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Operation on $\hat{\Omega}$-closed sets

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Abstract

This work is based on operation in a topological space. An operation has been extended to the class of $\hat{\Omega}$-open sets. The new class of $\gamma_{\hat{\Omega}}$-open sets has been introduced and two kinds of closures such as, $\gamma_{\hat{\Omega}}Cl$ and $(\hat{\Omega}Cl)_\gamma$ are studied. Necessary basic properties have been derived. Moreover, $\hat{\Omega}$-regular operation on $\hat{\Omega}O(X,\tau)$ has been introduced in which intersection of any two $\gamma_{\hat{\Omega}}$-closed sets is $\gamma_{\hat{\Omega}}$-closed. Also three types of separation axioms are defined and few results on them have been derived.

Keywords

$\gamma_{\hat{\Omega}}$-open set, $\gamma_{\hat{\Omega}}Cl$, $(\hat{\Omega}Cl)_\gamma$, $\hat{\Omega}$-regular operation, $\hat{\Omega}$-open operation, $\gamma_{\hat{\Omega}}$-T_i spaces $(i = 0,1,2)$.

AMS Subject Classification

54A05, 54A10, 54D10.

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Contents

1. Introduction

Generalized open sets play a vital role in research area of General Topology. Levine [7] introduced the concept of semi-open sets in topology. In 1987, Bhattacharyya and Lahiri [3] used semi-open sets to define the notion of semi-generalized closed sets. Kasahara [8] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α-continuous functions with α-closed graphs and α-compact spaces. Jankovic [5] introduced the concept of α-closure of a set in X via α-operation and investigated further characterizations of a function with α-closed graph. Later Ogata [10] defined and studied the concept of γ-open sets and applied it to investigate operation-functions and operation-separation.Recently several researchers developed many concepts of operation γ in a space X. Krishnan, Gangster and Balachandran [9] introduced and studied the concept of the operation γ on the class of all semiopen sets of (X,τ) and defined the notion of semi γ-open sets and investigated some of their properties. An, Cuong and Maki [1] defined and investigated an operation γ on the class of all preopen sets of (X,τ) and introduced the notion of pre-γ-open sets and developed some of their properties. Asaad [2] defined the notion of an operation γ on the class of generalized open sets in (X,τ) and studied some of its applications. Recently, the concept of $\hat{\Omega}$-closed set was introduced and investigated by Lellis Thivagar et al. [6]. In this paper, the concept of an operation γ has been extended to the class of $\hat{\Omega}$-open sets and it leads to the introduction of the notion of $\gamma_{\hat{\Omega}}$-open sets on a topological spaces (X,τ). Furthermore, some basic properties of $\gamma_{\hat{\Omega}}$-Closures have been derived. In last Section, $\gamma_{\hat{\Omega}}$-T_i spaces where $i \in \{0,1,2\}$ are introduced and investigated using the operation γ on $\tau_{\hat{\Omega}}$.

2. Preliminaries

In this section, some definitions and results that are used in this work have been dealt. Throughout this paper, (X,τ) or X represents a topological space on which no separation axioms are assumed, unless otherwise mentioned.

Definition 2.1. *[7] A subset A of a topological space (X,τ) is called a* **semi-open** *set if $A \subseteq cl(int(A))$. $SO(X)$ denotes the set of all semi-open sets in (X,τ).It's complement is known as a* **semi-closed** *set on X.*

Definition 2.2. *([11], Definition 2.2) A subset A of X is called a δ-closed set in a topological space (X,τ) if $A = \delta cl(A)$,* *where $\delta cl(A) = \{x \in X : int(cl(U)) \cap A \neq \emptyset, U \in O(X,x)\}$. The complement of a δ-closed set in (X,τ) is called a δ-open set in (X,τ). The set of all δ-closed sets in X is denoted by $\delta C(X)$. From[4], lemma 3, $\delta cl(A) = \bigcap\{F \in \delta C(X) : A \subseteq F\}$ and from corollary 4, $\delta cl(A)$ is a δ-closed set for a subset A in a topological space (X,τ)*

Definition 2.3. *([6], Definition 3.1) Let (X,τ) be a topological space. A is said to be* **$\hat{\Omega}$-closed** *set if $\delta cl(A) \subseteq U$ when $A \subseteq U$, where U is a semi-open subset of X. The complement of $\hat{\Omega}$-closed set is $\hat{\Omega}$-open set.*

Definition 2.4. *([6], Definition 5.1) Let A be a subset of a topological space (X,τ). Then* **$\hat{\Omega}$-closure** *of A is defined to be the intersection of all $\hat{\Omega}$-closed sets containing A and it is denoted by $\hat{\Omega}cl(A)$. That is $\hat{\Omega}cl(A) = \bigcap\{F : A \subseteq F$ and $F \in \hat{\Omega}C(X)\}$. Always $A \subseteq \hat{\Omega}cl(A)$.*

