Abstract

The zero divisor graph $\Gamma(R)$ of a commutative ring $R$ is a graph whose vertices are non-zero zero divisors of $R$ and two vertices are adjacent if their product is zero. The characteristic polynomial of matrix $M$ is defined as $|\lambda I - M|$ and roots of the characteristic polynomial are known as eigenvalues of $M$. We investigate eigenvalues and characteristic polynomials for some zero divisor graphs.

Keywords

Zero-divisor Graph, Adjacency Matrix, Characteristic Polynomial, Eigenvalue, Energy.

AMS Subject Classification

05C25, 05C50.

1. Introduction

The concept of zero divisor graph of commutative ring $R$ was introduced by Beck [5] in 1988. In last two decades the zero divisor graph is extensively studied by many researchers [2–4, 8, 9]. For any matrix $M$, the characteristic polynomial is defined as $|\lambda I - M|$ and roots of the characteristic polynomial are called eigenvalues of $M$. The concept of energy of graph was introduced by Gutman [6] in 1978. The study of energy of zero divisor graph was first initiated by Ahmadi and Nezhad [1] for the ring $\mathbb{Z}_n$ for $n = p^2$ and $n = pq$, where $p$ and $q$ are distinct primes. The adjacency matrix and eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for $n = p^3$ and $n = p^2q$ was studied by Reddy et al. [10].

In this paper, we study the energy and characteristic polynomial of zero divisor graph $\Gamma(\mathbb{Z}_n)$ for $n = p^4$ and zero divisor graphs obtained from direct product of rings. Throughout this paper we consider the commutative ring $R$ with unity. If $R$ is a ring then $\mathcal{Z}(R)$ and $\mathcal{Z}^*(R)$ denote the set of zero divisors and set of non-zero zero divisors of the ring $R$ respectively. The zero divisor graph of a ring $R$, denoted as $\Gamma(R)$, is a graph whose vertices are the non-zero zero divisors and two vertices are adjacent if and only if their product is zero. We use $M(\Gamma(R))$ to denote the adjacency matrix of $\Gamma(R)$ and $E(\Gamma(R))$ for the energy, defined as sum of modulus of eigenvalues of graph, of $\Gamma(R)$ and the matrix with all entries 1 will be denoted as $J$.

2. Main Results

Proposition 2.1. [7] Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be any matrix. Then $|M| = |A||D - CA^{-1}B|$

Theorem 2.2. Let $n = p^4$ with $p$ any prime. If $\lambda$ is any nonzero eigenvalue of $\Gamma(\mathbb{Z}_n)$ then,

$$\lambda^3 - \lambda^2 (p^2 - 1) - \lambda p^2 (p - 1)^2 + p^3 (p - 1)^3 = 0$$

Proof. Let $n = p^4$. Then the set of non-zero zero divisors of $\mathbb{Z}_n$ is $\mathcal{Z}^*(\mathbb{Z}_n) = \{p, 2p, 3p, ..., (p^3 - 1)p\}$. We partition the set $\mathcal{Z}^*(\mathbb{Z}_n)$ as $\mathcal{Z}^*(\mathbb{Z}_n) = A \cup B \cup C$, where $A = \{k1p \mid k1 = 1, 2, 3, ..., p^2 - 1\}$ and $p \not| k1$, $B = \{k2p^2 \mid k2 = 1, 2, 3, ..., p^2 - 1\}$ and $p \not| k2$, and $C = \{k3p^3 \mid k3 = 1, 2, 3, ..., p - 1\}$. Then $|A| = p^3 - p^2$, $|B| = p^2 - p$, and $|C| = p - 1$. Since the elements of $A$ and $B$ are not adjacent, we get the zero matrices of order $p^3 - p^2$, $(p^3 - p^2) \times (p^2 - p)$ and $(p^2 - p) \times (p^3 - p^2)$. As the elements of $A$ and $C$ are adjacent implies we get a matrix of ones of order $(p^3 - p^2) \times (p - 1)$. Similarly get the matrices corresponding to $B \& B$, $B \& C$, $C \& C$, and $C \& B$. The matrix corresponding to $B \& C$ is a matrix of all ones.
C & A.C & C. Hence we get the adjacency matrix by considering the elements of A first, then B and then C as

$$M(\Gamma(Z_p^\ast)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where $O$ is the zero matrix and $J$ is the matrix of ones. Let $\lambda$ be any eigenvalue of $M(\Gamma(Z_p^\ast))$. Then

$$\lambda I - M = \begin{bmatrix} \lambda I - O & 0 \\ 0 & \lambda I - J \end{bmatrix} = 0$$

Let $T_1 = \begin{bmatrix} \lambda I - O \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} -J \end{bmatrix}$, $T_3 = \begin{bmatrix} -J \end{bmatrix}$ and $T_4 = \begin{bmatrix} -J \end{bmatrix}$.

