Introduction of color class dominating sets in graphs

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Abstract
Let G = (V, E) be a graph. In this paper, we define a new graph parameter called color class domination number of G. A color class dominating set of G is a proper coloring χ of G with the extra property that every color class in χ is dominated by a vertex in G. A color class dominating set is said to be a minimal color class dominating set if no proper subset of χ is a color class dominating set of G. The color class domination number of G is the minimum cardinality taken over all minimal color class dominating sets of G and is denoted by γ_c(G). Here we also obtain γ_c(G) for Path graph, Cycle graph, Helm graph, Flower graph, Sunflower graph, Gear graph and Sunlet graph.

Keywords
Chromatic number, Domination number, Color class Dominating set, Color class domination number.

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1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [3].

Let G = (V, E) be a graph of order p. The open neighborhood N(v) of a vertex v ∈ V(G) consists of the set of all vertices adjacent to v. The closed neighborhood of v is N[v] = N(v) ∪ {v}. For a set S ⊆ V, the open neighborhood N(S) is defined to be ∪v∈SN(v) and the closed neighborhood of S is N[S] = N(S) ∪ S. For any set H of vertices of G, the induced sub graph <H> is the maximal sub graph of G with vertex set H.

A subset S of V is called a dominating set if every vertex in V − S is adjacent to some vertex in S. A dominating set is a minimal dominating set if no proper subset of S is a dominating set of G. The domination number γ(G) is the minimum cardinality taken over all minimal dominating sets of G. A γ-set is any minimal dominating set with cardinality γ. A proper coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by χ(G).

The join G₁ + G₂ of Graphs G₁ and G₂ with disjoint vertex sets V₁ and V₂ and edge sets E₁ and E₂ is the graph union G₁ ∪ G₂ together with each vertex in V₁ is adjacent to every vertices in V₂. A path on n vertices denoted by Pₙ, is a connected graph with all but two vertices have degree 2 and V(Pₙ) = {vᵢ/1 ≤ i ≤ n} with vᵢvᵢ₊₁ ∈ E(Pₙ) for i < n. A cycle graph is a graph on n ≥ 3 vertices containing a single cycle through all vertices and is denoted by Cₙ. The Complete graph Kₙ has every pair of p vertices adjacent. A wheel graph on n + 1 vertices is denoted by W₁,n = K₁ + Cₙ. The helm graph Hₙ is the graph obtained from a wheel graph W₁,n by adjoining a pendant edge at each vertex of the cycle Cₙ. The flower graph Fₙm is the graph obtained from a helm graph by joining each pendant vertex to the central vertex of the helm. The Sunflower graph Sfₙ is the resultant graph obtained from the flower graph of wheel W₁,n by adding pendant edges to the
A color class dominating set is said to be a minimal color wheel graph. Let $G$ be a graph of order $n$ or $\gamma_C$ is a proper coloring of $G$. Then for each color class $\mathcal{C}$ is dominated by a vertex in $G$. A color class dominating set is said to be a minimal color class dominating set if no proper subset of $\mathcal{C}$ is a color class dominating set of $G$. The color class dominating number of $G$ is the minimum cardinality taken over all minimal color class dominating sets of $G$ and is denoted by $\gamma_{\mathcal{C}}(G)$. This concept is illustrated by the following example.

**Theorem 2.2.** Let $G$ be a graph of order $p$ without isolated vertices. Then

(i) $\chi(G) \leq \gamma_{\mathcal{C}}(G)$

(ii) $\max\{\chi(G), \gamma(G)\} \leq \gamma_{\mathcal{C}}(G) \leq p$.

