Total dominator chromatic number of $P_m \times C_n$

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Abstract
A total dominator coloring of a graph $G$ with $\delta(G) \geq 1$ is a proper coloring of $G$ with the extra property that every vertex in $G$ properly dominates a color class. The total dominator chromatic number of $G$ is the minimum number of colors needed in a total dominator coloring of $G$, denoted by $\chi_{td}(G)$. In this paper, we obtain total dominator chromatic number of $P_m \times C_n$.

Keywords
Total dominator chromatic number, ladder graph, grid graph and $P_m \times C_n$.

AMS Subject Classification
05C15, 05C69.

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1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definition of graph theory as found in [1]. Let $G = (V,E)$ be a graph of order $n$ with $\delta(G) \geq 1$. The open neighborhood $N(v) = \{u \in V(G) / uv \in E(G)\}$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The path and cycle of order $n$ are denoted by $P_n$ and $C_n$ respectively. For any two graphs $G$ and $H$, we define the cartesian product, denoted by $G \times H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices $(u_1, v_1)$ and $(u_2, v_2)$ iff either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$. $P_m \times C_n$ is defined as the cartesian product of path and cycle. A grid graphs can be defined as $P_m \times P_n$ where $m,n \geq 2$.

A subset $S$ of $V$ is called a total dominating set if every vertex in $V$ is adjacent to some vertex in $S$. The total dominating set is minimal total dominating set if no proper subset of $S$ is a total dominating set of $G$. The total domination number $\gamma_t$ is the minimum cardinality taken over all minimal total dominating set of $G$. A $\gamma_t$-set is any minimal total dominating set with cardinality $\gamma_t$.

A proper coloring of $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices have different colors. The chromatic number, $\chi(G)$, is the minimum number of colors in a proper coloring of $G$. A total dominator coloring of a graph $G$ is a proper coloring of $G$ with the extra property that every vertex in $G$ properly dominates a color class. The total dominator chromatic number of $G$ is the minimum number of colors needed in a total dominator coloring of $G$ denoted by $\chi_{td}(G)$. This concept was introduced by A. Vijayalekshmi in [2]. This notion is also referred as a Smarandachely $k$-dominator coloring of $G(k \geq 1)$ and was introduced by A. Vijayalekshmi in [4]. For an integer $k \geq 1$, a Smarandachely $k$-dominator coloring of $G$ is a proper coloring of $G$ such that every vertex in $G$ properly dominates a $k$ color class. The smallest number of colors for which there exist a Smarandachely $k$-dominator coloring of $G$ is called the Smarandachely $k$-dominator chromatic number of $G$, and is denoted by $\chi_{td}^k(G)$. For further details on this theory and on its applications, we advice the reader to refer [6–9].

In a proper coloring $\mathcal{C}$ of $G$, a color class of $\mathcal{C}$ is a set consisting of all those vertices assigned the same color. Let $\mathcal{C}^1$ be a minimal $td$-coloring of $G$. We say that a color class $c_i \in \mathcal{C}^1$ is called a non-dominated color class $(n-d$ color class) if it is not dominated by any vertex of $G$. These color classes are also called repeated color classes.
2. Preliminaries

In this segment, we remember the critical [3] theorem which is quite helpful in our research. For the subsequent observation the total dominator chromatic number of a ladder graphs has been identified.

**Theorem 2.1.** [3] Let G be $P_n$ or $C_n$. Then
\[
\chi_{td}(G) = \begin{cases} 
2\left\lfloor \frac{n}{6} \right\rfloor + 2, & \text{if } n \equiv 0 \text{ (mod 6)} \\
2\left\lfloor \frac{n-2}{6} \right\rfloor + 4, & \text{otherwise.}
\end{cases}
\]

**Theorem 2.2.** [3] For every $n \geq 2$, the total dominator chromatic number of a ladder graph $L_n$ is
\[
\chi_{td}(L_n) = \begin{cases} 
2\left\lfloor \frac{n}{6} \right\rfloor + 2, & \text{if } n \equiv 0 \text{ (mod 6)} \\
2\left\lfloor \frac{n-2}{6} \right\rfloor + 4, & \text{otherwise.}
\end{cases}
\]

In this paper, we obtain the least value for total dominator chromatic number for $P_n \times C_n$.

