MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(03)**(2023), 303–311.
http://doi.org/10.26637/mjm1103/007

θ_f-Approximations via fuzzy proximity relations: Semigroups in digital images

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***R*eceived 24 September 2021; *A*ccepted 11 June 2023**

Abstract. This article introduces θ_f-approximations of sets in fuzzy proximal relator space where $\theta \in [0,1)$. θ_f-approximation provides a more sensitive approach for the upper approximations of subsets or subimages. θ_f-approximation of a subimage are given with an example in digital images. Furthermore, θ_f-approximately groupoid and semigroup in fuzzy proximal relator space are introduced.

AMS Subject Classifications: 08A05, 68Q32, 54E05.

Keywords: Proximity space, relator space, descriptive approximation, approximately semigroup.

Contents

1. Introduction

Efremovič discovered the proximity spaces in 1951 [2]. He defined proximity space using with proximity relation for proximity of arbitrary subsets of a set. In [10], one can find a list of publications on proximity spaces. A proximity measure is a measure of the closeness or nearness between two nonempty sets.

A relator is a set of binary relations on a nonempty set X that is denoted by $\mathcal{R}$. A relator space is defined as the pair $(X, \mathcal{R})$. In 2016, Peters introduced the concept of proximal relator space $(X, \mathcal{R}_\delta)$ where $\mathcal{R}_\delta$ is a family of proximity relations on X [14].

Zadeh defined fuzzy sets in 1965, which he interpreted as a generalization of set. A fuzzy set A in a universe X is a mapping $A : X \to [0,1]$ [23]. For some applications of fuzzy sets please see [16, 17, 19]. Fuzzy similarity measure between fuzzy sets are given in [22]. Fuzzy similarity measure between sets using with fuzzy proximity relation $\mu_{\mathcal{R}}$ and fuzzy proximal relator space $(X, \mu_{\mathcal{R}})$ are introduced in [11]. Studies in the field of algebraic topology were also discussed with a different perspective on these issues, and semitopological δ-groups were published in 2023 [7].

Fuzzy similarity measures and fuzzy proximity relations are useful tools for applications in the applied sciences such as digital image processing and computer vision. A digital image endowed with fuzzy proximity

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relation $\mu_{\mathcal{R}}$ is fuzzy proximal relator space. Therefore one can working on pixels in digital images to obtain classifications or approximations.

Let $A \subseteq X$. A descriptively upper approximation of A is defined with

$$\Phi^* A = \{x \in X \mid x\delta_\Phi A\}.$$

It means, descriptively upper approximation of A consists of all elements in X that only have exactly the same properties as elements in A. But sometimes more sensitive calculations may be needed.

For more sensitive approach to compute upper approximations of subsets, we can also consider elements with somewhat similar properties, even if they do not have the same properties. To do this, the concept of fuzzy set is one of the most effective mathematical tool. Therefore, θ_f-approximations of sets in fuzzy proximal relator space are developed. Main advantages of this study is that it effectively uses the concepts of fuzzy sets, proximity relations and upper approximation of sets together.

In section 2, definitions of Efremovič proximity, set description, descriptively near sets, descriptively upper approximation of sets, fuzzy proximity relation and fuzzy proximal relator space are given.

In section 3, θ_f-approximations of sets in fuzzy proximal relator space are introduced, where $\theta \in [0,1)$. θ_f-approximation provides a more sensitive approach for the upper approximations of subsets or subimages. θ_f-approximation of a subimage are given with an example in digital images. Furthermore, θ_f-approximately groupoid and semigroup in fuzzy proximal relator space are introduced.

2. Preliminaries

Definition 2.1. *[2, 3] Let X be a nonempty set and δ be a relation on $P(X)$. δ is called an Efremovič proximity that satisfy following axioms:*

(A_1) *$A \ \delta \ B$ implies $B \ \delta \ A$,*
(A_2) *$A \ \delta \ B$ implies $A \neq \emptyset$ and $B \neq \emptyset$,*
(A_3) *$A \cap B \neq \emptyset$ implies $A \ \delta \ B$,*
(A_4) *$A \ \delta \ (B \cup C)$ iff $A \ \delta \ B$ or $A \ \delta \ C$,*
(A_5) *$\{x\} \ \delta \ \{y\}$ iff $x = y$,*
(A_6) *$A \ \underline{\delta} \ B$ implies $\exists E \subseteq X$ such that $A \ \underline{\delta} \ E$ and $E^c \ \underline{\delta} \ B$*
for all $A, B, C \in P(X)$ and all $x, y \in X$. Efremovič proximity relation is denoted by δ_E.