Then by Proposition 2.1, $|\lambda I - M(\Gamma(Z_p^\ast))| = 0$.

Now by straightforward calculation we get $|T_1| = \lambda^{p^3 - p^4 - 1}(\lambda - (p^2 - p))$.

$$T_1^{-1} = \frac{1}{\lambda - (p^2 - p)} \begin{bmatrix} I \\ O \end{bmatrix}$$

and $T_3 T_1^{-1} T_2 = \left( \begin{bmatrix} \begin{pmatrix} \lambda - (p^2 - p) \end{pmatrix} \right) \left( \begin{bmatrix} \lambda - (p^2 - p) \end{pmatrix} \right) \left( \begin{bmatrix} \lambda - (p^2 - p) \end{pmatrix} \right) \right) = 0$

Thus characteristic polynomial is

$$\lambda^{p^3 - 4} \left( \lambda^3 - \lambda^2 (p^2 - 1) - \lambda p^2 (p^2 - 2) + p^3 (p - 1)^3 \right) = 0$$

Hence we get $\lambda^3 - \lambda^2 (p^2 - 1) - \lambda p^2 (p^2 - 2) + p^3 (p - 1)^3 = 0$ for any non zero eigenvalue $\lambda$ of $M(\Gamma(Z_p^\ast))$.

Theorem 2.3. Let $Z_p \times Z_p$ be a ring with $p$ be any prime then $E(\Gamma(Z_p \times Z_p)) = 2(p - 1)$.

**Proof.** Being $Z_p \times Z_p$ ring, $Z_p$ has no non-zero divisor implies $Z_0^\ast(\Gamma(Z_p \times Z_p)) = A \cup B$, where $A = \{ (0, kp) | k = 1, 2, 3, ..., p - 1 \}$ and $B = \{ (kp, 0) | k = 1, 2, 3, ..., p - 1 \}$, moreover $|Z_0^\ast(\Gamma(Z_p \times Z_p))| = 2(p - 1)$, $|A| = p - 1$ and $|B| = p - 1$. Since every element of A is adjacent with every element of B, we get the matrix $M_1$ with all entries ones of order $(p - 1) \times (p - 1)$. Hence we get the adjacency matrix by considering the elements of A first and then B as

$$M(\Gamma(Z_p \times Z_p)) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$|\lambda I - M(\Gamma(Z_p \times Z_p))| = 0$$

by simple calculation we get $|\lambda I - M(\Gamma(Z_p \times Z_p))| = 0$.

$$E(\Gamma(Z_p \times Z_p)) = 2(p - 1)$$

Theorem 2.4. Let $Z_p \times Z_p$ be a ring with $p$ be any prime. If $\lambda$ is any non-zero eigenvalue of the adjacency matrix $M(\Gamma(Z_p \times Z_p))$ then $\lambda$ satisfies the equation $\lambda^4 + \lambda^3 (p - 1) + \lambda^2 (2(p - 1)^2) + \lambda (p(p - 1)^3) + p(p - 1)^3 = 0$.

**Proof.** Let $Z_p \times Z_p$ be a ring. Note that $Z_p$ has no non-zero zero divisor and in $Z_p^{\ast}$ the non-zero zero divisors are multiples of $p$. Hence $Z^\ast(\Gamma(Z_p \times Z_p)) = A_1 \cup A_2 \cup A_3 \cup A_4$ where $A_1 = \{(x_1, kp) | x_1 \in Z_p^{\ast} \}$, $A_2 = \{(0, x_2) | x_2 \in Z_p^{\ast} \}$, $A_3 = \{(x_1, 0) | x_1 \in Z_p^{\ast} \}$ and $A_4 = \{(0, kp) | k = 1, 2, 3, ..., p - 1 \}$.