Remark 2.5. *([6], Remark 5.2) From the definition and Theorem 4.16, arbitrary intersection of $\hat{\Omega}$-closed sets in a topological space (X,τ) is $\hat{\Omega}$-closed set in (X,τ), $\hat{\Omega}cl(A)$ is the smallest $\hat{\Omega}$-closed set containing A.*

Theorem 2.6. *([6], Theorem 5.3) Let A be any subset of a topological space (X,τ). Then, A is a $\hat{\Omega}$-closed set in (X,τ) if and only if $A = \hat{\Omega}cl(A)$.*

Theorem 2.7. *([6], Theorem 5.11) In a topological space (X,τ), for $x \in X$, $x \in \hat{\Omega}cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $\hat{\Omega}$-open set U containing x.*

Definition 2.8. *[8] Let (X,τ) be a topological space. An operation γ on the topology τ is a mapping from $\tau \to P(X)$ such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \tau \to P(X)$.*

Notation 2.9.

i) *$U \in \hat{\Omega}O(X,x)$ denotes the set of all $\hat{\Omega}$-open sets in (X,τ)containing x.*

ii) *$\hat{\Omega}O(X,\tau)$ or $\hat{\Omega}O(X)$ or $\tau_{\hat{\Omega}}$ denotes the set of all $\hat{\Omega}$-open sets in a topological space (X,τ).*

iii) *The closure (res.interior,complement)of A is denoted by $cl(A)$ (res.int(A), A^c).*

iv) *$SO(X)$ denotes the set of all semi-open sets in a topological space (X,τ).*

3. Operation on $\hat{\Omega}O(X,\tau)$

Definition 3.1. *A function γ : $\hat{\Omega}O(X,\tau)\to$ P(X) is called an* **operation** *on $\hat{\Omega}O(X,\tau)$, if $U \subseteq \gamma(U)$ for every set $U \in \hat{\Omega}O(X,\tau)$.*

Remark 3.2. *For any operation $\gamma : \hat{\Omega}O(X,\tau) \to P(X), \gamma(X) = X$, and $\gamma(\emptyset) = \emptyset$.*

Definition 3.3. *A non-empty subset A of X is called* **$\gamma_{\hat{\Omega}}$-open set** *if for each $x \in A$, there exists an $\hat{\Omega}$-open set U such that $x \in U$ and $\gamma(U) \subseteq A$. The complement of $\gamma_{\hat{\Omega}}$-open set is $\gamma_{\hat{\Omega}}$-closed set. Assume that the empty set $\emptyset$ is always $\gamma_{\hat{\Omega}}$-open for any operation γ on $\hat{\Omega}O(X,\tau)$. $\tau_{\gamma_{\hat{\Omega}}}$ denotes the set of all $\gamma_{\hat{\Omega}}$-open sets on (X,τ). $\tau_{\gamma_{\hat{\Omega}}} = \{\emptyset\}\bigcup\{A/$ for each $x \in A$ there exists an $\hat{\Omega}$-open set $U \ni x \in U$ and $\gamma(U) \subseteq A.\}$*

Example 3.4. *$X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,b,c\},\{a,b,d\},X\}$, $\hat{\Omega}O(X) = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. γ is defined by $\gamma(\emptyset) = \emptyset, \gamma(\{a\}) = \{a\}, \gamma(\{b\}) = \{a,b\}, \gamma(\{a,b\}) = \{a,b,d\}$, $\gamma(X) = X$. Here, γ is an operation on $\tau_{\hat{\Omega}}$; $\{\emptyset,\{a\},\{a,b\},X\}$ are $\gamma_{\hat{\Omega}}$-open sets.*

Theorem 3.5. *Arbitrary union of $\gamma_{\hat{\Omega}}$-open sets is a $\gamma_{\hat{\Omega}}$-open set in a topological space X.*

Proof. Let $\{A_\alpha\}_{\alpha \in J}$ be any family of $\gamma_{\hat{\Omega}}$-open sets in a space (X,τ). Let $A = \bigcup_{\alpha\in J} A_\alpha$ and $x \in A$ be arbitrary. Then $x \in A_\alpha$ for some $\alpha \in J$. By the definition of $\gamma_{\hat{\Omega}}$-open, there exist $U \in \hat{\Omega}O(X,x)$ such that $\gamma(U) \subseteq A_\alpha \subseteq \bigcup_{\alpha\in J} A_\alpha = A$. Therefore, A is $\gamma_{\hat{\Omega}}$-open. □

Remark 3.6. *Arbitrary intersection of $\gamma_{\hat{\Omega}}$-closed sets is a $\gamma_{\hat{\Omega}}$-closed set in a topological space X.*

Example 3.7. *The intersection of any two $\gamma_{\hat{\Omega}}$-open sets is not necessarily an $\gamma_{\hat{\Omega}}$-open set in (X,τ). Let $X = \{a,b,c\}$ and $P(X) = \hat{\Omega}O(X,\tau)$. Define an operation $\gamma : \hat{\Omega}O(X,\tau) \to P(X)$ as follows. For every $U \in \hat{\Omega}O(X,\tau)$*

$$\gamma(U) = \begin{cases} U & \text{for } U \neq \{a\} \\ \{a,b\} & \text{for } U = \{a\} \end{cases}$$

Here $\{a,b\}$ and $\{a,c\}$ are $\gamma_{\hat{\Omega}}$-open sets but $\{a\}$ is not a $\gamma_{\hat{\Omega}}$-open set.