Definition 2.2. *[9] Let X be a nonempty set and δ be a relation on $P(X)$. δ is called a Lodato proximity that satisfy the axioms $(A_1) - (A_5)$ and*
(A_7) $A \ \delta \ B$ and $\{b\} \ \delta \ C$ $(\forall b \in B)$ implies $A \ \delta \ C$ for all $A, B, C \in P(X)$. Lodato proximity relation is denoted by $\delta_{\mathcal{L}}$.

Let X be a nonempty set and $\mathcal{R}$ be a set of relations on X. $\mathcal{R}$ and $(X, \mathcal{R})$ is called a relator and a relator space, respectively [20]. Let $\mathcal{R}_\delta$ be a family of proximity relations on X. Then $(X, \mathcal{R}_\delta)$ is a proximal relator space. As in [14], $\mathcal{R}_\delta$ contains proximity relations such as basic proximity $\delta_{\mathcal{B}}$ [18], Efremovič proximity δ_E [2, 3], Lodato proximity $\delta_{\mathcal{L}}$ [9], Wallman proximity δ_ω [21], descriptive proximity δ_Φ [12, 15].

In a discrete space, a non-abstract point has a location and features. Features can be measured using probe functions [8]. Let X be a nonempty set of non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$.

In this space, a function $\Phi : X \to \mathbb{R}^n$, $\Phi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$ is an object description represents a feature vector of $x \in X$ where each $\varphi_i : X \to \mathbb{R}$ is a probe function $(1 \leq i \leq n)$ that describes feature of a non-abstract point such as pixel in a digital image.

Throughout this work, nonempty set of non-abstract points X was considered. Efremovič proximity δ_E [3] and descriptive proximity δ_Φ in defining a descriptive proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$ were considered. Also, instead of the notions proximal relator space and fuzzy proximal relator space, the terms PR-space and FPR-space were used briefly, respectively.

Definition 2.3. *[10] Let Φ be an object description and $A \subseteq X$. Then the set description of A is defined as*

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.4. *[10, 13] Let $A, B \subseteq X$. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

Definition 2.5. *[12] Let $\delta_\Phi \in \mathcal{R}_{\delta_\Phi}$ and $A, B \subseteq X$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called a descriptively near B and denoted by $A\delta_\Phi B$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset$, then $A \underline{\delta}_\Phi B$ reads A is descriptively far from B.*

Definition 2.6. *[22] Let X be an universal set and $\mathcal{F}(X)$ be a class of all fuzzy sets of X. A function $\mu : \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow [0,1]$ is called a fuzzy similarity measure if the following axioms satisfy:*
(μ_1) $\mu(A, \emptyset) = 0$ and $\mu(A, A) = 1$,
(μ_2) $\mu(A, B) = \mu(B, A)$,
(μ_3) $A \subseteq B \subseteq C$ implies $\mu(A, B) \geq \mu(A, C)$ and $\mu(B, C) \geq \mu(A, C)$
for all $A, B, C \in \mathcal{F}(X)$.

Definition 2.7. *[11] Let $(X, \mathcal{R})$ be a PR-space,*

$$\begin{aligned} \mu_{\mathcal{R}} : \quad \mathcal{P}(X) \times \mathcal{P}(X) &\longrightarrow [0,1] \\ (A, B) &\longmapsto \mu_{\mathcal{R}}(A, B) \end{aligned}$$