Then $|A_1| = (p - 1)^2$, $|A_2| = (p^2 - p)$, $|A_3| = (p - 1)$, $|A_4| = (p - 1)$. Since no element of $A_1$ is adjacent with element of $A_1$, $A_2$ and $A_3$, we get the zero matrices of order $(p - 1)^2 \times (p - 1)^2$, $(p - 1)^2 \times (p - 2)$ and $(p - 1)^2 \times (p - 1)^2$ respectively. Also we get zero matrices corresponding to $A_2$ and $A_3$ and $A_2$ and $A_4$. And every element of $A_1$ is adjacent with every element of $A_4$ implies we get a matrix of order $(p - 1)^2 \times (p - 1)$ whose all entries are ones. Similarly we get matrices of ones corresponding to $A_4$ and $A_4$; and $A_3$ and $A_4$. Hence we get the adjacency matrix by considering the ele-
ments of $A_1$ first, then $A_2$, then $A_3$ and then $A_4$ as

$$M(\Gamma(Z_p \times Z_p)) = \begin{bmatrix} O & O & O & R_1 \\ O & O & R_2 & O \\ O & R_1^T & O & R_3 \\ R_1 & O & R_3^T & R_4 \end{bmatrix}$$

where $O$ is the zero matrix and $R_i$ is the matrix of ones for $i = 1, 2, 3, 4$.

Let $\lambda$ be any non-zero eigenvalue of $M(\Gamma(Z_p \times Z_p))$, then

$$|\lambda I - M(\Gamma(Z_p \times Z_p))| = |\begin{bmatrix} \lambda I & O & O & R_1 \\ O & \lambda I & R_2 & O \\ O & R_1^T & \lambda I & R_3 \\ R_1 & O & R_3^T & \lambda I - R_4 \end{bmatrix}| = 0$$

Let $T_1 = \lambda I$, $T_2 = \begin{bmatrix} 0 & -R_1 \\ -R_2 & 0 \end{bmatrix}$ and $T_4 = \begin{bmatrix} \lambda I & -R_3 \\ -R_3^T & \lambda I - R_4 \end{bmatrix}$

Then by Proposition 2.1 $|\lambda I - M(\Gamma(Z_p \times Z_p))| = |T_1 T_2 T_3 T_4^{-1} T_2| = 0$.

Since $T_1$ is a scalar matrix of order $(p-1)(2p-1)$, we get $|T_1| = \lambda^{(p-1)(2p-1)}$.

And $T_3 T_1^{-1} T_2 = \begin{bmatrix} \frac{p^2 - p J}{\lambda} & O \\ 0 & (p-1) J \end{bmatrix}$

So $|T_4 - T_3 T_1^{-1} T_2| = \left| \begin{array}{cc} \lambda I - \frac{p^2 - p J}{\lambda} & O \\ O & \lambda - (\lambda - (p-1) J)^{(p-1) J} \end{array} \right| = \lambda^{n(2p-1)-5}(\lambda^4 + \lambda^3 (p-1) + \lambda^2 (2p-1) + \lambda p(p-1)^3) + p(p-1)^5$.

Therefore $|\lambda I - M(\Gamma(Z_p \times Z_p))| = \lambda^{n(2p-1)-5}(\lambda^4 + \lambda^3 (p-1) + \lambda^2 (2p-1) + \lambda p(p-1)^3) + p(p-1)^5$ which is the characteristic polynomial of $\Gamma(Z_p \times Z_p)$.

Since $\lambda \neq 0$, we get $\lambda^4 + \lambda^3 (p-1) + \lambda^2 (2p-1) + \lambda p(p-1)^3 + p(p-1)^5 = 0$

\[ \text{Theorem 2.5. Let } Z_p \times Z_{pq} \text{ be a ring with } p, q \text{ be distinct primes. Then the spectra of zero divisor graph of } Z_p \times Z_{pq} \text{ is Spec}(\Gamma(Z_p \times Z_{pq})) = \left\{ (p-1) \pm \sqrt{4q-3}, \frac{1}{2} (p-1) \right\} \text{ for } i = 1, 2, 3, 4, \text{ where } \lambda_i \text{ is the solution of the equation } \lambda^4 - \lambda^3 (p-1) - \lambda^2 (2p-1)(q-1) + \lambda(p-1)^3(q-1) + (p-1)^4(q-1)^2 = 0 \]

