Ideals and IWI-ideals of residuated lattice Wajsberg algebras

A. Ibrahim and R. Shanmugapriya

Abstract
In this paper, we study WI-ideal of residuated lattice Wajsberg algebra and investigate some of their properties. Also, we announce the concept of implicative WI-ideal (IWI-ideal) of residuated lattice Wajsberg algebra. Further, we inspect some of its characterizations and attain some properties of residuated lattice H-Wajsberg algebra.

Keywords
Wajsberg algebra; Lattice Wajsberg algebra; Residuated lattice Wajsberg algebra; Residuated lattice H-Wajsberg algebra; WI-ideal; Lattice ideal; Ideal; Implicative WI-ideal.

AMS Subject Classification
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1. Introduction

Mordchaj Wajsberg [1] introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrens [2]. Residuated lattices were announced by Ward and Dilworth [3]. Ibrahim and Shajitha Begum [4] and [5] introduced the notions of Wajsberg implicative ideal (WI-ideal), ideals and implicative WI-ideals of lattice Wajsberg algebras and also investigated their properties with suitable illustrations. The authors [6],[7] and [8] introduced the notion of Wajsberg implicative ideal (WI-ideal) and Fuzzy Wajsberg Implicative ideal (FWI-ideal) of residuated lattice Wajsberg algebras.

In this paper, we consider ideal of residuated lattice Wajsberg algebra and investigate some related properties. Also, we introduce the notion of IWI-ideal of residuated lattice Wajsberg algebra. Further, we investigate some of its characterizations and obtain some properties of residuated lattice H-Wajsberg algebra.

2. Preliminaries

In this section, we recall some basic definitions and properties which are helpful to develop our main results.

Definition 2.1 ([2]). Let \( \mathcal{R}, \to, \ast, 1 \) be an algebra with a binary operation \( "\to" \) and a quasi complement \( "\ast" \) is called a Wajsberg algebra. Then if it satisfied the following axioms for all \( x,y,z \in \mathcal{R} \),

\( (i) \quad 1 \to x = x \)
\( (ii) \quad (x \to y) \to y = ((y \to z) \to (x \to z)) = 1 \)
\( (iii) \quad (x \to y) \to y = (y \to x) \to x \)
\( (iv) \quad (x^\ast \to y^\ast) \to (y \to x) = 1 \).

Definition 2.2 ([2]). A Wajsberg algebra \( \mathcal{R}, \to, \ast, 1 \) satisfied the following axioms for all \( x,y,z \in \mathcal{R} \),

\( (i) \quad x \to x = 1 \)
(ii) If \((x \to y) = (y \to x) = 1\) then \(x = y\)

(iii) \(x \to 1 = 1\)

(iv) \((x \to (y \to x)) = 1\)

(v) If \((x \to y) = (y \to z) = 1\) then \(x \to z = 1\)

(vi) \((x \to y) \to ((z \to x) \to (z \to y)) = 1\)

(vii) \(x \to (y \to z) = y \to (x \to z)\)

(viii) \(x \to 0 = x \to 1^\ast = x^\ast\)

(ix) \((x^\ast)^\ast = x\)

(x) \((x^\ast \to y^\ast) = y \to x\).

**Definition 2.3** ([2]). A Wajsberg algebra \(R\) is called a lattice Wajsberg algebra, if it satisfied the following conditions for all \(x, y, z \in R\),

(i) The partial ordering \(\leq\) on a lattice Wajsberg algebra, such that \(x \leq y\) if and only if \(x \to y = 1\)

(ii) \(x \lor y = (x \to y) \to y\)

(iii) \(x \land y = ((x^\ast \to y^\ast) \to y^\ast)^\ast\).

Thus, \((R, \lor, \land, \ast, 0, 1)\) is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

**Proposition 2.4** ([2]). A lattice Wajsberg algebra \((R, \to, \ast, 1)\) satisfied the following axioms for all \(x, y, z \in R\),

(i) If \(x \leq y\) then \(x \to z \geq y \to z\) and \(z \to x \leq z \to y\)

(ii) \(x \leq y \to z\) if and only if \(y \leq x \to z\)

(iii) \((x \land y)^\ast = (x^\ast \land y^\ast)\)

(iv) \((x \land y)^\ast = (x^\ast \lor y^\ast)\)

(v) \((x \lor y) \to z = (x \to z) \land (y \to z)\)

(vi) \(x \to (y \land z) = (x \to y) \land (x \to z)\)

(vii) \((x \to y) \lor (y \to x) = 1\)

(viii) \(x \to (y \lor z) = (x \to y) \lor (x \to z)\)

(ix) \((x \land y) \to z = (x \to z) \lor (y \to z)\)

(x) \((x \land y) \lor z = (x \lor z) \land (y \lor z)\)

(xi) \((x \land y) \to z = (x \to y) \to (x \to z)\).