be a fuzzy relation and $A, B \subseteq X$. Then $\mu_{\mathcal{R}}$ is called a fuzzy proximity relation if it satisfies the following axioms:
$(\mu_{\mathcal{R}})_1$ $\mu_{\mathcal{R}}(A, \emptyset) = 0$,
$(\mu_{\mathcal{R}})_2$ $\mu_{\mathcal{R}}(A, B) = \mu_{\mathcal{R}}(B, A)$,
$(\mu_{\mathcal{R}})_3$ $\mu_{\mathcal{R}}(A, B) \neq 0$ implies A is fuzzy proximal to B,
$(\mu_{\mathcal{R}})_4$ $\mu_{\mathcal{R}}(A, B \cup C) \neq 0$ implies $\mu_{\mathcal{R}}(A, B) \neq 0$ or $\mu_{\mathcal{R}}(A, C) \neq 0$
for all $A, B, C \in \mathcal{P}(X)$.
The set of all fuzzy proximity relations on $\mathcal{P}(X)$ is denoted by $\mathcal{P}_{\mu_{\mathcal{R}}}(X)$. Therefore $\mu_{\mathcal{R}}(A, B)$ is called a fuzzy proximity measure of A with B.
If $\mu_{\mathcal{R}}(A, B) > 0$, then A is fuzzy proximal to B. Also, if $\mu_{\mathcal{R}}(A, B) > \theta$, then A is θ-fuzzy proximal to B for $\theta \in (0, 1)$.

Definition 2.8. *[11] Let $(X, \mathcal{R})$ be a PR-space and $\mu_{\mathcal{R}}$ be a fuzzy proximity relation. Then $(X, \mathcal{R}, \mu_{\mathcal{R}})$ is called a FPR-space and shortly denoted by $(X, \mu_{\mathcal{R}})$.*

3. θ_f-Approximations and θ_f-Approximately Semigroups

Definition 3.1. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and $A \subseteq X$. A θ_f-approximation of A is determined with*

$$A^{\theta}_{\mu_{\mathcal{R}}} = \bigcup_{\mu_{\mathcal{R}}(A,B) > \theta} B,$$

where $B \in \mathcal{P}(X)$ and $\theta \in [0, 1)$.

For clarify the mechanism of θ_f-approximation please see Example 3.3.

Example 3.2. *Let X be a digital image and x, y be pixels of X. Probe functions $\varphi(x) = (R_x, G_x, B_x)$ and $\varphi_y = (R_y, G_y, B_y)$ are represent the RGB codes of pixels x, y. Let*

$$\begin{aligned} \mu_{\mathcal{R}} : X \times X &\longrightarrow [0,1] \\ (x, y) &\longmapsto \mu_{\mathcal{R}}(x, y) = \tfrac{|765 - D_{x,y}|}{765} \end{aligned}$$

be a fuzzy relation where

$$D_{x,y} = \sqrt{2\,\Delta R^2 + 4\,\Delta G^2 + 3\,\Delta B^2}$$

is a weighted Euclidean distance of pixels with respect to RGB such that $\Delta R = R_x - R_y$, $\Delta G = G_x - G_y$ *and* $\Delta B = B_x - B_y$. *In the definition of* $\mu_{\mathcal{R}}$, 765 *is the maximum value of* $D_{x,y}$.

Furthermore, fuzzy relationship between x *and* $y \cup z$ *means that*

$$\mu_{\mathcal{R}}(x, y \cup z) = \frac{|765 - \min\{D_{x,y}, D_{x,z}\}|}{765}$$

for all $x, y, z \in X$.

Now lets show that $\mu_{\mathcal{R}}$ *is a fuzzy proximity relation.*

$(\mu_{\mathcal{R}})_1$ *Since there is no similarity between* $x \in X$ *and* $\emptyset$, *it is clear that* $\mu_{\mathcal{R}}(x, \emptyset) = 0$.

$(\mu_{\mathcal{R}})_2$ $\mu_{\mathcal{R}}(x, y) = \mu_{\mathcal{R}}(y, x)$ *by* $D_{x,y} = D_{y,x}$ *for all* $x, y \in X$.

$(\mu_{\mathcal{R}})_3$ *Obviously* $\mu_{\mathcal{R}}(x, y) \neq 0$ *implies* A *is fuzzy proximal to* B.

$(\mu_{\mathcal{R}})_4$ *Let* $\mu_{\mathcal{R}}(x, y) = 0$ *and* $\mu_{\mathcal{R}}(x, z) = 0$ *for all* $x, y, z \in X$. *Then* $\mu_{\mathcal{R}}(x, y) = \frac{|765 - D_{x,y}|}{765} = 0$, *that is,* $D_{x,y} = 765$. *Similarly* $D_{x,z} = 765$. *Hence* $D_{x,y} = D_{x,z} = 765$ *and so* $\min\{D_{x,y}, D_{x,z}\} = 765$. *Thus* $\mu_{\mathcal{R}}(x, y \cup z) = \frac{|765 - \min\{D_{x,y}, D_{x,z}\}|}{765} = 0$. *Therefore* $\mu_{\mathcal{R}}(x, y \cup z) \neq 0$ *implies* $\mu_{\mathcal{R}}(x, y) \neq 0$ *or* $\mu_{\mathcal{R}}(x, z) \neq 0$ *for all* $x, y, z \in X$.

