq U$ is called $\delta$-closed [36] if $M = cl(\delta(M))$ where $cl(\delta(M)) = \{p \in U : int(cl(G)) \cap M \neq \phi, G \in \tau \}$ and $p \in G$. The complement of a $\delta$-closed set is called $\delta$-open.

Definition 2.2. A set $M \subseteq U$ is called pre-$\delta$-closed [21] (resp, $b$-closed [1], regular-$\delta$-closed [33], semi-$\delta$-closed [19] and $\alpha$-closed [22] if $cl(int(M)) \subseteq M$ (resp, $cl(int(M)) \cap int(cl(M)) \subseteq M$, $M = cl(int(M))$, $int(cl(M)) \subseteq M$ and $cl(int(cl(M))) \subseteq M$).

Definition 2.3. A set $M \subseteq U$ is called $\delta$-pre-closed [27] (resp, $e^*$-closed [13], $\delta$-semiclosed [26] and $a$-closed [14]) if $cl(int(\delta(M))) \subseteq M$ (resp, $cl(int(\delta(M))) \subseteq M$, $int(cl(\delta(M))) \subseteq M$ and $cl(int(cl(\delta(M)))) \subseteq M$).

Definition 2.4. A set $M \subseteq U$ is called:
(i) $\delta_{gp}$-closed [7] (resp, gpr-closed [15] and gp-closed [20]) if $pcl(M) \subseteq G$ whenever $M \subseteq G$ and $G$ is $\delta$-open (resp, regular open and open) in $U$.
(ii) $g\delta$s-closed [5] if $sc(M) \subseteq G$ whenever $M \subseteq G$ and $G$ is $\delta$-open in $U$.

Definition 2.5. A function $f: (U, \tau) \rightarrow (V, \sigma)$ is said to be:
(i) almost contra continuous [3] (resp, contra $R$-map [11], $\delta$-continuous [23], almost contra super-continuous [12], almost...
contra pre-continuous [10], almost contra gp-continuous, almost contra gpr-continuous [4] and almost contra gδs-continuous [6] if g−1(N) is closed (resp, regular closed, δ-open, δ-closed, pre-closed, gp-closed, gpr-closed and gδs-closed) in (X, τ) for every N ∈ RO(Y).

(ii) contra continuous [9] (resp, contra pre-continuous [16], contra gp-continuous [35] and contra gpr-continuous [18]) if g−1(N) is closed (resp, pre-closed, gp-closed and gpr-closed) in U for every N ∈ σ.

(iii) perfectly-continuous [24] (resp, almost perfectly-continuous [29]) if g−1(N) ∈ CO(U) for every N ∈ σ (resp, RO(Y)).

(iv) R-map [8] if g−1(N) ∈ RO(U) for every N ∈ RO(U).

Definition 3.2. Let \( f: U \to V \) be a locally indiscrete space, then the following diagram for a function \( f: U \to V \):

\[
\begin{array}{cccccc}
1 & \to & 2 & \to & 3 & \to & 4 & \to & 5 & \to & 6 & \to & 7 \\
& & & & & & & & & & & & & & 8
\end{array}
\]

Definition 3.1. A function \( f: U \to V \) is called almost contra \( \delta gp \)-continuous if the inverse image of every regular open set of \( V \) is \( \delta gp \)-closed in \( U \).

Theorem 3.2. The following are equivalent for \( f: U \to V \):

(i) \( f \) is almost contra \( \delta gp \)-continuous.

(ii) For every \( M \in \text{RC}(V) \), \( f^{-1}(M) \in \delta GPO(U) \).

Proof. Clear.

Remark 3.3. From Definitions 2.5 and 3.1, we have the following diagram for a function \( f: U \to V \):

\[
\begin{array}{cccccc}
1 & \to & 2 & \to & 3 & \to & 4 & \to & 5 & \to & 6 & \to & 7 \\
& & & & & & & & & & & & & & 8
\end{array}
\]

Notation: 1-contra R-map. 2-almost contra super-continuous. 3-almost contra continuity. 4-almost contra pre continuity. 5-almost contra gp-continuity. 6-almost contra \( \delta gp \)-continuity. 7-almost contra gpr-continuity. 8- contra \( \delta gp \)-continuous.

None of these implications are reversible.

Example 3.4. Consider \( (U, \tau) \) and \( (V, \eta) \) where \( U = \{a, b, c, d\} \), \( V = \{a, b, c, d\} \), \( \tau = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\} \} \), \( \eta = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \} \).