\[ \text{Proof. Let } Z_p \times Z_{pq} \text{ be a ring with } p, q \text{ be distinct primes. Note that the set of zero divisors of } Z_p \times Z_{pq} \text{ is given by } Z^*(Z_p \times Z_{pq}) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6, \text{ where } A_1 = \{(x_1, k_2p) | x_1 \in Z_p^* \text{ and } k_1 = 1, 2, 3, ..., p-1\}, A_2 = \{(0, x_2) | x_2 \in Z_{pq}^* \text{ and } x_2 \text{ is non zero divisor}\}, A_3 = \{(x_1, k_2q) | x_1 \in Z_p^* \text{ and } k_2 = 1, 2, 3, ..., p-1\}, A_4 = \{(0, k_2g) | k_2 = 1, 2, 3, ..., p-1\}, A_5 = \{(x_1, 0) | x_1 \in Z_p^* \text{ and } A_6 = \{(0, k_1p) | k_1 = 1, 2, 3, ..., q-1\}. \]

Then $|A_1| = (p-1)(q-1)$, $|A_2| = (p-1)(q-1)$, $|A_3| = (p-1)^2$, $|A_4| = (p-1)$, $|A_5| = (p-1)$, $|A_6| = (q-1)$. Since all the elements of $A_1$ is adjacent with all the elements of $A_4$, we get the matrix of ones. Similarly we get the matrices of ones corresponding to $A_2$ and $A_3$ and $A_5$ and $A_6$. As no element of $A_1$ is adjacent with element of $A_5$ and $A_6$, we get zero matrix. Similarly we get the zero matrices corresponding to rest of the pairs.

Hence we get adjacency matrix of zero divisor graph of $Z_p \times Z_{pq}$ by considering $A_1$ first, then $A_2$, then $A_3$, then $A_4$, and then $A_5$ and $A_6$ as $M(\Gamma(Z_p \times Z_{pq})) = \begin{bmatrix} |O|_{(p-1)(p+2q-3)(2p-1)(2q-3)} & |S_1|_{(p-1)(p+2q-3)(2p+q-3)} & |S_2|_{(p+2q-3)(2p+q-3)} \end{bmatrix}$, where $O$ is the zero matrix of order $(p-1)(p+2q-3)$ and $\lambda I = \lambda - O$.

\[ \text{Now let } \lambda I = \lambda - O, T_1 = S_1, T_2 = S_2^T \text{ and } T_4 = \lambda I - S_2. \]

Then by Proposition 2.1 $|\lambda I - M(\Gamma(Z_p \times Z_{pq}))| = |T_1 T_2 T_3 T_4^{-1} T_2| = 0$.

Since $T_1$ is a scalar matrix of order $(p-1)(p+2q-3)$, we get $|T_1| = \lambda^{(p-1)(p+2q-3)}$ and $T_3 T_1^{-1} T_2 = \begin{bmatrix} \frac{(p-1)(q-1)}{\lambda} & O & O \\ O & \frac{(p-1)(q-1)}{\lambda} & O \\ O & O & \frac{(p-1)^2}{\lambda} \end{bmatrix}$.

Therefore $|\lambda I - M(\Gamma(Z_p \times Z_{pq}))| = \lambda^{(p-1)(p+2q-3) + (p-1)^2(q-1)^2}$ which is characteristic polynomial. Hence the proof.

\[ \text{Theorem 2.6. Let } Z_p \times Z_{pq} \text{ be a ring with } p \text{ be any prime. If } \lambda \text{ is any non-zero eigenvalue of the adjacency matrix } M(\Gamma(Z_p \times Z_{pq})) \text{ then } \lambda \text{ satisfies the equation } \lambda^4 + \lambda^3 (p-1) - \lambda^2 (2p-1)(q-1) + \lambda(p-1)^3(q-1) + (p-1)^4(q-1)^2 = 0 \]

\[ \text{Proof. Let } Z_p \times Z_{pq} \text{ be a ring. Note that } Z_p \text{ has no non-zero zero divisor and in } Z_{pq}^*, \text{ the non-zero zero divisors are multiples of } p \text{ and } p^2. \]