Consequently, $\mu_{\mathcal{R}}$ *is a fuzzy proximity relation from Definition 2.7.*

Example 3.3. *Let* X *be a digital image consists of* 16 *pixels as in Fig. 1. Also, digital image* X *endowed with fuzzy proximity relation* $\mu_{\mathcal{R}}$ *from Example 3.2 is a* FPR*-space by Definition 2.8.*

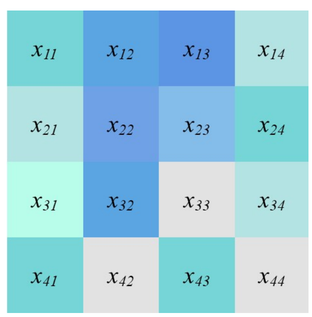

Figure 1: Digital image X

In digital image X, *a pixel* x_{ij} *is an element at position* (i, j) *(row and column). Table 1 lists the RGB codes for each pixel.*

Table 1. RGB codes for each pixel in X.

	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}
Red	117	91	91	180	180	110	132	117
Green	213	165	149	227	227	161	188	213
Blue	215	227	227	228	228	230	234	215

	x_{31}	x_{32}	x_{33}	x_{34}	x_{41}	x_{42}	x_{43}	x_{44}
Red	183	91	226	180	117	226	117	226
Green	253	165	226	227	213	226	213	226
Blue	233	227	226	228	215	226	215	226

From Example 3.2, values of fuzzy proximity relation $\mu_{\mathcal{R}}$ *are given in Table 2.*

Table 2. Values of fuzzy proximity relation $\mu_{\mathcal{R}}$.

$\mu_{\mathcal{R}}$	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	x_{41}	x_{42}	x_{43}	x_{44}
x_{11}	1	0.863	0.823	0.874	0.874	0.859	0.916	1	0.834	0.863	0.794	0.874	1	0.794	1	0.794
x_{12}	0.863	1	0.958	0.769	0.769	0.963	0.902	0.862	0.714	1	0.704	0.769	0.863	0.704	0.863	0.704
x_{13}	0.823	0.958	1	0.738	0.738	0.952	0.872	0.824	0.679	0.958	0.679	0.738	0.874	0.915	0.874	0.915
x_{14}	0.874	0.769	0.738	1	1	0.784	0.864	0.874	0.931	0.769	0.915	1	0.874	0.915	0.874	0.915
x_{21}	0.874	0.769	0.738	1	1	0.784	0.864	0.874	0.931	0.769	0.915	1	0.874	0.915	0.874	0.915
x_{22}	0.859	0.963	0.952	0.784	0.784	1	0.918	0.859	0.724	0.963	0.726	0.784	0.859	0.726	0.859	0.726
x_{23}	0.916	0.902	0.872	0.864	0.864	0.918	1	0.917	0.806	0.902	0.799	0.864	0.917	0.799	0.917	0.799
x_{24}	1	0.862	0.824	0.874	0.874	0.859	0.917	1	0.834	0.863	0.794	0.874	1	0.794	1	0.794
x_{31}	0.834	0.714	0.679	0.931	0.931	0.724	0.806	0.834	1	0.714	0.893	0.931	0.834	0.893	0.834	0.893
x_{32}	0.863	1	0.958	0.769	0.769	0.963	0.902	0.863	0.714	1	0.704	0.769	0.863	0.703	0.863	0.704
x_{33}	0.794	0.704	0.679	0.915	0.915	0.726	0.799	0.794	0.893	0.704	1	0.915	0.778	1	0.794	1
x_{34}	0.874	0.769	0.738	1	1	0.784	0.864	0.874	0.931	0.769	0.915	1	0.874	0.915	0.874	0.915
x_{41}	1	0.863	0.824	0.874	0.874	0.859	0.917	1	0.834	0.863	0.778	0.874	1	0.905	0.884	0.794
x_{42}	0.794	0.704	0.679	0.915	0.915	0.726	0.799	0.794	0.893	0.703	1	0.915	0.905	1	0.794	1
x_{43}	1	0.863	0.824	0.874	0.874	0.859	0.917	1	0.834	0.863	0.794	0.874	0.884	0.794	1	0.794
x_{44}	0.794	0.704	0.679	0.915	0.915	0.726	0.799	0.794	0.893	0.704	1	0.915	0.794	1	0.794	1

