Connected edge Detour global domination number of a graph

A. Punitha Tharani and A. Ferdina

Abstract
In this paper, we introduce the concept of connected edge detour global domination number of a graph. A subset $D$ of the vertex set $V(G)$ of a connected graph $G$ is called a connected edge detour global dominating set if $D$ is an edge detour global dominating set and the induced subgraph $< D >$ is connected. The connected edge detour global domination number $\gamma_{cedg}(G)$ of $G$ is the minimum cardinality taken over all connected edge detour global dominating sets in $G$. A connected edge detour global dominating set of cardinality $\gamma_{cedg}(G)$ is called a $\gamma_{cedg}$-set of $G$. We determine $\gamma_{cedg}(G)$ for some standard and special graphs and its properties are studied.

Keywords
Edge detour global domination number, connected edge detour global domination number.

AMS Subject Classification
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1. Introduction

By a graph $G = (V, E)$, we consider a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $n, m$ respectively. Edge Detour Global Dominating graphs were introduced and studied by Punitha Tharani and Ferdina [12]. For underlying definition and results, see references [1-14].

Theorem 1.1. For any connected graph of order $n \geq 2$. Then, $2 \leq dn(G) \leq \gamma_{edg}(G) \leq n$.

Theorem 1.2. Let $G$ be a graph of order $n$. Then $\gamma_{edg}(G) = n$ iff $G$ contains only end and full vertices.

Theorem 1.3. For the path graph $P_n$, $\gamma_{edg}(P_n) = \left[ \frac{n-1}{2} \right] + 2, n \geq 5$.

Theorem 1.4. For the complete graph $K_n$, $\gamma_{edg}(K_n) = n, n \geq 2$.

2. Connected Edge Detour Global Domination Number of a Graph

Definition 2.1. A subset $D$ of $V$ of a connected graph $G = (V, E)$ is called a connected edge detour global dominating set of $G$ if $D$ is an edge detour global dominating set and the induced subgraph $< D >$ is connected. The Connected edge detour global domination number $\gamma_{cedg}(G)$ of $G$ is the minimum cardinality taken over all connected edge detour global dominating sets in $G$. A connected edge detour global dominating set of cardinality $\gamma_{cedg}(G)$ is called a $\gamma_{cedg}$-set of $G$.

Example 2.2. Consider the graph $G$ given in Figure 1.
Here, $D_1 = \{v_1, v_4, v_6\}, D_2 = \{v_1, v_4, v_5\}, D_3 = \{v_1, v_3, v_5\}$ are $\gamma_{edg}$-sets of $G$ and so $\gamma_{edg}(G) = 3$. Now $D_5 = \{v_1, v_2, v_3, v_4\}, D_6 = \{v_1, v_2, v_3, v_6\}, D_7 = \{v_1, v_2, v_5, v_6\}$ are $\gamma_{cedg}$-set of $G$. Then $\gamma_{cedg}(G) = |D_5| = |D_6| = |D_7| = 4$.
Theorem 2.4. Let $G$ be a connected graph of order $n$. Then $2 ≤ \gamma_{edg}(G) ≤ \gamma_{cedg}(G) ≤ n$.

Proof. Let $D$ be an edge detour global dominating set. Every vertex in $D$ needs at least two vertices so that $\gamma_{edg}(G) ≥ 2$. Again, every connected edge detour global dominating set is an edge detour global dominating set, $\gamma_{edg}(G) ≥ \gamma_{cedg}(G)$ since the set of all vertices of $G$ is always a connected edge detour global dominating set. Therefore $n ≥ \gamma_{cedg}(G)$. Hence $2 ≤ \gamma_{edg}(G) ≤ \gamma_{cedg}(G) ≤ n$.

Remark 2.5. For a connected graph $G$ with $n ≥ 2$,

(i) $\gamma_{edg}(G) ≤ \gamma_{cedg}(G)$.
(ii) $\gamma_{cedg}(G) ≤ \gamma_{cedg}(G)$.
(iii) Strict inequality is also true in the above relation.
(iv) From the above Example $2.2 n = 6, \gamma_{edg}(G) = 3, \gamma_{cedg}(G) = 4$, the bound (Theorem 2.4) is sharp.

Observation 2.6. (i) Path $P_n$ of order $n(n ≥ 2), \gamma_{cedg}(P_n) = |V(P_n)|$.
(ii) Cycle $C_n$ of order $n(n ≥ 3), \gamma_{cedg}(C_n) = |V(C_n)| - 2$.
(iii) Complete graph $K_n$ of order $n(n ≥ 2), \gamma_{cedg}(K_n) = |V(K_n)|$.
(iv) Complete bipartite graph $K_{m,n}$.
\[ \gamma_{cedg}(K_{m,n}) = \begin{cases} 2 & \text{if } m = n = 1 \\ |V(K_{m,n})| - m + 1 & \text{if } n ≥ 2, m = 1 \\ 3 & \text{if } m,n ≥ 2 \end{cases} \]
(v) Star graph $K_{1,n}, \gamma_{cedg}(K_{1,n}) = |V(K_{1,n})|$.
(vi) Bistar graph $B_{n,n}, \gamma_{cedg}(B_{n,n}) = 2n + 2$.
(vii) Wheel graph $W_n(n ≥ 5), \gamma_{cedg}(W_n) = 3$.

Theorem 2.7. Every $\gamma_{cedg}$-set of a connected graph $G$ contains all the pendant vertices of $G$.

Proof. Let $D$ be a connected edge detour global dominating set of $G$. Then every set $D$ contains all the pendant vertices, since the pendant edges lie only in the detour joining the corresponding pendant vertex with some other vertex.