\[ Z^*(Z_p \times Z_{pq}) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6, \text{ where } A_1 = \{(0, x_2) | x_2 \in Z_p^* \text{ and } x_2 \text{ is non zero divisor}\}, A_2 = \{(x_1, k_2p) | x_1 \in Z_p^* \text{ and } k_1 = 1, 2, 3, ..., p-1\}, A_3 = \{(x_1, k_2q) | x_1 \in Z_p^* \text{ and } k_2 = 1, 2, 3, ..., p-1\}, A_4 = \{(0, k_2g) | k_2 = 1, 2, 3, ..., p-1\}, A_5 = \{(x_1, 0) | x_1 \in Z_p^* \text{ and } A_6 = \{(0, k_1p) | k_1 = 1, 2, 3, ..., q-1\}. \]
\[ \{(x_1, k_1 p) \mid x_1 \in \mathbb{Z}_p \wedge k_1 = 1, 2, 3, \ldots, p^2 - 1 \text{ and } p \nmid k_1 \}, A_4 = \{(x_1, 0) \mid x_1 \in \mathbb{Z}_p \}, A_5 = \{(0, k_1 p) \mid k_2 = 1, 2, 3, \ldots, p^2 - 1 \text{ and } p \nmid k_1 \} \text{ and } A_6 = \{(0, k_2 p^2) \mid k_1 = 1, 2, 3, \ldots, p - 1 \}. \]

Then \[ |A_1| = p^2(p - 1), |A_2| = (p - 1)^2, |A_3| = p(p - 1)^2, |A_4| = (p - 1)^2, |A_5| = p(p - 1), \text{ and } |A_6| = (p - 1). \] Since all the elements of \(A_1\) are adjacent with all the elements of \(A_4\), we get the matrix of ones. Similarly we get the matrices of ones corresponding to \(A_2 \& A_5, A_2 \& A_6, A_3 \& A_6, A_4 \& A_2; A_4 \& A_6; A_5 \& A_6, A_6 \& A_2, A_3, A_5, A_6, \) we get zero matrix. Similarly we get the zero matrices corresponding to rest of the pairs.

Hence we get adjacency matrix of zero divisor graph of \(\mathbb{Z}_p \times \mathbb{Z}_{pq}\) by considering \(A_1\) first, then \(A_2, A_3, A_4\), then \(A_5\) and then \(A_6\) as \(M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq})) = \begin{bmatrix} O & | & | & | & | & | & S_1 & | & S_2 \end{bmatrix} \) where \(O\) is the zero matrix of order \((p - 1)(2p - 1) \times (p - 1)(2p - 1)\) and

\[ S_1 = \begin{bmatrix} J & O & O \\ O & J & J \\ O & O & J \end{bmatrix}, S_2 = \begin{bmatrix} O & J & J \\ J & O & J \\ J & J & J \end{bmatrix}. \]

Let \(\lambda\) be any non-zero eigenvalue of \(M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))\), then \(\lambda S_1 - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))I = 0\).

Now let \(T_1 = \lambda I - O, T_2 = S_1, \) then \(T_1 = \lambda I - S_2\).

Then by Proposition 2.1 \(\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))I = |T_1| T_2 = T_3 T_4 T_5\).

Since \(T_1\) is a scalar matrix of order \((p - 1)(2p - 1)\), we get \(|T_1| = \lambda((p - 1)(2p - 1))\) and

\[ T_3 T_1^{-1} T_2 = \begin{bmatrix} \frac{p^2(p - 1)}{\lambda} & O & O \\ O & \frac{(p - 1)^2}{\lambda} & O \\ O & O & \frac{(p - 1)(p^2 + p + 1)}{\lambda} \end{bmatrix} \]

and \(|T_4 - T_3 T_1^{-1} T_2| = \lambda((p - 1)(p^2 + 4)(\lambda^4 + \lambda^3(p - 1) - \lambda^2(p - 1)^2) = \lambda(p - 1)^3((p^2 - p + 1) + p^2(p - 1)^2)\).

Therefore \(|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))I| = \lambda((p - 1)(2p - 1) + (p - 1)(p^2 + 1))(\lambda^4 + \lambda^3(p - 1) - \lambda^2(p - 1)^2) = \lambda((p - 1)^3 + (p^2 - p + 1) + p^2(p - 1)^2)\).

Since \(\lambda \neq 0\), we get \(\lambda^4 + \lambda^3(p - 1) - \lambda^2(p - 1)^2 = \lambda(p - 1)^3((p^2 - p + 1) + p^2(p - 1)^2) - \lambda^2(p - 1)^3(2p - 1)\).

\[ \text{Conclusion} \]

We have explored the concept of graph energy in the context of zero divisor graphs and obtained characteristic equations for various graphs. We have also investigated the energy of the graph \(\mathbb{Z}_p \times \mathbb{Z}_p\) (where \(p\) is prime.)