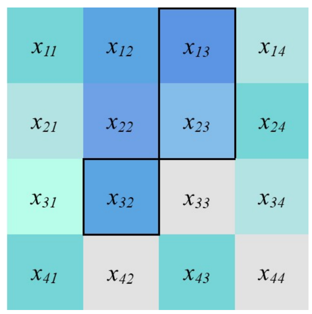

Figure 2: Subimage S

Let $S = \{x_{13}, x_{23}, x_{32}\}$ be a subimage of X as in Fig. 2 and $\theta = 0.92$. From Definition 3.1, θ_f-approximation of S is

$$S^{\theta}_{\mu_{\mathcal{R}}} = \bigcup_{\mu_{\mathcal{R}}(S,x)>\theta} x = \{x_{13}, x_{23}, x_{32}, x_{12}, x_{22}\}$$

where $x \in X$. Hence θ_f-approximation of subimage S consists of θ-fuzzy proximal pixels with S as in Fig. 3.

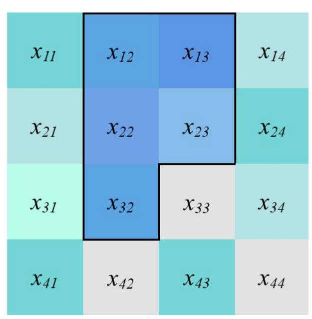

Figure 3: θ_f-approximation of subimage S

Lemma 3.4. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A \subseteq X$ and $\theta \in [0, 1)$. Then the following statements hold:*
(i) $\emptyset^{\theta}_{\mu_{\mathcal{R}}} = \emptyset$,
(ii) $A \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$,
(iii) $X^{\theta}_{\mu_{\mathcal{R}}} = X$.

Proof. It is straightforward. ∎

Theorem 3.5. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A, B \subseteq X$ and $\theta \in [0, 1)$. Then the following statements hold:*

(i) If $A \subseteq B$, then $B^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$,

(ii) $\left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}} = A^{\theta}_{\mu_{\mathcal{R}}}$,

(iii) $A^{\theta}_{\mu_{\mathcal{R}}} \cap B^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$,

(iv) $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}} \cup B^{\theta}_{\mu_{\mathcal{R}}}$.

Proof. (i) Let $A \subseteq B$ and $x \in B^{\theta}_{\mu_{\mathcal{R}}}$ where $\theta \in [0, 1)$. Then $\mu_{\mathcal{R}}(B, x) > \theta$ and hence $\mu_{\mathcal{R}}(A, x) > \theta$ since $A \subseteq B$. Thus $x \in A^{\theta}_{\mu_{\mathcal{R}}}$. Therefore $B^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$.

(ii) It is clear that $\left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$ from (i). Also, $A^{\theta}_{\mu_{\mathcal{R}}} \subseteq \left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}}$ by Lemma 3.4 (ii) and so $\left(A^{\theta}_{\mu_{\mathcal{R}}}\right)^{\theta}_{\mu_{\mathcal{R}}} = A^{\theta}_{\mu_{\mathcal{R}}}$.

(iii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (i) $A^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$ and $B^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$. Thus $A^{\theta}_{\mu_{\mathcal{R}}} \cap B^{\theta}_{\mu_{\mathcal{R}}} \subseteq (A \cap B)^{\theta}_{\mu_{\mathcal{R}}}$.