Proposition 3.8. *Every $\gamma_{\hat{\Omega}}$-open set is $\hat{\Omega}$-open in a space X.*

Proof. Let A be any $\gamma_{\hat{\Omega}}$-open subset of X. Let $x \in A$ be arbitrary. Then there exists $\hat{\Omega}$-open set U_x containing x such that $U_x \subseteq \gamma(U_x) \subseteq A$. Then $\bigcup\{U_x/x \in A\} = A$. By ([6], Theorem 4.16), A is $\hat{\Omega}$-open subset of X. □

Remark 3.9. *From Example 3.7, every $\hat{\Omega}$-open is not necessarily $\gamma_{\hat{\Omega}}$-open as $\{a\} \in \hat{\Omega}O(X)$ and $\{a\} \notin \tau_{\gamma_{\hat{\Omega}}}$. It turns out to find a space in which $\hat{\Omega}O(X) = \tau_{\gamma_{\hat{\Omega}}}$.*

Definition 3.10. *A space (X,τ) with an operation γ on $\hat{\Omega}O(X,\tau)$ is called* **$\gamma_{\hat{\Omega}}$-regular** *if for each $x \in X$ and for each $U \in \hat{\Omega}O(X,x)$, there exists an $\hat{\Omega}$-open set V such that $x \in V$ and $\gamma(V) \subseteq U$.*

Example 3.11. *$X = \{a,b,c,d\}, \tau = \{\emptyset,\{b,c\},\{a,b,c\},\{b,c,d\},X\}$, $\hat{\Omega}O(X) = \{\emptyset,\{b\},\{c\},\{b,c\},X\}$. γ is defined by $\gamma(\emptyset) = \emptyset, \gamma(\{b\}) = \{b\}, \gamma(\{c\}) = \{c\}, \gamma(\{b,c\}) = \{b,c,d\}, \gamma(X) = X$. Here, the operation γ on $\tau_{\hat{\Omega}}$ is $\gamma_{\hat{\Omega}}$-regular.*

Theorem 3.12. *Let (X,τ) be a topological space and $\gamma : \hat{\Omega}O(X,\tau) \to P(X)$ be an operation on $\hat{\Omega}O(X,\tau)$. Then the following conditions are equivalent:*

i) Every $\hat{\Omega}$-open set is $\gamma_{\hat{\Omega}}$-open set.

ii) X is an $\gamma_{\hat{\Omega}}$-regular space.

iii) For every $x \in X$ and for every $U \in \hat{\Omega}O(X,x)$, there exists an $\gamma_{\hat{\Omega}}$-open set V of (X,τ) containing x such that $V \subseteq U$.

Proof. $i) \Rightarrow ii)$ Let $x \in X$ be arbitrary and $U \in \hat{\Omega}O(X,x)$. By hypothesis, there exists $V \in \hat{\Omega}O(X,x)$ such that $\gamma(V) \subseteq U$.
$ii) \Rightarrow iii)$ Let x be any point of X and $U \in \hat{\Omega}O(X,x)$. By hypothesis, there exists $\hat{\Omega}$-open set V such that $x \in V$ and $\gamma(V) \subseteq U$. Again apply hypothesis to the set V. Then, there exists $\hat{\Omega}$-open set $V_1 \in \hat{\Omega}O(X,x)$ such that $\gamma(V_1) \subseteq V$. Then, V is $\gamma_{\hat{\Omega}}$-open set containing x such that $V \subseteq U$.
$iii) \Rightarrow i)$ Let U be any $\hat{\Omega}$-open set in X and $x \in U$ be arbitrary. By hypothesis, there exists $\gamma_{\hat{\Omega}}$-open set V_x containing x such that $V_x \subseteq U$. By Theorem 3.5, $U = \bigcup\limits_{x\in U} V_x$ is $\gamma_{\hat{\Omega}}$-open. □

Definition 3.13. *Let (X,τ) be any topological space.* **an operation** *γ on $\hat{\Omega}O(X,\tau)$ is called* **$\hat{\Omega}$-open** *if for each $x \in X$ and for every $U \in \hat{\Omega}O(X,x)$, there exists an $\gamma_{\hat{\Omega}}$-open set V containing x such that $V \subseteq \gamma(U)$.*

