On vertex integer-magic spectra of Caterpillar graphs

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Abstract
Consider any graph \( G = (V(G), E(G)) \) and \( k \) be any positive integer. Then a graph \( G \) is said to be \( \mathbb{Z}_k \)-vertex magic graph if there exist a map \( l : V(G) \rightarrow \mathbb{Z}_k - \{0\} \) such that for any vertex \( v \in V(G) \), sum of the labels of vertices in the open neighborhood of \( v \) is a constant, i.e., \( \omega(v) = \sum_{u \in N(v)} f(u) = \mu, \forall v \in V(G) \). The set \( VIM(G) = \{ k \in \mathbb{Z}^+ | G \text{ is } \mathbb{Z}_k \text{-vertex magic} \} \) is called vertex integer magic spectrum. In this paper, we determine \( VIM \) of caterpillar, super caterpillar and extended super caterpillar graphs.

Keywords
Super Caterpillar, Extended Super Caterpillar, vertex integer magic spectrum.

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1. Introduction

Consider \( G = (V(G), E(G)) \) with vertex set \( V(G) \) and edge set \( E(G) \) be a finite, simple, connected, undirected graph. We refer Hernstein [2] for group theoretic concepts and Bondy & Murthy [1] for graph-theoretic terminology. Throughout this paper \( \mathbb{Z}_k \) denote the group of integer modulo \( k \), where \( k \geq 3 \) unless otherwise mentioned and degree of a vertex \( v \) by \( \text{deg}(v) \). we denote \([a,b]\) for set of all integer \( x \) such that \( a \leq x \leq b \).

Any labeling \( f : E(G) \rightarrow A - \{0\}, \) where \( A \) be an abelian group is said to be \( A \)-magic labeling of \( G \) if the induced map \( f^+ : V(G) \rightarrow A \) defined by \( f^+(u) = \sum_{(u,v) \in E(G)} f(u,v) \) is a constant map and the graph \( G \) is called \( A \)-magic graph[10]. If \( A = \mathbb{Z}_k \), then \( G \) is \( \mathbb{Z}_k \)-magic graph and the set \( \{ k \in \mathbb{Z}^+ | G \text{ is } \mathbb{Z}_k \text{-magic} \} \) is called integer magic spectrum.


Definition 1.1. [12] For any abelian group \( A \), a mapping \( l : V(G) \rightarrow A - \{0\} \) is said to be \( A \)-vertex magic labeling of \( G \) if there exists an element \( \mu \) of \( A \) such that \( \omega(v) = \sum_{u \in N(v)} f(u) = \mu, \forall v \in V(G) \), where \( N(v) \) is the open neighborhood of \( v \). A graph \( G \) that admits such a labeling is called A-vertex magic graph. If \( A = \mathbb{Z}_k \), then \( G \) is called \( \mathbb{Z}_k \) vertex magic graph and set of all positives integers for which \( G \) is \( \mathbb{Z}_k \) vertex magic is called vertex integer magic spectrum.

Definition 1.2. [13] Supercaterpillar graph is a graph which formed by several arbitrary caterpillar graph \( Ti, i = 0, 1, \ldots, m \), and one of the end backbone vertex in each caterpillar \( Ti \) be
joined with root vertex $v_c$, by an edge.

**Definition 1.3.** [13] Let there be any numbers of mp caterpillars. These caterpillars are grouped in p groups each having m caterpillars. Let backbones of each group i of caterpillars be connected to vertex $v_c$, that is connected to root vertex $v_c$. Then the resulting tree is called an extended super-caterpillar.

**Theorem 1.4.** [8] A Graph $G$ is $Z_2$- magic if and only if every vertex of $G$ has same parity.

**Theorem 1.5.** [14] The system of congruences $x \equiv b_i (mod m_i)$, $m_i \neq 0$, $i \in [1, p]$ has solution if $gcd(m_i, m_j) | (b_i - b_j)$, $\forall i, j \in [1, p]$.

### 2. Main Results

A Graph $G$ is $Z_2$- vertex magic iff each vertex $v \in V(G)$ has same parity. This result is straight forward and obtaining similar results for $Z_k, k > 2$ seems to be difficult. Caterpillar graph is a graph in which elimination of all end vertices produces a path graph. In this present work, we investigate the integer vertex magic spectrum of special class of trees such as caterpillar, super caterpillar, extended super caterpillar.