(iv) Because of $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}}$ and $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq B^{\theta}_{\mu_{\mathcal{R}}}$ from (i). Therefore $(A \cup B)^{\theta}_{\mu_{\mathcal{R}}} \subseteq A^{\theta}_{\mu_{\mathcal{R}}} \cup B^{\theta}_{\mu_{\mathcal{R}}}$. ∎

Theorem 3.6. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A_i \subseteq X$ $(i = 1, 2, \cdots, n)$, $n \in \mathbb{N}$ and $\theta \in [0, 1)$. Then the following statements hold:*

(i) $\bigcap\limits_i (A_i)^{\theta}_{\mu_{\mathcal{R}}} \subseteq \left(\bigcap\limits_i A_i\right)^{\theta}_{\mu_{\mathcal{R}}}$,

(ii) $\left(\bigcup\limits_i A_i\right)^{\theta}_{\mu_{\mathcal{R}}} \subseteq \bigcup\limits_i (A_i)^{\theta}_{\mu_{\mathcal{R}}}$.

Theorem 3.7. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A \subseteq X$ and $\theta_i \in [0, 1)$ $(i = 1, 2, \cdots, n)$, $n \in \mathbb{N}$. Then*

(i) If $\theta_1 > \theta_2$, then $A^{\theta_1}_{\mu_{\mathcal{R}}} \subseteq A^{\theta_2}_{\mu_{\mathcal{R}}}$,

(ii) If $\theta_1 > \theta_2 > \cdots > \theta_n$, then $A^{\theta_1}_{\mu_{\mathcal{R}}} \subseteq A^{\theta_2}_{\mu_{\mathcal{R}}} \subseteq \cdots \subseteq A^{\theta_n}_{\mu_{\mathcal{R}}}$.

Proof. (i) Let $\theta_1 > \theta_2$ and $x \in A^{\theta_1}_{\mu_{\mathcal{R}}}$. Then $\mu_{\mathcal{R}}(A, x) > \theta_1$ and so $\mu_{\mathcal{R}}(A, x) > \theta_2$ since $\theta_1 > \theta_2$. Hence $x \in A^{\theta_2}_{\mu_{\mathcal{R}}}$. Consequently, $A^{\theta_1}_{\mu_{\mathcal{R}}} \subseteq A^{\theta_2}_{\mu_{\mathcal{R}}}$.

(ii) It is easily obtained from (i). ∎

Theorem 3.8. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space, $A \subseteq X$, $\theta_i \in [0, 1)$, $n \in \mathbb{N}$ and $\bigwedge\limits_i \theta_i = \alpha$, $\bigvee\limits_i \theta_i = \beta$. Then*

(i) $\bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\alpha}_{\mu_{\mathcal{R}}}$,

(ii) $\bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\beta}_{\mu_{\mathcal{R}}}$.

Proof. (i) Let $x \in \bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. Then $x \in A^{\theta_i}_{\mu_{\mathcal{R}}}$ and so $\mu_{\mathcal{R}}(A, x) > \theta_i$ for at least i. Hence $\mu_{\mathcal{R}}(A, x) > \alpha$ from $\bigwedge\limits_i \theta_i = \alpha$. Thus $x \in A^{\alpha}_{\mu_{\mathcal{R}}}$. Therefore $\bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} \subseteq A^{\alpha}_{\mu_{\mathcal{R}}}$. Similarly, we can show that $A^{\alpha}_{\mu_{\mathcal{R}}} \subseteq \bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. As a results, $\bigcup\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\alpha}_{\mu_{\mathcal{R}}}$ for all i.

(ii) Let $x \in \bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. Then $x \in A^{\theta_i}_{\mu_{\mathcal{R}}}$ and so $\mu_{\mathcal{R}}(A, x) > \theta_i$ for all i. Hence $\mu_{\mathcal{R}}(A, x) > \beta$ from $\bigvee\limits_i \theta_i = \beta$. Thus $x \in A^{\beta}_{\mu_{\mathcal{R}}}$. Therefore $\bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} \subseteq A^{\beta}_{\mu_{\mathcal{R}}}$. Similarly, we can show that $A^{\beta}_{\mu_{\mathcal{R}}} \subseteq \bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}}$. Consequently, $\bigcap\limits_i A^{\theta_i}_{\mu_{\mathcal{R}}} = A^{\beta}_{\mu_{\mathcal{R}}}$ for all i. ∎

Definition 3.9. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and let "$\cdot$" be a binary operation defined on X. $G \subseteq X$ is called a θ_f-approximately groupoid in FPR-space if $x \cdot y \in G^{\theta}_{\mu_{\mathcal{R}}}$ for all $x, y \in G$.*