Example 3.14. $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,b,c\},\{a,b,d\},X\}$, $\hat{\Omega}O(X) = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. *$\gamma$ is defined by* $\gamma(\emptyset)=\emptyset, \gamma(\{a\})=\{a\}, \gamma(\{b\})=\{a,b\}, \gamma(\{a,b\})=\{a,b,d\}$, $\gamma(X)=X$. *Here,the operation γ on $\tau_{\hat{\Omega}}$ is $\hat{\Omega}$-open.*

Definition 3.15. *Let (X,τ) be any topological space. An operation γ on $\hat{\Omega}O(X,\tau)$ is called* **$\hat{\Omega}$-regular** *if for each $x \in X$ and for every pair of sets $U_1,\ U_2 \in \hat{\Omega}O(X,x)$, there exists a set $V \in \hat{\Omega}O(X,x)$ such that $\gamma(V) \subseteq \gamma(U_1) \cap \gamma(U_2)$.*

Example 3.16. $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a,b\},X\}$, $\hat{\Omega}O(X) = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. *$\gamma$ is defined by* $\gamma(\emptyset)=\emptyset, \gamma(\{a\}) = \{a,b\}, \gamma(\{b\}) = \{a,b\}, \gamma(\{a,b\}) = \{a,b\}, \gamma(X) = X$. *Here, an operation γ on $\tau_{\hat{\Omega}}$ is $\hat{\Omega}$-regular.*

Proposition 3.17. *Intersection of any two $\gamma_{\hat{\Omega}}$-open sets is a $\gamma_{\hat{\Omega}}$-open in a $\hat{\Omega}$-regular operation on $\hat{\Omega}O(X,\tau)$.*

Proof. Let U and V be any two $\gamma_{\hat{\Omega}}$-open sets in X. Let $x \in U \cap V$ be any point. Then, $x \in U$ and $x \in V$. By the definition, there exists $U_1 \in \hat{\Omega}O(X,x)$ such that $\gamma(U_1) \subseteq U$. Similarly for the set V, there exists $U_2 \in \hat{\Omega}O(X,x)$ such that $\gamma(U_2) \subseteq V$. Now $\gamma(U_1)\cap\gamma(U_2) \subseteq U \cap V$. By hypothesis, there exists $\hat{\Omega}$-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1)\cap\gamma(U_2) \subseteq U \cap V$. Hence $U \cap V$ is $\gamma_{\hat{\Omega}}$-open subset of X. □

Remark 3.18. *By Proposition 3.17, the family of all $\gamma_{\hat{\Omega}}$-open sets satisfy the axioms topology provided an operation γ is a $\hat{\Omega}$-regular.*

4. Basic properties of Closures

Definition 4.1. *Let γ be an operation on $\hat{\Omega}O(X,\tau)$. then for any subset A of X, $\gamma_{\hat{\Omega}}$*-**closure** *is denoted by $\gamma_{\hat{\Omega}}Cl(A)$, defined as $\gamma_{\hat{\Omega}}Cl(A) = \bigcap\{F / A \subseteq F, X\setminus F \in \tau_{\gamma_{\hat{\Omega}}}\}$. It follows that $A \subseteq \gamma_{\hat{\Omega}}Cl(A)$, $\gamma_{\hat{\Omega}}Cl(\emptyset)=\emptyset$ and $\gamma_{\hat{\Omega}}Cl(X)=X$. Moreover, $\gamma_{\hat{\Omega}}Cl(A)$ is $\gamma_{\hat{\Omega}}$-closed as any intersection of $\gamma_{\hat{\Omega}}$-closed sets is $\gamma_{\hat{\Omega}}$-closed.*

Proposition 4.2. *A is $\gamma_{\hat{\Omega}}$-closed if and only if $\gamma_{\hat{\Omega}}Cl(A) = A$ for any subset A of X.*

Proof. It follows straight forward from the definition. □

Theorem 4.3. *Let A be any subset of a topological space (X,τ) and γ be an operation on $\hat{\Omega}O(X,\tau)$. Then, $x \in \gamma_{\hat{\Omega}}Cl(A)$ iff every $\gamma_{\hat{\Omega}}$-open set containing x meets A.*

Proof. Necessary: Let A be any subset of (X,τ). Assume that there exists $\gamma_{\hat{\Omega}}$-open set U containing x such that $U \cap A = \emptyset$. Then U^c is an $\gamma_{\hat{\Omega}}$-closed set such that $A \subseteq U^c$. By the definition of $\gamma_{\hat{\Omega}}Cl(A)$, $A \subseteq \gamma_{\hat{\Omega}}Cl(A) \subseteq U^c$. Now $x \notin U^c$ implies $x \notin \gamma_{\hat{\Omega}}Cl(A)$.
Sufficiency: Assume that $x \notin \gamma_{\hat{\Omega}}Cl(A)$. Then, there exists a $\gamma_{\hat{\Omega}}$-closed set F such that $A \subseteq F$ and $x \notin F$. Now, F^c is an $\gamma_{\hat{\Omega}}$-open set containing x such that $A \cap F^c = \emptyset$. □