**Proof.** Since $\gamma_{\mathcal{C}}$-coloring of $G$ is a proper coloring, $\chi(G) \leq \gamma_{\mathcal{C}}(G)$. Now, let $\gamma_{\mathcal{C}}$ be a $\gamma_{\mathcal{C}}$-coloring of $G$. Then for each color class $\mathcal{C}_i$, $1 \leq i \leq \gamma_{\mathcal{C}}(G)$, there exist a vertex $v_i \in V$ such that $\mathcal{C}_i$ is dominated by $v_i$. Let $S = \{v_1, v_2, \ldots, v_{\gamma_{\mathcal{C}}(G)}\}$, where $v_i \in \mathcal{C}_i$, $1 \leq i \leq \gamma_{\mathcal{C}}(G)$. Now, we have to show that $S$ is a $\gamma$-set. Let $y \in V - S$. Then $y \in \mathcal{C}_i$ for some $i$, $1 \leq i \leq \gamma_{\mathcal{C}}(G)$. By the definition of $\gamma_{\mathcal{C}}$-coloring of $G$, $y$ is adjacent to the vertex $v_i$ of $S$. Then $S$ is a $\gamma$-set. Therefore $\gamma(G) \leq \gamma_{\mathcal{C}}(G)$. Since $G$ is a graph of order $p$, $G$ can be colored with at most $p$ colors. Hence, $\max\{\chi(G), \gamma(G)\} \leq \gamma_{\mathcal{C}}(G) \leq p$. 

**Proposition 2.3.** For the Wheel graph $W_{1,n}$, $n \geq 3$,

$$\gamma_{\mathcal{C}}(W_{1,n}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

**Theorem 2.4.** Let $G$ be $P_n$ or $C_n$. Then for $n > 3$,

$$\gamma_{\mathcal{C}}(P_n) = \gamma_{\mathcal{C}}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{4} \\ n/2 + 1 & \text{if } n \equiv 2 \pmod{4} \\ n/2 + 3 & \text{if } n \equiv 1, 3 \pmod{4} \end{cases}$$

**Proof.** Let $V(P_n) = \{v_i \mid 1 \leq i \leq n\}$ and $v_iv_{i+1} \in E(P_n)$ for $i < n$. Let $n > 4$. Let $\mathcal{C}$ be a $\gamma_{\mathcal{C}}$-coloring of $P_n$. We consider three cases.

Case (i): $n \equiv 0 \pmod{4}$. For $1, 2, \ldots, n/4$, let $H_i = < v_{4i-3}, v_{4i-2}, v_{4i-1}, v_{4i}>$ be the vertex induced sub graph of $P_n$. Then for each $i, 1 \leq i \leq n/4$, assign two distinct colors, say, $2i-1, 2i$ to the vertices $v_{4i-3}, v_{4i-2}, v_{4i-1}$, and $v_{4i}$ respectively. Assign color $i + 1$ to the central vertex. Case (ii): $n \equiv 2 \pmod{4}$. Since $n - 2 \equiv 0 \pmod{4}$, $P_{n-2}$ is obtained from $P_{n-2}$ followed by $P_2$. So $\gamma_{\mathcal{C}}(P_n) = \gamma_{\mathcal{C}}(P_{n-2}) + \gamma_{\mathcal{C}}(P_2) = n/2 + 1$.

Case (iii): $n \equiv 1, 3 \pmod{4}$. When $n \equiv 1 \pmod{4}$, since $n - 1 \equiv 0 \pmod{4}$, $P_n$ is obtained from $P_{n-1}$ followed by $P_1$. So $\gamma_{\mathcal{C}}(P_n) = \gamma_{\mathcal{C}}(P_{n-1}) + \gamma_{\mathcal{C}}(P_1) = [n/2] + 1$. When $n \equiv 3 \pmod{4}$, as above $\gamma_{\mathcal{C}}(P_n) = \gamma_{\mathcal{C}}(P_{n-3}) + \gamma_{\mathcal{C}}(P_3) = [n/2] + 1$. This $\gamma_{\mathcal{C}}$-coloring is true for $C_n$ also. 

**Theorem 2.5.** For the Helm graph $G = H_n$, $n \geq 3$, $\gamma_{\mathcal{C}}(H_n) = n$.

**Proof.** Let $H_n$ be a helm graph with

$$V(H_n) = \{v\} \cup \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\}.$$ 

Assign colors 1, 2 and $n$ to the vertices $\{v, u_n\}, \{u_1, v_2, v_n\}$ and $\{v_{n-1}, v\}$ respectively. Assign color $i(3 \leq i \leq n - 1)$ to the vertices $\{v_i, u_{i-1}\}$. The color classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_n$ are dominated by $v_n, v_1, v_{n-1}$ respectively. Also the color class $\mathcal{C}_3(3 \leq i \leq n - 1)$ dominated by the vertex $v_{i-1}$. Hence $\gamma_{\mathcal{C}}(H_n) = n$. 