3. Main Result

In this section, we present and establish the main results.

For our convenience, we denote $G_{m,n} = P_m \times C_n$ and let $D = \{v_{ij} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$.

**Lemma 3.1.** For every $n$, $\chi_{td}(G_{2,n}) = 2\left\lfloor \frac{n}{4} \right\rfloor + 2$.

**Proof.** Since the td-colouring of $G_{2,n}$ is same as $td$-colouring of $L_n$, $\chi_{td}(G_{2,n}) = \chi_{td}(L_n)$. From Theorem 2.2, we get
\[
\chi_{td}(G_{2,n}) = 2\left\lfloor \frac{m}{3} \right\rfloor + 2.
\]

**Illustration:** Consider $G_{2,11}$

![Figure 1](image1)

Therefore
\[
\chi_{td}(G_{2,11}) = 10.
\]

**Theorem 3.2.** If $m \equiv 0 \text{ (mod 3)}$, then $\chi_{td}(G_{m,n}) = \frac{mn}{3} + 2$.

**Proof.** Let $D = \{v_{ij} \mid 1 \leq i \leq m \text{ and } j = 2,5,8,\ldots,(n-1)\}$ be a unique $\gamma$-set of $G_{m,n}$. We assign $\frac{mn}{3} + 2$ to vertices of $D$. Set $S = V(G_{m,n}) - D$, we assign two repeated colors say 1,2 the vertices $v_{ij}$ and $v_{kl} \in S$ such that $|i-k| + |j-l| = 1$ and adjacent vertices in $S$ received different colors, we get a $td$-coloring of $G_{m,n}$.

So
\[
\chi_{td}(G_{m,n}) = \frac{mn}{3} + 2.
\]

**Illustration:** Consider $G_{6,10}$

![Figure 2](image2)

Therefore
\[
\chi_{td}(G_{6,10}) = 22.
\]

**Theorem 3.3.** For $m \equiv 0 \text{ (mod 3)}$ and $n \equiv 1,2 \text{ (mod 3)}$,
\[
\chi_{td}(G_{m,n}) = \begin{cases} 
\frac{mn}{2} + 2, & \text{if } n \text{ is even} \\
\frac{mn}{3} + 3, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** Let $D_1 = \{v_{ij} \mid i = 2,5,8,\ldots,(n-1)\}$ be a unique $\gamma$-set of $G_{m,n}$. We consider two cases.

Case (i): When $n$ is even. The td-coloring of $G_{m,n}$ is same as the td-coloring of Theorem 3.1. So $\chi_{td}(G_{m,n}) = \frac{mn}{3} + 2$.

Case (ii): When $n$ is odd. Assign $\frac{mn}{3}$ distinct colors say $4,5,6,\ldots,\frac{mn}{3},\frac{mn}{3}+1,\frac{mn}{3}+2,\frac{mn}{3}+3$ to vertices of $D_1$. Set $S_1 = V(G_{m,n}) - D_1$, we assign two repeated colors say 1,2 to the vertices $v_{ij}$ and $v_{kl} \in S_1$ such that $|i-k| + |j-l| = 1$ and adjacent vertices in $S_1$ received two different repeated colors. Now there several vertices in $S_1$, which are not received repeated either 1 or 2, we assign another repeated color say 3 to the those vertices in $S_1$, we get a td-coloring of $G_{m,n}$. So $\chi_{td}(G_{m,n}) = \frac{mn}{3} + 3$.