Proposition 4.4. *If A and B are any two subsets of the space X, then the following statements hold.*

i) $A \subseteq B$, then $\gamma_{\hat{\Omega}}Cl(A) \subseteq \gamma_{\hat{\Omega}}Cl(B)$

ii) $\gamma_{\hat{\Omega}}Cl(A\cap B) \subseteq \gamma_{\hat{\Omega}}Cl(A) \cap \gamma_{\hat{\Omega}}Cl(B)$

iii) $\gamma_{\hat{\Omega}}Cl(A) \cup \gamma_{\hat{\Omega}}Cl(B) \subseteq \gamma_{\hat{\Omega}}Cl(A\cup B)$

iv) $\gamma_{\hat{\Omega}}Cl(\gamma_{\hat{\Omega}}Cl(A)) = \gamma_{\hat{\Omega}}Cl(A)$.

Proof.

i) If U is any $\gamma_{\hat{\Omega}}$-open subset of X containing x, then by Theorem 4.3 and hypothesis $B \cap U \neq \emptyset$. Again by Theorem 4.3, $x \in \gamma_{\hat{\Omega}}$ Cl(B)

ii) Suppose that $x \notin (\gamma_{\hat{\Omega}}Cl(A)) \cap (\gamma_{\hat{\Omega}}Cl(B))$. Then, there are two possibilities such as, either $x \notin \gamma_{\hat{\Omega}}Cl(A)$ or $x \notin \gamma_{\hat{\Omega}}Cl(B)$. If $x \notin \gamma_{\hat{\Omega}}Cl(A)$, then by Theorem 4.3,there exists $\gamma_{\hat{\Omega}}$-open set U of X containing x such that $U \cap A = \emptyset$. It follows that U does not meet $A \cap B$. Again by Theorem 4.3, $x \notin \gamma_{\hat{\Omega}}Cl(A\cap B)$. Similarly for the case $x \notin \gamma_{\hat{\Omega}}Cl(B)$.

iii) Proof is analogous to that of *ii*).

iv) If follows from proposition 4.2.

□

Definition 4.5. *For a subset A of a topological space X, $\hat{\Omega}$-closure with respect to an operation γ is denoted by $(\hat{\Omega}Cl)_\gamma(A)$ and defined as $(\hat{\Omega}Cl)_\gamma(A) = \{x \in X \ / \ \gamma(U) \cap A \neq \emptyset$ for each $U \in \hat{\Omega}O(X,x)\}$. Always, $(\hat{\Omega}Cl)_\gamma(\emptyset) = \emptyset$ and $(\hat{\Omega}Cl)_\gamma(X) = X$.*

Lemma 4.6. *$(\hat{\Omega}Cl)_\gamma(A)$ is $\hat{\Omega}$-closed set in X for any subset A of X.*

Proof. Let A be any subset of X and $F = (\hat{\Omega}Cl)_\gamma(A)$. Always, $F \subseteq (\hat{\Omega}Cl)_\gamma(F)$. If $x \notin F$, then there exists $U \in \hat{\Omega}O(X,x)$ such that $\gamma(U) \cap F = \emptyset$. Then, $U \cap F = \emptyset$. By Theorem 2.7, $x \notin \hat{\Omega}Cl(F)$. Therefore, $F = \hat{\Omega}Cl(F)$. By Theorem 2.6, F is $\hat{\Omega}$-closed. □

Lemma 4.7. *In a topological space (X,τ) with an operation γ on $\hat{\Omega}O(X,\tau)$, the following statements hold for any two subsets A and B of X.*

i) $A \subseteq (\hat{\Omega}Cl)_\gamma(A) \subseteq \gamma_{\hat{\Omega}}Cl(A)$.

ii) *A is $\gamma_{\hat{\Omega}}$-closed if and only if $(\hat{\Omega}Cl)_\gamma(A) = A$.*

iii) *If $A \subseteq B$, $(\hat{\Omega}Cl)_\gamma(A) \subseteq (\hat{\Omega}Cl)_\gamma(B)$.*

iv) $(\hat{\Omega}Cl)_\gamma(A \cap B) \subseteq (\hat{\Omega}Cl)_\gamma(A) \cap (\hat{\Omega}Cl)_\gamma(B)$.

v) $(\hat{\Omega}Cl)_\gamma(A) \cup (\hat{\Omega}Cl)_\gamma(B) \subseteq (\hat{\Omega}Cl)_\gamma(A \cup B)$.