**Definition 2.5** ([3]). A residuated lattice \((R, \lor, \land, \ast, 0, 1)\) satisfied the following conditions for all \(x, y, z \in R\),

(i) \((R, \lor, \land, 0, 1)\) is a bounded lattice

(ii) \((R, \ast, 1)\) is commutative monoid

(iii) \(x \land y \leq z\) if and only if \(x \leq y \to z\).

**Proposition 2.6** ([3]). Let \((R, \lor, \land, \ast, 0, 1)\) be a residuated lattice. Then the following are satisfied for all \(x, y, z \in R\),

(i) \((x \land y) \to z = x \to (y \to z)\)

(ii) \((x \land y) \lor z = x \lor (y \lor z)\)

(iii) \(x \land y = y \land x\).

**Definition 2.7** ([2]). Let \((R, \lor, \land, \ast, 0, 1)\) be a lattice Wajsberg algebra. If a binary operation \(\ast\) on \(R\) satisfied \(x \ast y = (x \to y)^\ast\) for all \(x, y \in R\). Then \((R, \lor, \land, \ast, 0, 1)\) is called a residuated lattice Wajsberg algebra.

**Definition 2.8** ([5]). The lattice Wajsberg algebra \(R\) is called a lattice \(H\)-Wajsberg algebra, if it satisfied \(x \lor y \lor ((x \land y) \to z) = 1\) for all \(x, y, z \in R\).

In a lattice \(H\)-Wajsberg algebra \(R\), the following are hold:

(i) \(x \to (x \to y) = (x \to y)\)

(ii) \(x \to (y \to z) = (x \to y) \to (x \to z)\).

**Definition 2.9** ([7]). The residuated lattice Wajsberg algebra \(R\) is called a residuated lattice \(H\)-Wajsberg algebra if it satisfied \(x \lor y \lor ((x \land y) \to z) = 1\) for all \(x, y, z \in R\).

In a residuated lattice \(H\)-Wajsberg algebra \(R\), the following are hold:

(i) \(x \land y \in R\)

(ii) \((x \land y) \to (x \to y) = (x \land y)\)

(iii) \((x \land y) \lor (x \land z) \to (x \to z) = (x \land y) \to (x \to z)\), for all \(x, y, z \in R\).

**Proposition 2.10** ([5]). Let \(R\) is a lattice \(H\)-Wajsberg algebra, then the following equality are hold

\[(x \to y)^\ast \to z = (x \to z)^\ast \to (y \to z)^\ast\] for all \(x, y, z \in R\).

**Definition 2.11** ([2]). Let \(I\) be a non-empty subset of a lattice Wajsberg algebra \(R\). Then \(I\) is called a WI-ideal \(R\), if satisfied for all \(x, y \in R\),

(i) \(0 \in I\)

(ii) \((x \to y)^\ast \in I\) and \(y \in I\) imply \(x \in I\).

**Definition 2.12** ([6]). Let \(I\) be a non-empty subset of a residuated lattice Wajsberg algebras \(R\). Then \(I\) is called a WI-ideal \(R\), if it satisfied the following for all \(x, y \in R\),

(i) \(0 \in I\)

(ii) \(x \land y \in I\) and \(y \in I\) imply \(x \in I\)

(iii) \((x \to y)^\ast \in I\) and \(y \in I\) imply \(x \in I\).

**Definition 2.13** ([2]). Let \(R\) be a lattice. An ideal \(I\) of \(R\) is a nonempty subset of \(R\) is called a lattice ideal, if it satisfied the following axioms for all \(x, y \in R\),
Proof. Let \( T \) be an ideal of \( \mathcal{R} \). From (ii) of Definition 2.15 shows that \( T \) satisfies (i) of Definition 2.13. Now, \[
(x \lor y)^* \otimes y = ([((x \to y) \to y)^* \to y^*)^*]
\]
From (ii) of Definition 2.3
\[
= [y \to ((x \to y) \to y)]^*[ From (ii) of Definition 2.2]
\]
\[
= [y \to ((y \to x) \to x)]^*[ From (ii) of Definition 2.1]
\]
\[
= [(y \to (y \lor x))^*][ From (ii) of Definition 2.3]
\]
\[
= ([((y \to y) \to (y \to x)) \to (y \to x))]^* [ From (ii) of Definition 2.3]
\]
\[
= ([y \to (y \to (y \to x))] \to (y \to x))^* [ From (ii) of Proposition 2.4]
\]
\[
= ([y \to (y \to x)] \to (y \to x))^* [ From (i) of Definition 2.3]
\]
\[
= (y \to (y \to x))^* \to (y \to x)^* [ From (ii) of Proposition 2.4]
\]
\[
= (y \to x) \to [(y \to y) \to (y \to x)] [ From (x) of Definition 2.2]
\]
\[
= T \subseteq T[ From (iv) of Definition 2.2]
\]
And
\[
(x \lor y)^* \to y = (((x \to y) \to y)^* \to y)
\]
From (i) of Definition 2.3
\[
= ([y \to ((x \to y) \to y)]^* \to y)
\]
\[
= (y^* \to (y^* \to x^*)) \to y
\]
\[
= [(y^* \to (y^* \to x^*)) \to y][ From (iii) of Definition 2.1]
\]
\[
= ([x^\land y^*] \to y)[ From (iii) of Definition 2.3]
\]
\[
= (x^\lor y^*) \to y)[ From (iv) of Proposition 2.4]
\]
\[
= (x^* \to y) \land (y^* \to y)[ From (v) of Proposition 2.4]
\]
\[
= x^* \to y \in T.
\]
Thus, we get
\[
(x \lor y)^* \otimes y = 1 \in T, (x \lor y)^* \to y = x^* \to y \in T.
\]
Since \( T \) is an ideal, \[
(x \lor y)^* \otimes y \in T, (x \lor y)^* \to y \in T
\]
imply \( x \lor y \in T \) and \( y \in T \). From Definition 2.13, we have \( T \) is a lattice ideal.