**Illustration:** Consider $G_{6,10}$

![Figure 3](image3)
Theorem 3.4. If $m \equiv 1 \pmod{3}$ then

$$\chi_{td}(G_{m,n}) = \begin{cases} \frac{(m-1)n}{3} + \left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod{4} \\ \frac{(m-1)n}{3} + \frac{n}{4} + 4 & \text{if } n \equiv 1 \pmod{4} \\ \frac{(m-1)n}{3} + \frac{n+2}{4} + 2 & \text{if } n \equiv 2 \pmod{4} \\ \frac{(m-1)n}{3} + \frac{n+2}{4} + 3 & \text{otherwise.} \end{cases}$$

Proof. Since $m - 1 \equiv 0 \pmod{3}$, $G_{m,n}$ is obtained by $G_{m-1,n}$ is followed by $P_1$. In a $td$-coloring of $G_{m,n}$, $\chi_{td}(G_{m,n}) = \chi_{td}(G_{m-1,n}) + \chi_{td}(P_1)$. Also the used repeated colors are same the $td$-coloring of $P_1$. So $\chi_{td}(G_{m,n}) = \chi_{td}(G_{m-1,n}) + \chi_{td}(P_1) - 2$. By Theorem 2.1, we get

$$\chi_{td}(G_{m,n}) = \begin{cases} \frac{(m-1)n}{3} + \left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod{4} \\ \frac{(m-1)n}{3} + \frac{n}{4} + 4 & \text{if } n \equiv 1 \pmod{4} \\ \frac{(m-1)n}{3} + \frac{n+2}{4} + 2 & \text{if } n \equiv 2 \pmod{4} \\ \frac{(m-1)n}{3} + \frac{n+2}{4} + 3 & \text{otherwise.} \end{cases}$$

Illustration: Consider $G_{4,10}$

Therefore $\chi_{td}(G_{4,10}) = 18$.

Illustration: Consider $G_{4,11}$

Therefore $\chi_{td}(G_{4,11}) = 20$.

Theorem 3.5. If $m \equiv 2 \pmod{3}$, then

$$\chi_{td}(G_{m,n}) = \begin{cases} \frac{(m-2)n}{3} + \frac{n}{4} + 2 & \text{if } n \text{ is even} \\ \frac{(m-2)n}{3} + \frac{n}{4} + 3 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Given $m - 2 \equiv 0 \pmod{3}$. We consider two cases.

Case (i): When $n$ is even. We have $G_{m,n}$ is obtained by $G_{m-2,n}$ followed by $P_2$. From Theorem 3.4, $\chi_{td}(G_{m,n}) = \chi_{td}(G_{m-2,n}) + \chi_{td}(P_2) - 2$. By Theorem 3.3 and Lemma 3.1, we get

$$\chi_{td}(G_{m,n}) = \frac{(m-2)n}{3} + \frac{n}{4} + 2.$$ 

Case (ii): When $n$ is odd. We have $G_{m,n}$ is obtained by $G_{m-2,n}$ followed by $P_2$. From Theorem 3.4, $\chi_{td}(G_{m,n}) = \chi_{td}(G_{m-2,n}) + \chi_{td}(P_2) - 2$.

By Theorem 3.3 and Lemma 3.1, we get

$$\chi_{td}(G_{m,n}) = \frac{(m-2)n}{3} + \frac{n}{4} + 3.$$ 

Thus

$$\chi_{td}(G_{m,n}) = \begin{cases} \frac{(m-2)n}{3} + \frac{n}{4} + 2 & \text{if } n \text{ is even} \\ \frac{(m-2)n}{3} + \frac{n}{4} + 3 & \text{if } n \text{ is odd.} \end{cases}$$

Illustration: Consider $G_{5,8}$

Therefore $\chi_{td}(G_{5,8}) = 16$. 

Illustration: Consider $G_{6,7}$

Therefore $\chi_{td}(G_{6,7}) = 17$. 

Figure 4

Figure 5

Figure 6

Figure 7
Illustration: Consider $G_{5,7}$

![Graph Illustration]

Therefore

$$\chi_{td}(G_{5,7}) = 16.$$ 

4. Conclusion

In this paper, we obtain total dominator chromatic number of $P_m \times C_n$.

References