**Example 2.6.**
Theorem 2.7. (i) If $G$ is a flower graph $F_{ln}$, $n \geq 3$, then $$\gamma_k(F_{ln}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

(ii) If $G$ is a sunflower graph $S_{fn}$, $n \geq 3$, then $$\gamma_k(S_{fn}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

Proof. (i)
By the definition of flower graph, $F_{ln}$ is obtained from a helm graph by joining each pendant vertex to the central vertex. Let $V(F_{ln}) = \{v_1, v_2, \ldots, v_{2n+1}\}$, where $v_1$ be the central vertex, $v_i(2 \leq i \leq n+1)$ be the vertices on the cycle $C_n$ and $v_i(n+2 \leq i \leq 2n+1)$ be the vertices on the pendant edges of $H_n$ such that $v_i(2 \leq i \leq n+1)$ is adjacent to $v_{n+i}$ and $v_1$. We consider two cases:

Case (i): $n$ is even. Let $C = \{C_1, C_2, C_3\}$ be a proper coloring of $F_{ln}$ in which $C_1 = \{v_1\}$, $C_2 = \{v_2, v_4, v_6, \ldots, v_{n}\} \cup \{v_{n+3}, v_{n+5}, \ldots, v_{2n+1}\}$, $C_3 = \{v_3, v_5, \ldots, v_{n}\} \cup \{v_{n+2}, v_{n+4}, \ldots, v_{2n+1}\}$. Then the color class $C_1$ is dominated by the vertex $v_2$ and the color classes $C_2$ and $C_3$ are dominated by the vertex $v_1$. Therefore, the coloring $C$ is a $\gamma_k$-coloring of $F_{ln}$ and hence

$$\gamma_k(F_{ln}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

(ii)
Let $G$ be a sunflower graph $S_{fn}$. Then $G$ is a flower graph with pendant edges attached to the central vertex. As in Theorem (2.7(i)), we assign the same proper coloring of $F_{ln}$ with color $2$ to the pendant vertices $\{v_{2n+2}, v_{2n+3}, \ldots, v_{3n+1}\}$ and we get the $\gamma_k$-coloring of $S_{fn}$. Hence

$$\gamma_k(S_{fn}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2.8. The gear graph $G_n$ has $\gamma_k(G_n) = \lceil \frac{n}{2} \rceil + 1$.

Proof. Let $V(G_n) = \{u\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$, where $v$ is the central vertex and $\deg(u_i) = 3$ and $\deg(v_j) = 2, 1 \leq i \leq n$. Assign distinct colors say, $1, 2, 3 \leq i \leq \lceil \frac{n}{2} \rceil$ to the vertices $\{v_{2i-1}, v_{2i}\}$ respectively. Also assign distinct colors say, $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil + 1$ to the vertices $\{u_1, u_2, \ldots, u_{n}\}$ and $\{v, v_n\}$ when $n$ is odd and $\{u_1, u_2, \ldots, u_n\}$ and $\{v, v_{n-1}, v_n\}$ when $n$ is even respectively, we get a $\gamma_k$-coloring. Hence,

$$\gamma_k(G_n) = \lceil \frac{n}{2} \rceil + 1.$$ 

Example 2.9.

Theorem 2.10. The Sunlet graph $SC_n$ has $\gamma_k(SC_n) = n$.

Proof. Let $V(SC_n) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ with $\deg(u_i) = 3(1 \leq i \leq n)$ and $\deg(v_j) = 1(1 \leq i \leq n)$. We consider two cases:

Case (i). When $n$ is even, assume color $i$, where $i = 1, 3, 5, \ldots, n-1$ to the vertices $\{u_i, v_{i+1}\}$ and color $j$, where $j = 2, 4, \ldots, n$ to the vertices $\{u_j, v_{j-1}\}$ respectively, we get the $\gamma_k$-coloring of $SC_n$.

Case (ii). When $n$ is odd, assign colors $1, 2$ and $n$ to the vertices $\{u_1, v_n\}, \{u_2, u_n, v_1\}$ and $\{v_{n-1}\}$ respectively. Also assign color $i(3 \leq i \leq n-1)$ to the vertices $\{u_i, v_{i-1}\}$, we get a $\gamma_k$-coloring. Thus $\gamma_k(SC_n) = n$. 

\[\square\]
Example 2.11.

![Figure 6. $n$ odd and $n$ even](image)

### References


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