On cartesian product of commutative, self-distributive and transitive BE-algebra

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Abstract
In this paper we develop the idea of cartesian product of BE-algebras. Furthermore we introduced the cartesian product on commutative, self-distributive and transitive BE-algebras.

Keywords
BE-algebra, commutative BE-algebra, self-distributive BE-algebra, transitive BE-algebra.

AMS Subject Classification
06F35, 03G25, 08A30, 03B52.

1. Introduction
After the introduction of the concepts of BCK and BCI algebras ([4,5]) by K. Iseki in 1966, some more systems of similar type have been introduced and discussed by a number of authors in the last two twenty years. K. H. Kim and Y. H. Yon studied dual BCK algebra and M.V. algebra in 2007 ([6]). It is known that BCK-algebras is a proper subclass of BCI-algebras. There are so many generalizations of BCK/BCI-algebras, such as BCH-algebras ([9]), dual BCK-algebras ([6]), d-algebras ([3]), etc. In ([2]), H. S. Kim and Y. H. Kim introduced the concept of BE-algebra as a generalization dual BCK-algebra. A. Walendziak ([1]) introduced the notion of commutative BE-algebras and discussed some of its properties.

2. preliminaries

Definition 2.1. Let (A;*, 1) be a system of type (2,0) consisting of a non-empty set A, a binary operation “*” and a fixed element 1. The system (A; *, 1) is called a BE-algebra ([2,7,8]) if the following conditions are satisfied:

(i) a*a = 1
(ii) a*1 = 1
(iii) 1*a = a
(iv) a*(b*c) = b*(a*c), ∀a, b, c ∈ A.

Note 2.2. In any BE-algebra we can define a binary relation “≤” as a ≤ b if and only if a*b = 1, ∀a, b, ∈ A.

Lemma 2.3. In a BE-algebra the following identities are true [2]:

1. a*(b*a) = 1
2. a*((a*b)*b) = 1.

Definition 2.4. Let (A;*, 1) be a BE-algebra. An element a ∈ A is said to commute with b ∈ A if (a*b) = (b*a) a. If this condition is true for all a, b ∈ A, then (A;*, 1) is called a commutative BE-algebra [1].

Definition 2.5. A BE-algebra (A;*, 1) is said to be self distributive if a*(b*c) = (a*b)*(a*c), ∀a, b, c ∈ A

Definition 2.6. A BE-algebra (A;*, 1) is said to be transitive [10] if for any a, b, c ∈ A,

b*c ≤ (a*b)*(a*c).
3. Cartesian Product of BE-algebras

In this section we study the properties of Cartesian product of BE-algebras.

Theorem 3.1. Let \((A; *, 1)\) be a system consisting of a non-empty set \(A\), a binary operation "*" and a distinct element 1. Let \(B = A \times A = \{(a_1, a_2) : a_1, a_2 \in A\}\). For \(u, v \in B\) with \(u = (a_1, a_2), v = (b_1, b_2)\), we define an operation "\(\circ\)" in \(B\) as

\[
 u \circ v = (a_1 * b_1, a_2 * b_2).
\]

Then \((B, \circ, (1, 1))\) is a BE-algebra iff \((A; *, 1)\) is a BE-algebra.

Proof. Suppose that \((B, \circ, (1, 1))\) be a BE-algebra. Let \(a \in A\) and we choose \(u = (a, 1) \in B\). Then

1. \(u \circ u = (1, 1) \Rightarrow (a * a, 1 * 1) = (1, 1)
\[ \Rightarrow a * a = 1, \text{ since } 1 * 1 = 1. \]

2. \(u \circ (1, 1) = (1, 1) \Rightarrow (a * 1, 1 * 1) = (1, 1)
\[ \Rightarrow a * 1 = 1. \]

3. \((1, 1) \circ u = u \Rightarrow (1 * a, 1 * 1) = (a, 1)
\[ \Rightarrow 1 * a = a. \]

4. Let \(a, b, c \in A\) and we choose \(u = (a, 1), v = (b, 1), \) and \(w = (c, 1).\) Then
\[
 u \circ (v \circ w) = v \circ (u \circ w)
\]
\[
 \Rightarrow (a * (b * c), 1 * (1 * 1)) = (b * (a * c), 1 * (1 * 1))
\]
\[
 \Rightarrow a * (b * c) = b * (a * c).
\]

This proves that \((A; *, 1)\) is a BE-algebra. Conversely suppose that \((A; *, 1)\) is a BE-algebra. Let \(u = (a_1, a_2) \in B\). Then

1. \(u \circ u = (a_1, a_2) \circ (a_1, a_2)
\[ = (a_1 * a_1, a_2 * a_2)
\[ = (1, 1). \]

2. \(u \circ (1, 1) = (a_1, a_2) \circ (1, 1)
\[ = (a_1 * 1, a_2 * 1)
\[ = (1, 1). \]

3. \((1, 1) \circ u = (1, 1) \circ (a_1, a_2)
\[ = (1 * a_1, 1 * a_2)
\[ = (a_1, a_2)
\[ = u. \]

4. Let \(u = (a_1, a_2), v = (b_1, b_2), \) and \(w = (c_1, c_2)\) be any three elements of \(B\).

Then
\[
 u \circ (v \circ w) = (a_1, a_2) \circ ((b_1, b_2) \circ (c_1, c_2))
\[ = (a_1, a_2) \circ (b_1 * c_1, b_2 * c_2)
\[ = (a_1 * (b_1 * c_1), a_2 * (b_2 * c_2))
\[ = (b_1 * (a_1 * c_1), b_2 * (a_2 * c_2))
\[ = (b_1, b_2) \circ (a_1 * c_1, a_2 * c_2)
\[ = (b_1, b_2) \circ ((a_1, a_2) \circ (c_1, c_2))
\[ = v \circ (u \circ w).
\]

Hence \((B, \circ, (1, 1))\) be a BE-algebra.