Example 3.4. Consider a set \( \mathcal{R} = \{0, p, q, r, x, y, z, 1\} \). Define

**Table 1. Complement**

<table>
<thead>
<tr>
<th>x</th>
<th>x^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>p</td>
<td>x</td>
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<td>q</td>
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<td>z</td>
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<td>l</td>
<td>0</td>
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</tbody>
</table>

Then, a partial ordering “\( \leq \)” on \( \mathcal{R} \), such that \( 0 \leq a \leq b \leq c \leq 1 \)
Table 2. Implication

<table>
<thead>
<tr>
<th>→ 0</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>p</td>
<td>x</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>q</td>
<td>y</td>
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<tr>
<td>x</td>
<td>p</td>
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<tr>
<td>y</td>
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<td>1</td>
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<tr>
<td>I</td>
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<td>p</td>
<td>q</td>
<td>r</td>
<td>x</td>
<td>y</td>
<td>z</td>
</tr>
</tbody>
</table>

\( \Rightarrow (x \otimes y) \rightarrow y \in T, (x \rightarrow y)^* \rightarrow y \in T. \)

Therefore, \( x \otimes y \in T, (x \rightarrow y)^* \in T \) and \( y \in T \) imply \( x \in T. \)

Hence, we get \( T \) is a WI-ideal. \( \square \)

3.2 Properties of IWI-ideal of residuated lattice Wajsberg algebras

In this section, we introduce the concept of implicative WI-ideal \((\text{IWI-ideal})\) of residuated lattice Wajsberg algebra and we find some of its properties with illustrations.

Definition 3.7. Let \( I \) be a non-empty subset of residuated lattice wajsberg algebra \( \mathcal{A} \). Then, \( I \) is said to be a WI-IW idealf of \( \mathcal{A} \), if it satisfies the following axioms for all \( x, y \in \mathcal{A} \):

(i) \( 0 \in I \)

(ii) \( y \otimes z \in I \) and \( ((x \otimes y) \otimes z) \in I \) imply \( x \otimes z \in I \)

(iii) \( (y \rightarrow z)^* \in I \) and \( ((x \rightarrow y)^* \rightarrow z^*) \) imply \( (x \rightarrow z)^* \).

Proposition 3.8. If \( I \) is a IWI-IW -ideal of residuated lattice Wajsberg algebra \( \mathcal{A} \) then \( I \) is a WI -ideal of \( \mathcal{A} \).

Proof. Let \( I \) be a IWI-ideal of \( \mathcal{A} \), then \( 0 \in I, y \otimes z \in I \) (\( y \rightarrow z)^* \in I \) and \( (x \otimes y) \otimes z) \in I \) imply \( x \otimes z \in I \), \( (x \rightarrow y)^* \rightarrow z^* \) \( \in I \). If \( y \in I \) and \( x \otimes y \in I \), \( (x \rightarrow y)^* \in I \) for all \( x, y \in \mathcal{A} \), we have \( y \otimes 0 = (y \rightarrow 0)^* = (y \rightarrow 0)^* = 1^* = 0 \in I \) [ From Definition 2.8]

\( y \rightarrow 0)^* = (y^*)^* = y \in I \) [ From(ii) of Definition 2.9].

Now, \( (x \otimes y) \otimes z = ((x \rightarrow y)^* \rightarrow 0)^* \) [ From Definition 2.8]
\( = ((x \rightarrow x)^* \rightarrow 1)^* = (1^* \rightarrow 1)^* = (0 \rightarrow 1)^* = 1^* = 0 \in I \)

and
\( ((x \rightarrow y)^* \rightarrow 0)^* = (((x \rightarrow y)^*)^*)^* = (x \rightarrow y)^* \in I \) [ From(ii) of Definition 2.9].