Define \( f: (U, \tau) \to (V, \eta) \) by \( f(a) = q, f(b) = d, f(c) = s, f(d) = r \).

Clearly \( f \) is almost contra \( \delta gp \)-continuous but for \( \{d\} \in \text{RO}(V) \), \( f^{-1}({\{d\}}) = \{a\} \notin \delta GP(U) \). Therefore \( f \) is not almost contra \( \delta gp \)-continuous.

Example 3.12. Consider \( (U, \tau) \) and \( (V, \eta) \) where \( U = \{p, q, r, s\} \), \( V = \{a, b, c, d, \}, \tau = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\} \} \), \( \eta = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \} \).

Define \( f: (U, \tau) \to (V, \eta) \) by \( f(a) = p, f(b) = q, f(c) = r, f(d) = s \).

Clearly \( f \) is almost contra \( \delta gp \)-continuous but for \( \{a\} \in \text{RO}(V) \), \( f^{-1}({\{a\}}) = \{p\} \notin \delta GC(U) \). Therefore \( f \) is not almost contra \( \delta gp \)-continuous.

Theorem 3.13. Let \( U \) be a locally indiscrete space, then the following properties are equivalent:

(i) \( f: U \to V \) is almost contra \( \delta gp \)-continuous.

(ii) \( f: U \to V \) is almost contra \( \delta gp \)-continuous.

(iii) \( f: U \to V \) is almost contra \( \delta gp \)-continuous.

Remark 3.9. almost contra \( \delta gp \)-continuity and almost contra \( \delta gs \)-continuity are independent each other.

Example 3.10. In Example 3.4, \( f \) is \( \delta gp \)-continuous but it is not a contra \( \delta gp \)-continuous.

Example 3.11. Consider \( (U, \tau) \) and \( (V, \eta) \) as in Example 3.4, Define \( f: (U, \tau) \to (V, \eta) \) by \( f(p) = q, f(q) = s, f(r) = p \) and \( f(s) = r \). Then \( f \) is almost contra \( \delta gp \)-continuous but for \( \{q\} \in \text{RO}(V) \), \( f^{-1}({\{q\}}) = \{p\} \notin \delta GC(U) \). Therefore \( f \) is not almost contra \( \delta gp \)-continuous.
Theorem 3.14. (1) Let $U$ be extremely disconnected space, then every almost contra $\delta g\delta s$-continuous function $f: U \to V$ is almost contra $\delta gp$-continuous.

(2) Let $U$ be strongly irresolvable space. Then every almost contra $\delta gp$-continuous function $f: U \to V$ is almost contra $\delta g\delta s$-continuous.

Lemma 3.15. [34] The following are equivalent for any $M \subseteq U$:

1. $M$ is clopen.
2. $M$ is open and pre-closed.
3. $M$ is open and gp-closed.
4. $M$ is $\delta$-open and $\delta gp$-closed.
5. $M$ is regular-open and gpr-closed.
6. $M$ is regular-open and pre-closed.
7. $M$ is $\delta$-open and pre-closed.

Theorem 3.16. The following statements are equivalent:

1. $f: U \to V$ is almost perfectly continuous.
2. $f: U \to V$ is almost continuous and almost contra pre-continuous.
3. $f: U \to V$ is almost continuous and almost contra gp-continuous.
4. $f: U \to V$ is $\delta$-continuous and almost contra $\delta gp$-continuous.
5. $f: U \to V$ is R-map and almost contra gpr-continuous.
6. $f: U \to V$ is R-map and almost contra pre-continuous.
7. $f: U \to V$ is $\delta$-continuous and almost contra pre-continuous.

Theorem 3.17. Let $U$ be a $\delta gpT_{\frac{1}{2}}$-space. Then the following statements are equivalent:

1. $f: U \to V$ is almost contra pre-continuous.
2. $f: U \to V$ is almost contra gp-continuous.
3. $f: U \to V$ is almost contra $\delta gp$-continuous.

Theorem 3.18. Let $U$ be a preregular$T_{\frac{1}{2}}$-space. Then the following statements are equivalent:

1. $f: U \to V$ is almost contra pre-continuous.
2. $f: U \to V$ is almost contra gp-continuous.
3. $f: U \to V$ is almost contra $\delta gp$-continuous.
4. $f: U \to V$ is almost contra gpr-continuous.

