



Haar wavelet matrices for the numerical solution of system of ordinary differential equations

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Abstract

In this paper, the numerical solution of the system of ordinary differential equations by Haar wavelet method is presented. The interest is on solving the problem using the Haar wavelet basis due to its simplicity and efficiency in numerical approximations. The approach of Haar wavelet method for the numerical solution of system of equations is mentioned and the obtained numerical solution has been compared with exact solution. Also, the numerical results are presented for demonstrating the validity and applicability of the Haar wavelet method.

Keywords

System of differential equations, Haar wavelet method, numerical solution.

AMS Subject Classification

76D08, 65T60, 35A24, 65L10.

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1. Introduction

Nowadays, various fields of Science and engineering problems are analysed, understood by the help of mathematical tool called Differential Equations(DE). Due to the application of DE, in diverse fields. Many Researchers are attracted towards finding the solution of DE. In recent years, Wavelet based Numerical techniques has becoming the one of the most important and popular methods arising the diverse techniques of Numerical approximations, Because of the translation and dilation property of Wavelets. C.F Chen and H.C Haiao[4] have given useful contribution on solving system of Wavelet. Author[10-12] applied for the Haar wavelet in solved Partial differential equations and DE. According the lepik, The higher order derivative present in DE is approximated by Haar Wavelet. Now, by using the integration of Haar wavelets we

get the system of algebraic equations with unknown wavelet coefficients. When these systems are solved, we obtained the coefficients of some Haar wavelet series and this series gives us the wavelet solution. Lapik also shows the approximation is well good already with some coefficients. Further, author in some more papers given as the solution of integro-differential equations(IDE) and integral equations(IE) by using this method which is based on the operational matrices defined by him and previously by Chen-Hsiao. G. Hariharan et. al. [9] have given easy method for the numerical solution of partial differential equations(PDE). Vedat Suat Erturk and Shather Momani [16] have solving system of fractional differential equations(FDE) using differential transform method. Sachin Bhalekar and Varsha Daftardar – Gejji [15] given a system of nonlinear fractional equations with using new iterative method.

In the present paper, We introduced on solving system as linear differential equations by using a new direct computational methods we are presented. In this method, Higher order derivative present in the system is approximated by Haar wavelet in order to reduce the problems to set of algebraic equations with unknown wavelet coefficients[14]. The linear differential equations(LDE) are play main role in modeling numerous problems in Biology, Chemistry, physics and science of Engineering[2, 5]. Diverse of problems modelled system of LDE, Fractional differential equations, integral equations and PDE.

Several numerical techniques are used in order to find the solution of LDE do not have their exact solutions. Few of the numerical techniques are homotopy perturbation method[13], Adomain decomposition method[1], Variational iteration method have be used solving these problems, C. cattani [3]find the solution non linear PDE. These numerical techniques have their own limitations, In order to overcome those limitations we apply the Haar wavelet method to find the solution of LDE[6,7]. Since, This method has main advantage that we don't need to solve manually it is fully computer supported.

In this article, Haar wavelet operational matrix method is discussed in order to understand the physical behavior of LDE with initial conditions and the obtained solutions are compared with the exact solutions are shown in the plotted figure.

2. Haar Wavelet

In 1909, Alfred Haar proposed the first known wavelet called Haar wavelet[8], haar wavelet is piecewise constant "square-shaped" function in $t \in [0, 1)$.

$$\psi(t) = \begin{cases} 1, & \text{for } 0 \leq t < \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \quad (2.1)$$

The Haar Wavelet family, $t \in [0, 1)$ is defined as

$$h_i(t) = \begin{cases} 1, & \text{for } t \in [\alpha, \beta) \\ -1, & \text{for } t \in [\beta, \gamma) \\ 0, & \text{elsewhere} \end{cases} \quad (2.2)$$

This is also known as mother wavelet. Where,

$$\alpha = \frac{k}{m} \quad \beta = \frac{k+0.5}{m} \quad \text{and} \quad \gamma = \frac{k+1}{m}. \quad (2.3)$$

Where $m = 2^j$; $j = 0, 1, 2, 3, \dots, J$; $k = 0, 1, 2, \dots, m - 1$ and m, k are the integers, J indicates the maximum level resolution, and j is the resolution of wavelet, integr k is translation parameter, The index i in Eq. (2.2) is calculated using this formula $i = m + k + 1$. For $i = 1$, the function $h_1(x)$ is the scaling function, The family of Haar wavelet is defin as

$$h_1(t) = \begin{cases} 1 & \text{for } t \in [0, 1) \\ 0 & \text{elsewhere} \end{cases}$$