**Lemma 2.1.** Let $a$ be an element in $Z_k$. Then for any $n \geq 2$, $a = \sum_{i=1}^{n} g_i$ for some $g_i \in Z_k - \{0\}$.

**Theorem 2.2.** Any caterpillar graph $G$ is $Z_k$ vertex magic graph iff $G$ has no vertex of degree two.

**Proof.** Let $v \in V(G)$ such that $deg(v) = 2$. Then $\exists u_i, u_{i+1} \in V(G)$ such that $u_i \in N(u_{i+1})$ with $deg(u_i) = 2$ and $deg(u_{i+1}) > 2$. If not $G$ must be a path graph. Let $u_{i-1}$ be the other vertex adjacent to $u_i$, then $\omega(u_i) = l(u_{i+1}) + l(u_{i-1}) = \mu + l(u_{i-1}) = \mu$. Hence $l(u_{i-1}) = 0$, which is a contradiction.

Conversely, let $w_1, w_2, \ldots, w_n$ are the pendant vertices of $u_i$. It follows from lemma 2.1, $k - g = \sum_{i=1}^{n} g_i$, $0 = \sum_{i=1}^{n} h_i$ for $i \in [1, n]$. Now, define $l: V(T) \rightarrow Z_k - \{0\}$ by $l(u_i) = g$ for $i \in [1, n]$ $l(w_i') = g_i$, if $r \in [1, n_1], i \in [2, n-1]$ and $deg(u_i) \geq 4$ $l(w_i') = k - g$ if $i \in [2, n-1]$ and $deg(u_i) = 3$ $l(w_i') = h_i$, for $r \in [1, n_1]$ $l(w_i') = h_i$, for $r \in [1, n_2]$ Then $G$ is $Z_k$ vertex magic graph with magic constant $g$.

Clearly the above theorem holds for $k = 1$, since any integer can be written as sum of 2 or more nonzero integers. Thus any caterpillar graph $G$ is $Z$-vertex magic graph iff $G$ has no vertex with degree exactly 2.

**Corollary 2.3.** The vertex integer magic spectrum of a caterpillar graph $G$, $VIM(G) = \begin{cases} \phi & \text{if } G \text{ has vertex of degree 2} \\ N - \{2\} & \text{if } G \text{ has no vertex of degree 2} \end{cases}$

**Theorem 2.4.** Super caterpillar graph $G$ is $Z_k$ vertex magic graph iff $G$ has no vertex of degree 2 and $gcd(deg(v_c)) - 1, k) > 1$.

**Proof.** Let super caterpillar graph $G$ is $Z_k$ vertex magic graph with magic constant $\mu = g$. Suppose $G$ has atleast a vertex with degree two. Let $v \neq v_c$ such that $deg(v) > 2$. Then $\exists u_i \in V(G)$ such that $u_i \in N(u_{i+1})$ with $deg(u_i) = 2$ and $deg(u_{i+1}) > 2$. Suppose not, then $u_i$ be vertex of caterpillar $T_i$ which is a path. If $deg(u_{i+1}) > 2$ then $l(u_{i+1}) = g$.

Thus $l(u_{i-1}) = 0$, which contradicts the fact that $G$ is group vertex magic.

If $deg(v_c) = 2$ then $\omega(v_c) = l(u_1) + l(u_2) = 2g \neq g$.

Now, all vertices $u_i'$ of $T_i$ for $i \in [1, m]$ must have pendant vertex. Otherwise $u_i'$ is of degree 2. Then $l(u_i') = g$ for all $i \in [1, m]$. Let $l(u_i') = deg(v_c) = deg(T_i) = g = mg = g$. Therefore $o(g)$ divides both $m - 1$ and $k$. Hence $gcd(m - 1, k) > 1$, where $deg(T_i) = m$.

Conversely, we label all vertices of $T_i$ for $i \in [1, m]$ same as in above theorem 2.2 except the pendant vertices of $u_i'$ for $i \in [1, m]$. Label all pendant vertices of $u_i'$ such that $\omega(u_i') = g$ and $l(v_c) = g$. Now, if we choose the nonzero element $g$ in such a way that $o(g) = d$, where $d = gcd(deg(v_c)) - 1, k)$. Then $\omega(v_c) = deg(v_c) = g$. Thus $\omega(v) = g$ for all $v \in V(G)$.