Theorem 3.19. Let $U$ be a $T_{\delta gp}$-space. Then the following are equivalent:

1. $f: U \to V$ is almost contra continuous.
2. $f: U \to V$ is almost contra pre-continuous.
3. $f: U \to V$ is almost contra gp-continuous.
4. $f: U \to V$ is almost contra $\delta gp$-continuous.
5. $f: U \to V$ is almost contra gpr-continuous.

Theorem 3.20. The following properties are equivalent:

1. $f: U \to V$ is almost contra $\delta gp$-continuous.
2. for every $M \in \beta O(V)$, $f^{-1}(cl(M)) \in \delta GPO(U)$.
3. for every $M \in SO(V)$, $f^{-1}(cl(M)) \in \delta GPO(U)$.
4. for every $M \in PO(V)$, $f^{-1}(int(cl(M))) \in \delta GPC(U)$.
5. for every $M \in O(V)$, $f^{-1}(int(cl(M))) \in \delta GPC(U)$.
6. for every $F \in C(V)$, $f^{-1}(int(cl(F))) \in \delta GPO(U)$.

Proof. (1) $\rightarrow$ (2) Let $M \in \beta O(V)$, then $cl(M) \in RC(V)$. Then by (1), $f^{-1}(cl(M))$ is $\delta gp$-open in $U$.

(2) $\rightarrow$ (3) Obvious.

(3) $\rightarrow$ (4) Let $M \in PO(V)$. Then $\forall \int(cl(M))$ is regular closed and hence it is semi-open. By (3), $f^{-1}(\int(cl(M))) = f^{-1}(\int(cl(M))) = f^{-1}((\int(cl(M))) \in \delta GPO(U)$. Hence $f^{-1}(\int(cl(M))) \in \delta GPO(U)$.

(4) $\rightarrow$ (1) Let $H \in RO(V)$. Then $H \in PO(V)$. By (4), $f^{-1}(H) = f^{-1}(\int(H))$ is $\delta gp$-closed in $U$.

(5) $\rightarrow$ (1) Similar to (1) $\rightarrow$ (5).

(6) $\rightarrow$ (5) Similar to (1) $\rightarrow$ (5).

Lemma 3.21. [25] The following properties hold for any $M \subseteq U$:

1. $\alpha cl(M) = cl(M)$ for every $M \in \beta O(U)$.
2. $pcl(M) = cl(M)$ for every $M \in SO(U)$.
3. $scl(M) = int(cl(M))$ for every $M \in PO(U)$.

Theorem 3.22. The following statements are equivalent:

1. $f: U \to V$ is almost contra $\delta gp$-continuous.
2. for every $A \in \beta O(V)$, $f^{-1}(\alpha cl(A)) \in \delta GPO(U)$.
3. for every $A \in SO(V)$, $f^{-1}(pcl(A)) \in \delta GPO(U)$.
4. for every $A \in PO(V)$, $f^{-1}(scl(A)) \in \delta GPC(U)$.

Theorem 3.23. [7] Let $M \subseteq U$. Then $p \in \delta gpc(M)$ if and only if $H \cap M \neq \Phi$ for every $H \in \delta GPO(U,p)$.

Recall that for a set $M \subseteq U$, $rker(M) = \cap \{G \in RO(U): M \subseteq G \}$ where $rker(M)$ is called the kernel of $M$ [11].

Lemma 3.24. [11] For any sets $M,N \subseteq U$, the following hold:
Theorem 3.26. The following properties are equivalent:

(a) $p \in rker(M)$ if and only if $M \cap F = \emptyset$ for every $F \in RC(U,p)$

(b) $M \subseteq rker(M)$ and $M = rker(M)$ if $M \in RO(U)$

(c) If $M \subseteq N$, then $rker(M) \subseteq rker(N)$.

Definition 3.25. [34] A space $U$ is called $\delta gp$-additive if $\deltaGPC(U)$ is closed under arbitrary intersections.

Theorem 3.26. The following properties are equivalent:

(1) $f: U \to V$ is almost contra $\delta gp$-continuous and $U$ is $\delta gp$-additive.

(2) For each $p \in U$ and each $N \in RC(V,f(p))$, there exists an $M \in \deltaGPC(U,p)$ such that $f(M) \subseteq N$.

(3) For each $p \in U$ and each $B \in SO(V,f(p))$, there exists an $A \in \deltaGPC(U,p)$ such that $f(A) \subseteq cl(B)$.

(4) $f(\delta gpcl(C)) \subseteq rker(f(C))$ for any $M \subseteq U$.