Proof. *i)* Let $x \in A$ be arbitrary. If $U \in \hat{\Omega}O(X,x)$, then by [Theorem 5.11] $U \cap A \neq \emptyset$. Thus $\gamma(U) \cap A \neq \emptyset$. By the definition, $x \in (\hat{\Omega}Cl)_\gamma$(A). For another part, assume that $x \notin \gamma_{\hat{\Omega}}Cl(A)$. Then there exists $\gamma_{\hat{\Omega}}$-closed set F such that $A \subseteq F$ and $x \notin F$. Since every $\gamma_{\hat{\Omega}}$-closed set is $\hat{\Omega}$-closed, F is $\hat{\Omega}$-closed subset of X. Then, $F^c \in \hat{\Omega}O(X,x)$ such that $F^c \cap A = \emptyset$. Thus, $x \notin (\hat{\Omega}Cl)_\gamma(A)$.

ii) Assume that A is $\gamma_{\hat{\Omega}}$-closed. If $x \notin A$, then $x \in A^c = U(say)$. Now U is $\gamma_{\hat{\Omega}}$-open subset of X containing x. By the definition, there exists $V \in \hat{\Omega}O(X,x)$ such that$\gamma(V) \subseteq U = A^c$. Thus $\gamma(V) \cap A = \emptyset$ says $x \notin (\hat{\Omega}Cl)_\gamma(A)$

iii) Let $x \in (\hat{\Omega}Cl)_\gamma(A)$ and Let U be any $\hat{\Omega}$-open set containing x. By hypothesis, $\gamma(U) \cap A \neq \emptyset$ and hence $\gamma(U) \cap B = \emptyset$. Thus $x \in (\hat{\Omega}Cl)_\gamma(B)$.
Therefore, $(\hat{\Omega}Cl)_\gamma(A) \subseteq (\hat{\Omega}Cl)_\gamma(B)$.

iv) Suppose that $x \notin \left((\hat{\Omega}Cl)_\gamma(A)\right) \cap \left((\hat{\Omega}Cl)_\gamma(B)\right)$. Then, there are two possibilities such as, either $x \notin (\hat{\Omega}Cl)_\gamma(A)$ or $x \notin (\hat{\Omega}Cl)_\gamma(B)$. If $x \notin (\hat{\Omega}Cl)_\gamma(A)$, then there exists $\hat{\Omega}$-open set U of X containing x such that $\gamma(U) \cap A = \emptyset$. It follows that $\gamma(U)$ does not meet $A \cap B$. By the definition, $x \notin (\hat{\Omega}Cl)_\gamma(A \cap B)$. Similarly for the case $x \notin (\hat{\Omega}Cl)_\gamma(B)$.

v) Proof is analogous to that of *iv*).

□

Theorem 4.8. *If γ is an $\hat{\Omega}$-regular operation on $\hat{\Omega}O(X,\tau)$, then for any two subsets A, B of X the following results hold.*

i) $\gamma_{\hat{\Omega}}Cl(A) \cup \gamma_{\hat{\Omega}}Cl(B) = \gamma_{\hat{\Omega}}Cl(A \cup B)$.

ii) $(\hat{\Omega}Cl)_\gamma(A) \cup (\hat{\Omega}Cl)_\gamma(B) = (\hat{\Omega}Cl)_\gamma(A \cup B)$

Proof. *i)* Always $\gamma_{\hat{\Omega}}Cl(A) \cup \gamma_{\hat{\Omega}}Cl(B) \subseteq \gamma_{\hat{\Omega}}Cl(A \cup B)$. For other inclusion, if $x \notin \gamma_{\hat{\Omega}}Cl$(A) $\cup$ $\gamma_{\hat{\Omega}}Cl$(B), then there exist $\gamma_{\hat{\Omega}}$-open sets U and V containing x such that $A \cap U = \emptyset$ and $B \cap V = \emptyset$. By proposition 3.17, $U \cap V$ is $\gamma_{\hat{\Omega}}$-open set in X such that $(U \cap V) \cap (A \cup B) = \emptyset$. Therefore, $x \notin \gamma_{\hat{\Omega}}Cl(A \cup B)$.

ii) Always $(\hat{\Omega}Cl)_\gamma$(A) $\cup$ $(\hat{\Omega}Cl)_\gamma$(B) $\subseteq (\hat{\Omega}Cl)_\gamma(A \cup B)$. For other inclusion, if $x \notin (\hat{\Omega}Cl)_\gamma$(A) $\cup$ $(\hat{\Omega}Cl)_\gamma$(B), then $x \notin (\hat{\Omega}Cl)_\gamma$(A) and $x \notin (\hat{\Omega}Cl)_\gamma$(B). By the definition, $\gamma(U_1) \cap A = \emptyset = \gamma(U_2) \cap B$ for some $U_1, U_2 \in \hat{\Omega}O(X,x)$. By the definition of $\hat{\Omega}$-regular operation, there exists $V \in \hat{\Omega}O(X,x)$ such that $\gamma(V) \subseteq \gamma(U_1) \cap \gamma(U_2) \subseteq (A \cup B)$. Then, $(A \cup B) \cap \gamma(V) = \emptyset$ implies $x \notin (\hat{\Omega}Cl)_\gamma(A \cup B)$.