Theorem 2.8. Every $\gamma_{cedg}$-set of a connected graph $G$ contains all the vertices of $G$ has degree $n - 1$.

Proof. Let $w$ be a vertex of a connected graph $G$ has degree $n - 1$. Then the vertex $w$ belongs to every dominating set in the complement $G$ of $G$. Since $w$ is dominate itself in $G$. Then all the full vertices of $G$ belong to the global dominating set of $G$. Hence, every $\gamma_{cedg}$-set contains all the full vertices.

Theorem 2.9. Let $G$ be a connected graph of order $n ≥ 2$ and $D$ be a $\gamma_{cedg}$-set of $G$. Then for any cut vertex $x$ of $G$, every component of $G - x$ contains an element of $D$.

Proof. Let $x$ be a cut vertex of a connected graph $G$ and $D$ be a connected edge detour global dominating set. Let $H$ be one of the components of $G - x$. Suppose no vertex of $D$ belongs to $H$. Then any pendant vertex of $G$ does not belong to $H$ (by Theorem 2.7). Therefore, $H$ has at least one edge, say $u_t u_{t+1}$. Since $D$ is a $\gamma_{cedg}$-set, there exists vertices $u, w \in D$ such that $u_t u_{t+1}$ lies on some $u - w$ detour, $P; u = u_1, u_2, \ldots, u_t, u_{t+1}, \ldots, u_n = w$ in $G$ or both the ends $u_t$ and $u_{t+1}$ of the edge $u_t u_{t+1}$ are in $D$. Suppose that $u_t u_{t+1}$ lies on the detour $P$. Let $P_{t}$ be the subpath of $P$, say $u_t - u_1$ and $P_{t}$ be the subpath of $P$, say $u_t - w$. Since $x$ is a cut vertex of $G$, then $x$ belongs to both $P_{t}$ and $P_{t}$, so that $P$ is not a detour, which is a contradiction to the fact. Suppose that $u_t$ and $u_{t+1}$ are in $D$, then $H$ contains vertices of $D$, which is again a contradiction.

Theorem 2.10. Every $\gamma_{cedg}$-set of a connected graph $G$ contains all the cut vertices of $G$.

Proof. Let $x$ be a cut vertex of a connected graph $G$ of order $n ≥ 2$ and $D$ be a connected edge detour global dominating set of $G$. Then $G - x$ has more than one component, say $G_1, G_2, \ldots, G_i(i ≥ 2)$. Then $\gamma_{cedg}$-set $D$ contains at least one vertex from each component $G_k(1 ≤ k ≤ i)$ of $G - x$ (by Theorem 2.9). Since induced subgraph $<D>$ is connected it follows that $x ∈ D$.

Corollary 2.11. Every $\gamma_{cedg}$-set of a connected graph $G$ contains pendant vertices, full vertices and cut vertices of $G$.

Proof. The proof follows from Theorem 2.7, 2.8 and 2.10.

Corollary 2.12. For any tree $T$ of $n$ vertices, $\gamma_{edg}(T) = |V(T)|, n ≥ 2$.

Proof. The proof follows from Corollary 2.11.

Corollary 2.13. Let $G$ be any connected graph with $l$ pendant vertices, $m$ full vertices and $n$ cut vertices. Then $\max \{2, l + m + n\} ≤ \gamma_{edg}(G) ≤ n$.

Proof. The proof follows from Theorem 2.4 and Corollary 2.11.

Theorem 2.14. For $3 ≤ j ≤ n(\forall j, n \in \mathbb{Z})$, there exists a connected graph $G$ of order $n$ with $\gamma_{edg}(G) = j$. 
Proof. Case 1: If \( j = n \), let \( G = P_n \). Then by Observation 2.6 (i), \( \gamma_{ced}(G) = j \).

Case 2. If \( 3 < j < n \), let \( G = W_n \). Then by Observation 2.6 (vii), \( \gamma_{ced}(G) = j \).

Case 3. If \( 3 < j < n \), let \( G \) be a connected graph obtained from \( W_{n-j+3} \). Let \( V(G) = \{v, v_1, v_2, v_3, \ldots, v_{n-j+2}, w_1, w_2, \ldots, w_{j-3}\} \). The graph \( G \) is shown in Figure 2.

Let \( v_1, v_2, \ldots, v_{n-j+2} \) and \( w_1, w_2, \ldots, w_{j-3} \) be the new vertices which are joining to \( v_2 \). Now we have to prove that \( \gamma_{ced}(G) = j \). Then the set \( D = \{w_1, w_2, \ldots, w_{j-3}\} \) together with a cut vertex \( v_2 \) is a subset of every \( \gamma_{ced} \)-set of \( G \). It is clear that \( D \) is a global dominating set but not an edge detour set of \( G \). Let \( D' = D \cup \{v, v_{j-3}\} \). Then every edge of \( G \) lies on a detour joining a pair of vertices of \( D' \). Clearly, the set \( D' \) is \( \gamma_{edg} \)-set and \( < D' > \) is connected. Therefore, \( D' \) is a connected edge detour global dominating set of minimum cardinality,

\[
|D'| = |D \cup \{v, v_{j-3}\}| \\
= |D| + |\{v, v_{j-3}\}| \\
= |\{w_1, w_2, \ldots, w_{j-3}\}| + |v_2| + |\{v, v_{j-3}\}| \\
= j - 3 + 1 + 2 = j.
\]

Hence \( \gamma_{ced}(G) = j \).

References