(5) $\delta gpcl(f^{-1}(D)) \subseteq f^{-1}(rker(D))$ for any $D \subseteq V$.

Proof. (1)\to(2) Let $N \in RC(V)$ such that $f(p) \in N$, then $p \in f^{-1}(N)$. By hypothesis, $f^{-1}(N) \in \deltaGPC(U)$. Set $M = f^{-1}(N)$, then $f(M) = f(f^{-1}(N)) \subseteq N$.

(2)\to(3) Let $B \in SO(V)$ such that $f(p) \in B$, then $cl(B) \in RC(V)$. By hypothesis, $f^{-1}(cl(B)) \in \deltaGPC(U)$ and $p \in f^{-1}(cl(B))$. Set $A = f^{-1}(cl(B))$, then $f(A) = f(f^{-1}(cl(B))) \subseteq cl(B)$.

(3)\to(4) Let $C \subseteq U$. Suppose $p \notin f^{-1}[rker(f(C))]$. Then by Lemma 3.24, there exists a $D \in RC(V,f(p))$ such that $f(C) \cap D = \emptyset$, which implies that $f^{-1}[rker(f(C))] = \emptyset$. Then by (3), there exists a $G_p \in \deltaGPC(U)$ such that $f(G_p) \subseteq cl(D) = D$. Hence $f(C \cap f(G_p)) \subseteq f(C) \cap f(G_p) \subseteq f(C) \cap D = \emptyset$ which implies $C \cap G_p = \emptyset$. This shows that $p \notin \delta gpcl(C)$.

(4)\to(5) Let $D \subseteq V$, then $f^{-1}(D) \subseteq U$. By (4) and Lemma 3.24, $f(\delta gpcl(f^{-1}(D))) \subseteq rker(f(f^{-1}(D))) \subseteq rker(D)$. Thus $\delta gpcl(f^{-1}(D)) \subseteq f^{-1}(rker(D))$.

(5)\to(1): Let $H \in RO(V)$. Then by (5) and Lemma 3.24, $\delta gpcl(f^{-1}(H)) \subseteq f^{-1}(rker(H)) = f^{-1}(H)$ and hence $\delta gpcl(f^{-1}(H)) = f^{-1}(H)$. Since $U$ is $\delta gp$-additive, $f^{-1}(H) \in \deltaGPC(U)$.

Theorem 3.27. The following properties are equivalent:

(a) $f: U \to V$ is almost contra $\delta gp$-continuous.

(b) For every $N \in e^o O(V), f^{-1}(cl_{\delta}(N)) \in \deltaGPC(U)$.

(c) For every $N \in \delta SO(V), f^{-1}(cl_{\delta}(N)) \in \deltaGPC(U)$.

(d) For every $N \in \delta SO(V,f^{-1}(int(cl_{\delta}(N)))) \in \deltaGPC(U)$.

(e) For every $N \in O(V,f^{-1}(int(cl_{\delta}(N)))) \in \deltaGPC(U)$.

(f) For every $N \in C(V,f^{-1}(int(cl_{\delta}(N)))) \in \deltaGPC(U)$.

Proof. Similar to the proof of Theorem 3.20.

Lemma 3.28. [2] For a set $M \subseteq U$, the following properties hold:

(i) $a-cl(M) = cl_{\delta}(M)$ for every $M \in e^o O(U)$.

(ii) $\delta pcl(M) = cl_{\delta}(M)$ for every $M \in \delta SO(U)$.

(iii) $\delta scl(M) = int(cl_{\delta}(M))$ for every $M \in \delta PO(U)$.

Theorem 3.29. The following statements are equivalent:

(i) $f: U \to V$ is almost contra $\delta gp$-continuous.

(ii) For every $H \in e^o O(V), f^{-1}(a-cl(H)) \in \deltaGPC(U)$.

(iii) For every $H \in \delta SO(U), f^{-1}(\delta pcl(H)) \in \deltaGPC(U)$.

(iv) For every $H \in \delta PO(U), f^{-1}(\delta scl(H)) \in \deltaGPC(U)$.

Definition 3.30. [29] A space $U$ is said to be weakly $\delta gp$-continuous if for every open subset $H$ of $V$, $f^{-1}(cl(H)) \in \deltaGPC(U)$.

Definition 3.31. [29] A space $U$ is said to be endowed with an almost partition topology if $RC(U) = O(U)$.

Theorem 3.32. Every almost contra $\delta gp$-continuous function $f(U, \tau) \to (V, \sigma)$ is weakly $\delta gp$-continuous.