Corollary 3.2. If \((A; *, 1)\) and \((B; o, e)\) are two BE-algebra, then \(C = A \times B\) is also a BE-algebra under the operation defined as follows: For \(u = (a_1, b_1)\) and \(v = (a_2, b_2)\) in \(C\),

\[
 u \circ v = (a_1 * a_2, b_1 \circ b_2).
\]

Here the distinct element of \(C\) is \((1, e)\).

Note 3.3. The above result can be extended for finite number of BE-algebras.

Theorem 3.4. Let \((A; *, 1)\) be a BE-algebra and let \(B = A \times A\). Then

(a) \(B\) is commutative iff \(A\) is commutative.

(b) \(B\) is self distributive iff \(A\) is self distributive.

Proof. (a) First suppose that \(B\) is commutative. Let \(a\) and \(b\) be arbitrary elements of \(A\). We choose \(u = (a, 1)\) and \(v = (b, 1)\), since \(B\) is commutative, we have

\[
 (u \circ v) \circ v = (v \circ u) \circ u.
\]

This gives \(((a \circ b) \circ b, 1) = ((b \circ a) \circ a, 1)\), which in turns imply that

\[
 (a \circ b) \circ b = (b \circ a) \circ a.
\]

Hence \(A\) is commutative. Conversely suppose that \(A\) is commutative. Let \(u = (a_1, a_2)\) and \(v = (b_1, b_2)\) be any two arbitrary elements of \(B\). Then

\[
 u \circ (v \circ w) = ((a_1, a_2) \circ (b_1, b_2)) \circ (b_1, b_2)
\[ = (a_1 * b_1, a_2 * b_2) \circ (b_1, b_2)
\[ = ((a_1 \circ b_1) \circ (a_2 \circ b_2)) \circ (b_1, b_2)
\[ = ((b_1 \circ a_1) \circ (b_2 \circ a_2)) \circ (a_1, a_2)
\[ = (v \circ u) \circ u.
\]

Hence \(B\) is commutative.

(b) First suppose that \(B\) is self distributive. Let \(a, b\) and \(c\) be arbitrary elements of \(A\). We choose \(u = (a, 1), v = (b, 1)\) and \(w = (c, 1)\). Since \(B\) is self distributive, we have

\[
 u \circ (v \circ w) = (u \circ v) \circ (u \circ w).
\]
Theorem 3.5. Let \( (A; \ast, 1) \) be a BE-algebra and let \( B = A \times A \). Then \( B \) is transitive iff \( A \) is transitive.

Proof. Let \( (A; \ast, 1) \) be a BE-algebra and \( u = (a_1, b_1), v = (a_2, b_2), \) and \( w = (a_3, b_3) \) be any three arbitrary elements of \( B \). Then

\[
\left( v \circ w \right) \circ \left( \left( u \circ v \right) \circ \left( u \circ w \right) \right)
\]

\[
= (v \circ w) \circ ((a_1 \ast a_2, b_1 \ast b_2) \circ (a_1 \ast a_3, b_1 \ast b_3))
\]

\[
= (v \circ w) \circ ((a_1 \ast a_2) \circ (a_1 \ast a_3), (b_1 \ast b_2) \circ (b_1 \ast b_3))
\]

\[
= (a_2 \ast a_3, b_2 \ast b_3) \circ ((a_1 \ast a_2) \ast (a_1 \ast a_3), (b_1 \ast b_2) \ast (b_1 \ast b_3))
\]

\[
= ((a_2 \ast a_3) \ast (a_1 \ast a_2) \ast (a_1 \ast a_3), (b_2 \ast b_3) \ast (b_1 \ast b_2) \ast (b_1 \ast b_3))
\]

\[
= (1, 1).
\]

Therefore

\[
v \circ w \leq \left( u \circ v \right) \circ \left( u \circ w \right)\]

So \( B \) is transitive. Conversely assume that \( B \) be transitive. Let \( a, b, \) and \( c \) be three arbitrary elements of \( A \). We consider the elements \( u = (a, 1), v = (b, 1) \) and \( w = (c, 1) \) of \( B \). Since \( B \) is transitive, we have,

\[
v \circ w \leq (u \circ v) \circ (u \circ w)
\]

\[
\Rightarrow (v \circ w) \circ ((u \circ v) \circ (u \circ w)) = (1, 1)
\]

\[
(b \ast c) \ast ((a \ast b) \ast (a \ast c)), 1 = (1, 1)
\]

\[
\Rightarrow (b \ast c) \ast ((a \ast b) \ast (a \ast c)) = 1
\]

\[
\Rightarrow b \ast c \leq (a \ast b) \ast (a \ast c)
\]

Hence \( A \) is transitive. \( \square \)

References