Let we consider G is a θ_f-approximately groupoid with the operation "$\cdot$" in $(X, \mu_\mathcal{R})$, $g \in G$ and $A, B \subseteq G$. The subsets $g \cdot A, A \cdot g, A \cdot B \subseteq G^\theta_{\mu_\mathcal{R}} \subseteq X$ are described as follows:

$$g \cdot A = gA = \{ga | a \in A\},$$

$$A \cdot g = Ag = \{ag | a \in A\},$$

$$A \cdot B = AB = \{ab | a \in A, b \in B\}.$$

Definition 3.10. *Let $(X, \mu_\mathcal{R})$ be a FPR-space, "$\cdot$" be a binary operation on X and $S \subseteq X$. S is named a θ_f-approximately semigroup in FPR-space if the conditions mentioned below are obtained:*
(1) $x \cdot y \in S^\theta_{\mu_\mathcal{R}}$,
(2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *property satisfy on* $S^\theta_{\mu_\mathcal{R}}$
for all $x, y, z \in S$.

If θ_f-approximately semigroup has approximately identity element $e_\theta \in S^\theta_{\mu_\mathcal{R}}$ such that $x \cdot e_\theta = e_\theta \cdot x = x$ for all $x \in S$, then S is called a θ_f-approximately monoid in FPR-space. If $x \cdot y = y \cdot x$ property holds in $S^\theta_{\mu_\mathcal{R}}$ for all $x, y \in S$, then S is commutative θ_f-approximately semigroup in FPR-space.

Example 3.11. *Assume X is a 16 pixel digital image, as shown in Fig. 1 and $S = \{x_{13}, x_{23}, x_{32}\}$ be a subimage of X. From Example 3.3, θ_f-approximation of S is*

$$S^\theta_{\mu_\mathcal{R}} = \{x_{13}, x_{23}, x_{32}, x_{12}, x_{22}\}$$

where $\theta = 0.92$.
Let

$$\begin{aligned} \cdot : X \times X &\longrightarrow X \\ (x_{ij}, x_{kl}) &\longmapsto x_{ij} \cdot x_{kl} = x_{pr} \end{aligned}$$

be a binary operation on X such that $p = \min\{i, k\}$ and $r = \min\{j, l\}$.
By Definition 3.10, since
(1) $x_{ij} \cdot x_{kl} \in S^\theta_{\mu_\mathcal{R}}$,
(2) $(x_{ij} \cdot x_{kl}) \cdot x_{mn} = x_{ij} \cdot (x_{kl} \cdot x_{mn})$ *property satisfy on $S^\theta_{\mu_\mathcal{R}}$ for all $x_{ij}, x_{kl}, x_{mn} \in S$*
are satisfied, S is indeed a θ_f-approximately semigroup in FPR-space $(X, \mu_\mathcal{R})$ with " $\cdot$ ".
Also, since $x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$ for all $x_{ij}, x_{kl} \in S$ property satisfies in $S^\theta_{\mu_\mathcal{R}}$, S is a commutative θ_f-approximately semigroup.

Definition 3.12. *Let $(X, \mu_\mathcal{R})$ be a FPR-space, $S \subseteq X$ be a θ_f-approximately semigroup and $T \subseteq S$ ($T \neq \emptyset$). T is called a θ_f-approximately subsemigroup if T is a θ_f-approximately semigroup with the operation in S.*

Theorem 3.13. *Let S be a θ_f-approximately semigroup and $T \subseteq S$ ($T \neq \emptyset$). If $T^\theta_{\mu_\mathcal{R}}$ is a θ_f-approximately groupoid and $T^\theta_{\mu_\mathcal{R}} \subseteq S^\theta_{\mu_\mathcal{R}}$, then T is a θ_f-approximately subsemigroup of S .*

Proof. Since $T^\theta_{\mu_\mathcal{R}}$ is a θ_f-approximately groupoid, thus $x \cdot y \in T^\theta_{\mu_\mathcal{R}}$ for all $x, y \in T$. Furthermore, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property satisfies on $T^\theta_{\mu_\mathcal{R}}$ for all $x, y, z \in T$, since S is a θ_f-approximately semigroup and $T^\theta_{\mu_\mathcal{R}} \subseteq S^\theta_{\mu_\mathcal{R}}$. Consequently, T is a θ_f-approximately subsemigroup of S. ■