Proof. Let $H \in O(V)$, then $cl(H) \in RC(V)$. By hypothesis, $f^{-1}(cl(H))$ is $\delta gp$-open in $U$. Therefore $f$ is weakly $\delta gp$-continuous.

Conversely, let $\sigma$ be almost partition topology and $N \in RC(V)$. Then $N \in O(V)$. The weakly $\delta gp$-continuity of $f$ implies $f^{-1}(cl(N)) = f^{-1}(N) \in \deltaGPC(U)$.

Theorem 3.33. (i) If $f: U \to V$ is almost contra $\delta gp$-continuous and $g: V \to W$ is contra $R$-map, then $(g \circ f): U \to W$ is almost contra $\delta gp$-continuous.

(ii) If $f: U \to V$ is contra $\delta gp$-continuous and $g: V \to W$ is almost continuous, then $(g \circ f): U \to W$ is almost contra $\delta gp$-continuous.

(iii) If $f: U \to V$ is $\delta gp$-irresolute and $g: V \to W$ is almost contra $\delta gp$-continuous, then $(g \circ f): U \to W$ is almost contra $\delta gp$-continuous.

Proof. (i) Let $N \in RO(W)$. Then $g^{-1}(N) \in RO(V)$ since $g$ is contra $R$-map. The almost contra $\delta gp$-continuity of $f$ implies $f^{-1}[g^{-1}(N)] = (g \circ f)^{-1}(N) \subseteq \deltaGPC(U)$. Hence $g \circ f$ is almost contra $\delta gp$-continuous.

The proofs of (ii) and (iii) are similar to (i).

Definition 3.34. [35] A function $f: U \to V$ is called pre $\delta gp$-closed if $f(M) \in \deltaGPC(V)$ for every $M \in \deltaGPC(U)$.

Theorem 3.35. Let $f: U \to V$ be pre $\delta gp$-closed surjection and $g: V \to W$ be a function such that $g \circ f: U \to W$ is almost contra $\delta gp$-continuous, then $g$ is almost contra $\delta gp$-continuous.
Proof. Let B ∈ RO(W). Then \((g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))\) is \(\delta gp\)-closed in U. As \(f\) is pre \(\delta gp\)-closed surjection, \(f(f^{-1}(g^{-1}(B))) = (g^{-1}(B))\) is \(\delta gp\)-closed in V. Therefore g is almost contra \(\delta gp\)-continuous.

Theorem 3.36. If the graph function \(g: U \to \delta gp(V)\) is \(\delta gp\)-continuous, then \(g\) is almost contra \(\delta gp\)-continuous.

Proof. Let N ∈ RO(V), then \(U \times N \in RO(U \times V)\). The almost contra \(\delta gp\)-continuity of \(g\) implies \(f^{-1}(N) = g^{-1}(X \times N) \in \delta GP(U)\). Therefore f is almost contra \(\delta gp\)-continuous.

Recall that for a function \(f: U \to V\), the subset \(G_f = \{(x,f(x)) : x \in U\} \subset U \times V\) is said to be graph of \(f\).

Definition 3.37. A graph \(G_f\) of a function \(f: U \to V\) is said to be \(\delta gp\)-closed graph if for each \((p,q) \notin G_f\), there exist \(M \in \delta GPO(U,p)\) and \(N \in O(V,q)\) such that \((U \times V) \cap G_f = \phi\).

As a consequence of Definition 3.37 and the fact that for any subsets \(A \subseteq U\) and \(B \subseteq V\), \((A \times B) \cap G_f = \phi\) if and only if \(f(A) \cap B = \phi\), we have the following result.

Lemma 3.38. For a graph \(G_f\) of a function \(f: U \to V\), the following statements are equivalent:

(i) \(G_f\) is \(\delta gp\)-closed in \(U \times V\)
(ii) For each \((p,q) \notin G_f\), there exist \(M \in \delta GPO(U,p)\) and \(N \in O(V,q)\) such that \((U \times V) \cap G_f = \phi\).

Definition 3.39. A space \(U\) is called \(\delta gp\)-T1 space if for any pair of distinct points \(p, q \in U\), there exist \(G,H \in \delta GPO(U)\) such that \(p \in G, q \notin G\) and \(q \in H, p \notin H\).

Theorem 3.40. If \(f: U \to V\) is an injection, then \(U\) is \(\delta gp\)-T1.