Since \( I \) is a IWI-ideal of \( \mathcal{A} \). Which follows that \( x = (x^*)^* = x \otimes 0 = (x \rightarrow 0)^* = (x \rightarrow 1)^* = 1^* = 0 \in I \),
[ From Definition 2.8]
\( x = (x^*)^* = (x \rightarrow 0)^* = y^* = x \in I \). [ From(ii) of Definition 2.9].

Hence, \( I \) is a WI-IW -ideal of \( \mathcal{A} \). \( \square \)

Example 3.9. Consider a set \( \mathcal{A} = \{0, p, q, r, s, t, 1\} \). Define a partial ordering \( "\leq" \) on \( \mathcal{A} \), such that \( 0 \leq a \leq b \leq c \leq d \leq 1 \) with a binary operations \( \otimes \) and \( \rightarrow \) and a quasi complement \( "*" \) on \( \mathcal{A} \) as in following tables 3 and 4.

Define \( \vee \) and \( \wedge \) operations on \( \mathcal{A} \) as follows:
\( (x \vee y) = (x \rightarrow y) \rightarrow y \)
\( (x \wedge y) = (x^* \rightarrow y^*) \rightarrow (y \otimes x)^* \)

for all \( x, y \in \mathcal{A} \). Then, \( \mathcal{A} \) is a residuated lattice Wajsberg algebra. It is easy to verify that, \( I_2 = \{0, q, s, 1\} \) is a WI-IW -ideal of \( \mathcal{A} \).
Table 3. Complement

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Table 4. Complement

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</table>

**Proposition 3.10.** Every WI-ideal of a residuated lattice H-Wajsberg algebra is a IWI-ideal of $\mathcal{R}$.

**Proof.** Let $\mathcal{R}$ be a residuated lattice H-Wajsberg algebra and let $I$ be a WI-ideal of $\mathcal{R}$ for all $x, y, z \in \mathcal{R}$. Then we have

$$y \otimes z, (x \otimes y) \otimes z \in I, (y \rightarrow z)^*, ((x \rightarrow y)^* \rightarrow z)^* \in I$$

and

$$((x \rightarrow z)^* \rightarrow (y \rightarrow z)^*)^* = ((x \rightarrow y)^* \rightarrow z)^* \in I$$

[From Proposition 2.6]

Since $I$ is a WI-ideal of $\mathcal{R}$, $(x \rightarrow z)^* \in I$ hence, $I$ is a IWI-ideal of $\mathcal{R}$. □

**Proposition 3.11.** If $\mathcal{R}$ is a residuated lattice H-Wajsberg algebra if and only if every WI-ideal of $\mathcal{R}$ is a IWI-ideal of $\mathcal{R}$.

**Proof.** We can easily prove from Proposition 3.10. □

**Proposition 3.12.** Let $\mathcal{R}$ be a residuated lattice Wajsberg algebra and $I$ be a subset of $A$. Define $I^* = \{x^* / x \in I\}$ is a IWI-ideal of $\mathcal{R}$ if and only if $I^*$ is an implicative filter of $\mathcal{R}$.

**Proof.** Let $I$ be a IWI-ideal of $\mathcal{R}$, then $1 = 0^* \in I^*$, since $0 \in I$ for all $x, y, z \in \mathcal{R}$. If $x \otimes y, x \rightarrow y$ and $x \otimes (y \otimes z), x \rightarrow (y \rightarrow z) \in I^*$, then we have

$$(x \otimes y)^* = x \otimes y \in I, y^* = y \in I$$

and

$$(z^* \otimes (y^*)^*) = (z \otimes x) \otimes y = (x \otimes y) \otimes z \otimes x = x \otimes y \in I$$

[From (ii) of Proposition 2.6]

$$(z^* \otimes x^*) = (z \otimes x) \otimes y = (x \otimes y) \otimes z \otimes x = x \otimes y \in I$$

[From (ii) of Proposition 2.6]

$$(z^* \otimes y^*) = (z \otimes x) \otimes y = (x \otimes y) \otimes z \otimes x = x \otimes y \in I$$

[From (ii) of Proposition 2.6]

Thus, $I^*$ is an implicative filter of $\mathcal{R}$, equivalently $(x \otimes z) \in I^*$, $(x \rightarrow z)^* \in I^*$. Therefore, $I$ is a IWI-ideal of $\mathcal{R}$. □

### 4. Conclusion

In this paper, we have studied WI-ideal of residuated lattice Wajsberg algebra and investigated some of their properties. Also, we have announced the concept of implicative WI-ideal (IWI-ideal) of residuated lattice Wajsberg algebra. Further, we have inspected some of its characterizations and attained some properties of residuated lattice H-Wajsberg algebra.

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