Introducing notations are

$$p_{i, 1}(t) = \int_0^t h_i(t) dt \quad (2.4)$$

$$p_{i, v}(t) = \int_0^t h_{i, v-1}(t) dt \quad v = 2, 3. \quad (2.5)$$

Integrals can be evaluated by using Eq. (2.4) and the first one of them are given by

$$p_{i, 1}(t) = \begin{cases} t - \alpha, & \text{for } t \in [\alpha, \beta) \\ \gamma - t, & \text{for } t \in [\beta, \gamma) \\ 0, & \text{elsewhere} \end{cases} \quad (2.6)$$

$$p_{i, 2}(t) = \begin{cases} \frac{1}{2}(t - \alpha)^2 & \text{for } t \in [\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - t)^2 & \text{for } t \in [\beta, \gamma), \\ \frac{1}{4m^2} & \text{for } t \in [\gamma, 1), \\ 0 & \text{elsewhere.} \end{cases} \quad (2.7)$$

Any function $y(t) \in L^2([0, 1])$ can be expanded in Haar series

$$y(t) = \sum_{i=1}^8 a_i h_i(t), \quad (2.8)$$

where a_i , $i = 1, 2, \dots$ is the Haar co-efficient which is given by

$$a_i = 2^j \int_0^1 y(t) h_i(t) dt$$

Now, we can evaluate Haar wavelet coefficients so that integral square error can be reduce, and which is defined as follows

$$E = \int_0^1 \left[y(t) - \sum_{i=1}^m a_i h_i(t) \right]^2 dt, \quad m = 2^j, \quad j \in \{0\} \cup \mathbb{N}$$

If we terminate the the series expansion of $y(t)$ we get

$$y(t) \approx \sum_{i=1}^m a_i h_i(t) = a^T H$$

Where $m = 2^j$,

$$a^T = [a_1, a_2, a_3, \dots, a_m]$$

$$H = \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_m^T \end{bmatrix} = \begin{bmatrix} h_{1,0} & h_{1,1} & \cdots & h_{1,m} \\ h_{2,0} & h_{2,0} & \cdots & h_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m,0} & h_{m,1} & \cdots & h_{m,m} \end{bmatrix}$$

Where, $h_1^T, h_2^T, \dots, h_m^T$, indicates the Haar wavelet's discrete forms by the equation (3.4).

3. Haar Wavelet Matrices

The function $y(t)$ approximation can be made using Haar function,

$$y(t) = \sum_{i=1}^{2m} a_i h_i(t) \quad (3.1)$$



By definition of family of Haar Wavelet

$$h_i(t) = \begin{cases} 1, & \text{for } t \in [\alpha, \beta) \\ -1, & \text{for } t \in [\beta, \gamma) \\ 0, & \text{elsewhere} \end{cases} \quad (3.2)$$

We are considered the collocation points

$$x_j = \frac{j-0.5}{2M}, j = 1, 2, 3, \dots, 2M \quad (3.3)$$

The scaling function of Haar wavelet is given by

$$h_1(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

To express the general notation of Haar wavelet matrix as

$$H_m = [h_m(1/2m), h_m(3/2m), \dots, h_m(2m - 1/2m)].$$

Thus we have

$$H_1 = (1), \quad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

Similarly,

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad (3.4)$$

on Integrating the family of Haar Wavelet over the vector $(0, t)$ is given by

$$p_i(t) = \begin{cases} t - \alpha, & \text{for } t \in [\alpha, \beta) \\ \gamma - t, & \text{for } t \in [\beta, \gamma) \\ 0, & \text{elsewhere} \end{cases} \quad (3.5)$$

We are considered the collocation points from of the equation (3.3) in equation (3.5), Operational matrix p_m obtain the values are

$$p_1 = \left(\frac{1}{2}\right), p_2 = \frac{1}{4} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

$$p_4 = \frac{1}{16} \begin{pmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

$$p_8 = \begin{pmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.6)$$

4. Haar wavelet method

Suppose, consider the system LDE

$$\begin{aligned} D^{\alpha_1} y_1(t) &= f_1(t, y_1, y_2, \dots, y_n) \\ D^{\alpha_2} y_2(t) &= f_2(t, y_1, y_2, \dots, y_n) \\ &\vdots \\ &\vdots \\ D^{\alpha_n} y_n(t) &= f_n(t, y_1, y_2, \dots, y_n) \end{aligned}$$