**Corollary 2.5.** Let $G$ be a super caterpillar graph with $deg(v_c) = m$ and $m - 1 = p_1^{i_1} \cdot p_2^{i_2} \ldots p_s^{i_s}$, where $p_i$ for $i \in [1, s]$ be prime. Then,

$$VIM(G) = \begin{cases} \phi & \text{if } G \text{ has vertex of degree 2} \\ \bigcup_{i=1}^{s} p_i N - \{2\} & \text{if } G \text{ has no vertex of degree 2} \end{cases}$$

**Proof.** We have $deg(v_c) = m$, and for any $n \in N$, $gcd(m - 1, np_i) > 1$ for $i \in [1, s]$.
Theorem 2.6. Consider an extended super caterpillar graph $G$ without a vertex $v \neq v_1$ with $\deg(v) = 2$ is $\mathbb{Z}_k$-vertex magic graph if

1. $\deg(v_i)$ is same for all $i \in [1,p]$.
2. $\gcd(\deg(v_{c_1}), \deg(v_{c_2}), \ldots, \deg(v_{c_p}) - 1, k) > 1$.

Proof. Let $d > 1$ be $\gcd(\deg(v_{c_1}), \deg(v_{c_2}), \ldots, \deg(v_{c_p}) - 1, k)$. Then $\exists k \in \mathbb{Z}_k$ such that $\deg(v_i) = d$ for all $i \in [1,p]$. Now let all vertices as in (i) except root vertex $v_1$. Let $l(v_{c_1}) = g$ and $N(v_{c_1}) = \{v_{c_1} : i \in [1,p]\}$, and $\omega(v_{c_1}) = g$ for all $i \in [1,p]$. Now label all other vertices of each super caterpillar as same as theorem 2.4 such that $\omega(v) = g$ for all $v \in V(G)$. Thus $G$ is $\mathbb{Z}_k$-vertex magic graph with magic constant $\mu = g$.

Theorem 2.7. Consider an extended super caterpillar graph $G$ without a vertex $v \neq v_1$ with $\deg(v) = 2$ is $\mathbb{Z}_k$-vertex magic graph iff there exist some prime $q$ such that $q \mid k$ and $q \mid (\deg(v_{c_1}) - \deg(v_{c_2}))$ for all $i, j \in [1,p]$.

Proof. Label all vertices except $v_1$ as same as in theorem 2.6. Then $\omega(v) = g$ except $v \in N(v_1)$. Let $l(v_1) = x$, then $G$ is $\mathbb{Z}_k$-vertex magic graph iff the system of linear congruences $\deg(v_{c_i}) - 1 \equiv x \pmod{q}$ is solvable. Now by theorem 1.6, $G$ is $\mathbb{Z}_k$-vertex magic graph iff $\deg(v_{c_i}) - \deg(v_{c_j}) \equiv 0 \pmod{q}$ for all $i, j \in [1,p]$. Therefore $\omega(g) \mid (\deg(v_{c_i}) - \deg(v_{c_j}))$ for all $i, j \in [1,p]$. Thus $\exists$ prime $q$ such that $q \mid k$ and $q \mid (\deg(v_{c_i}) - \deg(v_{c_j}))$ for all $i, j \in [1,p]$, since $\omega(g) > 1$.

Corollary 2.8. Consider an extended Super caterpillar graph with no vertex $v \neq v_1$ such that $\deg(v) = 2$. Then,

$$VIM(G) = \begin{cases} [N-2] & \text{if } \deg(v_{c_1}) \text{ is same for all } i \in [1,p] \\ p_i[N-2] & \text{if } p_i \mid (\deg(v_{c_1}) - 1) \text{ for all } i \in [1,p] \\ q[N-2] & \text{if } q \mid (\deg(v_{c_1}) - \deg(v_{c_j})) \forall i, j \in [1,p] \end{cases} $$

3. Conclusion

In this paper, we give a characterization of $\mathbb{Z}_k$-vertex magic labeling of caterpillar, super Caterpillar and extended super caterpillars graphs. We also determined vertex integer magic spectrum of these graphs.

References


Figure 2. An extended super caterpillar graph.