□

Theorem 4.9. *Let A be any subset of a topological space (X,τ). If γ is an $\hat{\Omega}$-open operation on $\hat{\Omega}O(X,\tau)$, then the following statements are true.*

i) $(\hat{\Omega}Cl)_\gamma(A) = \gamma_{\hat{\Omega}}Cl(A)$

ii) $(\hat{\Omega}Cl)_\gamma\left((\hat{\Omega}Cl)_\gamma(A)\right) = (\hat{\Omega}Cl)_\gamma(A)$

iii) $(\hat{\Omega}Cl)_\gamma(A)$ *is $\gamma_{\hat{\Omega}}$-closed set in X.*

Proof. *i)* Always $(\hat{\Omega}Cl)_\gamma$(A)$\subseteq \gamma_{\hat{\Omega}}Cl(A)$. Let $x \notin (\hat{\Omega}Cl)_\gamma$(A). Then there exists an $U \in \hat{\Omega}O(X,x)$ such that $\gamma(U) \cap A = \emptyset$. By the choice of γ, there exists an $\gamma_{\hat{\Omega}}$-open set V containing x such that $V \subseteq \gamma(U)$. Now $V \cap A \subseteq \gamma(U) \cap A = \emptyset$. By Theorem 4.3, $x \notin \gamma_{\hat{\Omega}}Cl(A)$. So, $\gamma_{\hat{\Omega}}Cl(A) \subseteq (\hat{\Omega}Cl)_\gamma$(A). Hence $(\hat{\Omega}Cl)_\gamma$(A) $= \gamma_{\hat{\Omega}}Cl(A)$.

ii) By *i*) and Proposition 4.4.(iv), $(\hat{\Omega}Cl)_\gamma((\hat{\Omega}Cl)_\gamma$(A)) $=$ $(\hat{\Omega}Cl)_\gamma$(A).

iii) It follows from *i*) and by the result that $\gamma_{\hat{\Omega}}Cl(A)$ is $\gamma_{\hat{\Omega}}$-closed.

□

Theorem 4.10. *Let A be any subset of a topological space (X,τ) and γ be an operation on $\hat{\Omega}O(X,\tau)$. Then the following statements are equivalent.*

i) *A is $\gamma_{\hat{\Omega}}$-open set.*

ii) $(\hat{\Omega}Cl)_\gamma\ (X \setminus A) = X \setminus A$.

iii) $\gamma_{\hat{\Omega}}Cl(X \setminus A) = X \setminus A$.

iv) *$X \setminus A$ is $\gamma_{\hat{\Omega}}$-closed set.*

Proof. It follows from the definition. □

Theorem 4.11. *Let (X,τ) be a topological space and γ be an $\hat{\Omega}$-regular operation on $\hat{\Omega}O(X,\tau)$. Then $\gamma_{\hat{\Omega}}Cl(A) \cap U \subseteq \gamma_{\hat{\Omega}}Cl(A \cap U)$ holds for every $\gamma_{\hat{\Omega}}$-open set U and every subset A of X.*

Proof. Assume that $x \in \gamma_{\hat{\Omega}}Cl(A) \cap U$ for every $\gamma_{\hat{\Omega}}$-open set U and every subset A of X. Let V be any $\gamma_{\hat{\Omega}}$-open subset of X containing x. By Proposition 3.17, $U \cap V$ is $\gamma_{\hat{\Omega}}$-open set containing x. Since, $x \in \gamma_{\hat{\Omega}}Cl(A)$, $A \cap (U \cap V) \neq \emptyset$. That is, $(A \cap U) \cap V \neq \emptyset$. By Theorem 4.3, $x \in \gamma_{\hat{\Omega}}Cl(A \cap U)$. □

5. Separation axioms

Definition 5.1. *A topological space (X,τ) with an operation γ on $\hat{\Omega}O(X,\tau)$ is called $\gamma_{\hat{\Omega}}$-T_0 if for any two points x,y in X such that $x \neq y$ there exists an $U \in \hat{\Omega}O(X,\tau)$, such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.*

Definition 5.2. *A topological space (X,τ) with an operation γ on $\hat{\Omega}O(X,\tau)$ is called $\gamma_{\hat{\Omega}}$-T_1 if for any two points x,y in X such that $x \neq y$, there exist two $\hat{\Omega}$-open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.*

Definition 5.3. *A topological space (X,τ) with an operation γ on $\hat{\Omega}O(X,\tau)$ is called $\gamma_{\hat{\Omega}}$-T_2 if for any two points x,y in X such that $x \neq y$ there exist two $\hat{\Omega}$-open sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \emptyset$.*

Theorem 5.4. *Let γ be an $\hat{\Omega}$-open operation on $\hat{\Omega}O(X,\tau)$. Then (X,τ) is an $\gamma_{\hat{\Omega}}$-T_0 space iff $(\hat{\Omega}Cl)_\gamma(\{x\}) \neq (\hat{\Omega}Cl)_\gamma(\{y\})$ for every pair x,y of X with $x \neq y$.*