Proof. Let \(f: U \to U\) be an injection. Then \(f(x) \neq f(y)\) implies \(x \neq y\) for all \(x, y \in U\). Hence \(U\) is \(\delta gp\)-T1.

Theorem 3.41. If \(f: U \to V\) is a \(\delta gp\)-closed graph \(G_f\), then \(V\) is \(T_1\) if \(f\) is surjective.

Proof. Let \(f: U \to V\) be a \(\delta gp\)-closed graph \(G_f\), and \(f\) be surjective. Then \(f(U) = V\) and \(f^{-1}(Y) = U\) for any \(Y \subseteq V\). Hence \(U\) is \(T_1\).

Corollary 3.42. If \(f: U \to V\) is a \(\delta gp\)-closed graph \(G_f\), then \(f\) is bijective if and only if both \(U\) and \(V\) are \(\delta gp\)-T1.

Proof. Follows from Theorems 3.40 and 3.41.

Definition 3.43. [30] A space \(U\) is said to be weakly Hausdorff if every point of \(U\) is expressed by the intersection of regular closed sets of \(U\).

Theorem 3.44. If \(f: U \to V\) is an injection, then \(U\) is \(\delta gp\)-T1.

Proof. Let \(V\) be weakly Hausdorff and \(p, q \in V\) with \(p \neq q\). Then there exist \(A, B \in RO(V)\) such that \(f(p) \notin A, f(q) \notin B\). The almost contra \(\delta gp\)-continuity of \(f\) implies \(f^{-1}(A) = f^{-1}(B) \in \delta GPO(U)\) such that \(p \notin f^{-1}(A), q \notin f^{-1}(B)\). Since \(f\) is almost contra \(\delta gp\)-continuous, \(f^{-1}(A) \cap f^{-1}(B) = \phi\). Therefore \(U\) is \(\delta gb\)-normal.

Definition 3.45. A space \(U\) is said to be:

(i) \(\delta gp\)-connected if \(U\) is not the union of two disjoint non-empty \(\delta gp\)-open sets.
(ii) \(\delta gp\)-ultra connected if every two non-void \(\delta gp\)-closed subsets of \(U\) intersect.

Theorem 3.46. If \(f: U \to V\) is a surjective function, then \(V\) is \(\delta gp\)-connected.

Proof. (1) On the contrary assume that \(V\) is not a connected space, then there exist \(P \neq \phi\) and \(Q \neq \phi\) in \(O(V)\) such that \(P \cap Q = \phi\) and \(V = P \cup Q\). Also, \(P \cap Q \in CO(V)\). Since \(f\) is \(\delta gp\)-continuous, \(f^{-1}(P), f^{-1}(Q) \in \delta GPO(U)\), \(f^{-1}(P) \cap f^{-1}(Q) = \phi\). Also, \(U = f^{-1}(P) \cup f^{-1}(Q)\). This shows that \(U\) is not \(\delta gp\)-connected.

(ii) \(\delta gp\)-ultra connected if every two non-void \(\delta gp\)-closed subsets of \(U\) intersect.

Theorem 3.47. A space \(U\) is ultra normal, if every pair of disjoint closed sets can be separated by disjoint closed sets.

(i) [31] A space \(U\) is ultra Hausdorff if every pair of disjoint closed sets can be separated by disjoint closed sets.

(ii) \(\delta gp\)-normal if every pair of disjoint closed sets can be separated by disjoint \(\delta gp\)-open sets.

Theorem 3.50. If \(f: U \to V\) is an injection, then \(U\) is \(\delta gp\)-injective.

Proof. Let \(f\) be injective and \(p, q \in V\) with \(p \neq q\). Then \(f(p) \neq f(q)\). Since \(V\) is ultra Hausdorff, there exist \(M \in \delta GPO(U)\) and \(N \in O(V,q)\) such that \(p \in M, q \notin M\) and \(M \cap N = \phi\). Therefore \(U\) is \(\delta gp\)-Hausdorff.

Definition 3.49. A space \(U\) is ultra normal, if every pair of disjoint closed sets can be separated by disjoint closed sets.

(i) \(\delta gp\)-normal if every pair of disjoint closed sets can be separated by disjoint \(\delta gp\)-open sets.

Theorem 3.51. If \(f: U \to V\) is ultra Hausdorff and \(p, q \in V\) with \(p \neq q\). Then there exist \(M, N \in \delta GPO(U)\) such that \(p \in M, q \notin M\) and \(M \cap N = \phi\). Therefore \(U\) is \(\delta gb\)-normal.