Definition 3.14. *Let $(X, \mu_\mathcal{R})$ be a FPR-space, $S \subseteq X$ be a θ_f-approximately semigroup and $I \subseteq S$.*
(1) I is called a θ_f-approximately left ideal of S if $I^\theta_{\mu_\mathcal{R}}$ is a left ideal of S, i.e., $S(I^\theta_{\mu_\mathcal{R}}) \subseteq I^\theta_{\mu_\mathcal{R}}$.
(2) I is called a θ_f-approximately right ideal of S if $I^\theta_{\mu_\mathcal{R}}$ is a right ideal of S, i.e., $(I^\theta_{\mu_\mathcal{R}})S \subseteq I^\theta_{\mu_\mathcal{R}}$.
(3) I is called a θ_f-approximately bi-ideal of S if $I^\theta_{\mu_\mathcal{R}}$ is a bi-ideal of S, i.e., $(I^\theta_{\mu_\mathcal{R}})S(I^\theta_{\mu_\mathcal{R}}) \subseteq I^\theta_{\mu_\mathcal{R}}$.

Example 3.15. *In Example 3.11, let we use θ_f-approximately semigroup $S = \{x_{13}, x_{23}, x_{32}\}$. From Definition 3.14, obviously $S \subseteq S$ is a θ_f-approximately left ideal, θ_f-approximately right ideal and also θ_f-approximately bi-ideal of S.*

Theorem 3.16. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and $S \subseteq X$. If S is a semigroup in X, then S is a θ_f-approximately semigroup in FPR-space.*

Proof. Assume that $S \subseteq X$ be a semigroup. Using Lemma 3.4 (ii), $S \subseteq S^{\theta}_{\mu_{\mathcal{R}}}$ is obtained. Hence $x \cdot y \in S^{\theta}_{\mu_{\mathcal{R}}}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ condition is also accurate in $S^{\theta}_{\mu_{\mathcal{R}}}$ for all $x, y, z \in S$. After that S is a θ_f-approximately semigroup in FPR-space. ■

The Theorem 3.16 shows that θ_f-approximately semigroup is a generalization of a semigroup.

Theorem 3.17. *Let $(X, \mu_{\mathcal{R}})$ be a FPR-space and $S \subseteq X$. If I is a left (right) ideal of θ_f-approximately semigroup S and $\left(S^{\theta}_{\mu_{\mathcal{R}}}\right)\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$ ($\left(I^{\theta}_{\mu_{\mathcal{R}}}\right)\left(S^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$), then I is a θ_f-approximately left (right) ideal of S.*

Proof. Let we consider I be a left ideal of θ_f-approximately semigroup S, that is, $SI \subseteq I$. We know that $S \subseteq S^{\theta}_{\mu_{\mathcal{R}}}$. Hence, from the hypothesis $\left(S^{\theta}_{\mu_{\mathcal{R}}}\right)\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$,

$$\begin{aligned} S\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) &\subseteq \left(S^{\theta}_{\mu_{\mathcal{R}}}\right)\left(I^{\theta}_{\mu_{\mathcal{R}}}\right) \\ &\subseteq I^{\theta}_{\mu_{\mathcal{R}}}. \end{aligned}$$

As a results, $I^{\theta}_{\mu_{\mathcal{R}}}$ is a left ideal of S and so I is a θ_f-approximately left ideal of S. Also, It is obviously if I is a right ideal of θ_f*-approximately semigroup* S and $\left(I^{\theta}_{\mu_{\mathcal{R}}}\right)\left(S^{\theta}_{\mu_{\mathcal{R}}}\right) \subseteq I^{\theta}_{\mu_{\mathcal{R}}}$, I is a θ_f-approximately right ideal of S. ■

4. Conclusions

This work proposed θ_f-approximations of sets in fuzzy proximal relator space to provide more sensitive approach for approximations or clustering. From the examples and results, it was verified that θ_f-approximation is able to classify the pixels in digital images more precisely according to the selected $\theta \in [0, 1)$. Other results about θ_f-approximately algebraic structures provides a theoretical basis for further studies. Future studies should investigate the performance of this theory with experimental studies in any applied fields.

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