Proof. Let x,y be any two distinct points of an $\gamma_{\hat{\Omega}}$-T_0 space (X,τ). Then,there exists a $\gamma_{\hat{\Omega}}$-open set U such that $x \in U$ and $y \notin \gamma(U)$. Since γ is an $\hat{\Omega}$-open, there exists a $\gamma_{\hat{\Omega}}$-open set V such that $x \in V$ and $V \subseteq \gamma(U)$. Therefore, $y \in X \setminus \gamma(U) \subseteq X \setminus V$. Now $X \setminus V$ is an $\gamma_{\hat{\Omega}}$-closed set in (X,τ) such that $(\hat{\Omega}Cl)_\gamma(\{y\}) \subseteq X \setminus V$. Thus $(\hat{\Omega}Cl)_\gamma(\{x\}) \neq (\hat{\Omega}Cl)_\gamma(\{y\})$.
Conversely,if x,y are any two distinct points of X then, $(\hat{\Omega}Cl)_\gamma(\{x\}) \neq (\hat{\Omega}Cl)_\gamma(\{y\})$. Choose $z \in X$ such that $z \in (\hat{\Omega}Cl)_\gamma(\{x\})$, and $z \notin (\hat{\Omega}Cl)_\gamma(\{y\})$. If $x \in (\hat{\Omega}Cl)_\gamma(\{y\})$, then $(\hat{\Omega}Cl)_\gamma(\{x\}) \subseteq (\hat{\Omega}Cl)_\gamma(\{y\})$. That is, $z \in (\hat{\Omega}Cl)_\gamma(\{y\})$, which is a contradiction. So, $x \notin (\hat{\Omega}Cl)_\gamma(\{y\})$. Then, there exists an $\hat{\Omega}$-open set U containing x such that $\gamma(U) \cap \{y\} = \emptyset$. Now, $x \in U$ and $y \notin \gamma(U)$ that satisfy the condition of $\gamma_{\hat{\Omega}}$-T_0 space. □

Theorem 5.5. *The space (X,τ) is $\gamma_{\hat{\Omega}}$-T_1 if and only if for every point $x \in X$, $\{x\}$ is an $\gamma_{\hat{\Omega}}$-closed set.*

Proof. Let (X,τ) be a $\gamma_{\hat{\Omega}}$-T_1 space and x be any point of X. Then for any point $y \in X$ such that $x \neq y$, there exists an $\hat{\Omega}$-open set V_y such that $y \in V_y$ but $x \notin \gamma(V_y)$. Thus, $y \in \gamma(V_y) \subseteq X \setminus \{x\}$. This implies that $X \setminus \{x\} = \bigcup\{\gamma(V_y) : y \in X \setminus \{x\}\}$. Now $X \setminus \{x\}$ is $\gamma_{\hat{\Omega}}$-open set in (X,τ) and hence $\{x\}$ is $\gamma_{\hat{\Omega}}$-closed set in (X,τ).
Conversely, let $x,y \in X$ such that $x \neq y$. By hypothesis, we get $X \setminus \{y\}$ and $X \setminus \{x\}$ are $\gamma_{\hat{\Omega}}$-open sets such that $x \in X \setminus \{y\} = U$ (say) and $y \in X \setminus \{x\} = V$ (say). Therefore, there exist $\hat{\Omega}$-open sets U and V such that $x \in U, y \in V, \gamma(U) \subseteq X \setminus \{y\}$ and $\gamma(V) \subseteq X \setminus \{x\}$. So, $y \notin \gamma(U)$ and $x \notin \gamma(V)$. This implies that (X,τ) is $\gamma_{\hat{\Omega}}$-T_1. □

Theorem 5.6. *For any topological space (X,τ) and any operation γ on $\tau_{\hat{\Omega}}$, the following properties hold.*

i) Every $\gamma_{\hat{\Omega}}$-T_2 space is $\gamma_{\hat{\Omega}}$-T_1.

ii) Every $\gamma_{\hat{\Omega}}$-T_1 space is $\gamma_{\hat{\Omega}}$-T_0.

Proof. It follows from definitions. □

6. Conclusion

In this paper, an attempt has been made to define operation on the class of $\hat{\Omega}$-open sets. with the help of this operation,the new class of $\gamma_{\hat{\Omega}}$-open sets has been introduced and two kinds of closures such as, $\gamma_{\hat{\Omega}}Cl$ and $(\hat{\Omega}Cl)_\gamma$ studied. Their basic properties have been derived. Moreover, it is shown by an example that intersection of any two $\gamma_{\hat{\Omega}}$-closed sets is not necessarily a $\gamma_{\hat{\Omega}}$-closed but that holds in a $\hat{\Omega}$-regular operation on $\hat{\Omega}O(X,\tau)$ has been derived.

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