Where D^{α_j} is the derivative y_i of order α_j , $0 < \alpha_j \leq 1$, subject to the initial conditions

$$y_1(0) = c_1, \quad y_2(0) = c_2, \quad y_3(0) = c_3, \dots, y_n(0) = c_n$$

Suppose,

$$D_j^\alpha y_n(t) = \sum_{i=1}^m (a_i)_n h_i(t), \quad j = 1, 2, \dots, \quad n = 1, 2, \dots \quad (4.1)$$

After integrating above system of linear differential equations from 0 to t with respect to t, We have

$$y_n(t) = \sum_{i=1}^m (a_i)_n P_{i,1}(t) + c_j, \quad n = 1, 2, \dots \quad (4.2)$$

now we obtain the algebraic form of system of equation by introducing (4.1) and (4.2) in LDE. By solving this system we find the Haar coefficients $(a_i)_j$. from the equation (4.2) then we can find the approximate value of $y_n(t)$, $n = 1, 2, \dots$

5. Illustrative Example

In this section, we consider the system of LDE in order to apply the solution procedure as mentioned in the previous section. We solve the system of ordinary differential equations with initial conditions

$$y_1'(t) = y_3(t) - \cos t, \quad y_2(0) = 0, \quad (5.1)$$

$$y_2'(t) = y_3(t) - e^t, \quad y_2(0) = 0, \quad (5.2)$$

$$y_3'(t) = y_1(t) - y_2(t), \quad y_3(0) = 2, \quad (5.3)$$

Exact solution of the above system is $y_1(t) = e^t$, $y_2(t) = \sin t$, $y_3(t) = e^t + \cos t$

Suppose,

$$y_1'(t) = \sum_{i=1}^m (a_i)_1 h_i(t) = a_1^T H \quad (5.4)$$

$$y_2'(t) = \sum_{i=1}^m (a_i)_2 h_i(t) = a_2^T H \quad (5.5)$$

$$y_3'(t) = \sum_{i=1}^m (a_i)_3 h_i(t) = a_3^T H \quad (5.6)$$



Now we obtain the following equation by integrating the equations (5.4)-(5.6) from 0 to t with respect to t.

$$y_1(t) = \sum_{i=1}^m (a_i)_1 P_{i,1}(t) + 1 = a_1^T P_1 + 1 \quad (5.7)$$

$$y_2(t) = \sum_{i=1}^m (a_i)_2 P_{i,1}(t) = a_2^T P_1 \quad (5.8)$$

$$y_3(t) = \sum_{i=1}^m (a_i)_3 P_{i,1}(t) + 2 = a_3^T P_1 + 2 \quad (5.9)$$

Now substitute values from equations (5.4)-(5.9) in equations (5.1)-(5.3) then we get

$$a_1^T H - a_3^T P_1 - 2 + \cos t = 0 \quad (5.10)$$

$$a_2^T H - a_3^T P_1 - 2 + e^t = 0 \quad (5.11)$$

$$a_3^T H - a_1^T P_1 - 1 + a_2^T P_1 = 0 \quad (5.12)$$

We found after simplification of this equation (5.10)-(5.11) we get the result

$$a_1^T = a_3^T P_1 H^{-1} + F H^{-1} \quad (5.13)$$

$$a_2^T = a_3^T P_1 H^{-1} + G H^{-1} \quad (5.14)$$

Now from equations (5.12)-(5.14)

$$a_3^T = F H^{-1} P_1 H^{-1} - G H^{-1} P_1 H^{-1} + H^{-1} \quad (5.15)$$

Here G and F both of order $1 \times m$ represents the discrete values of $2 - e^t$ and $2 - \cos t$ respectively. with the help of equations (5.13)-(5.15) we can find the value of a_1^T , a_2^T and a_3^T . Then from the equations (5.7)-(5.9) we get the approximate value of $y_1(t)$, $y_2(t)$ and $y_3(t)$. Then find and substitute the values of a_i then we have $y_1(t)$, $y_2(t)$ and $y_3(t)$. after substituting the values, we obtain numerical solution then we compare with the exact solution is shown in the figure. The obtained solution quickly converges to exact solution as the number of collocation points increases.

6. Conclusion

In the present investigation, we discuss the operational matrices of Haar wavelet in order to find the numerical solution of system of linear ordinary differential equations with their initial conditions. also, we plotted the Haar wavelet solution and the exact solutions of the problem. From the obtained solution we conclude that the proposed method gives the accurate solution when the number of collocation points are increased. Hence the proposed method is very convenient to solve the system of linear differential equations arisen in the diverse fields of science and engineering.

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