Malaya Journal of Matematik, Vol. 9, No. 1, 195-198, 2021
https://doi.org/10.26637/MJM0901/0033

Color class dominations sets in various classes of graphs

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Abstract
Let $G=(V,E)$ be a graph. A color class dominating set of G is a proper coloring $\mathscr{C}$ of G with the extra property that every color class in $\mathscr{C}$ is dominated by a vertex in G. A color class dominating set is said to be a minimal color class dominating set if no proper subset of $\mathscr{C}$ is a color class dominating set of G. The color class domination number of G is the minimum cardinality taken over all minimal color class dominating sets of G and is denoted by $\gamma_\chi(G)$. Here we also obtain $\gamma_\chi(G)$ for Multi-star graph, Windmill graph, Barbell graph, Lollipop graph, Complete m-partite graph, Fan graph, Crown graph and Cocktail party graph.

Keywords
Chromatic number, domination number, color class dominating set, color class domination number.

AMS Subject Classification
05C15, 05C69.

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Article History: Received **14** November **2020**; Accepted **12** January **2021**

Contents

1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [4]. Let $G=(V,E)$ be a graph of order p. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consist of the set of all vertices adjacent to v.The closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\bigcup_{v\in S} N(v)$, and the closed neighborhood of S is $N[S]=N(S)\cup S$.
A subset S of V is called a dominating set if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set is minimal dominating set if no proper subset of S is a dominating set of G. The domination number $\gamma(G)$ is the minimum cardinality taken over all minimal dominating sets of G. A γ_- set is any minimal dominating set with cardinality γ. A proper coloring of G is an assignment of colors to the vertices of G, such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. A color class dominating set of G is a proper coloring $\mathscr{C}$ of G with the extra property that every color classes in $\mathscr{C}$ is dominated by a vertex in G. A color class dominating set is said to be a minimal color class dominating set if no proper subset of $\mathscr{C}$ is a color class dominating set of G. The color class domination number of G is the minimum cardinality taken over all minimal color class dominating sets of G and is denoted by $\gamma_\chi(G)$. This concept was introduced by A. Vijayalekshmi et all [2]. The join G_1+G_2 of Graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1\cup G_2$ together with each vertex in V_1 is adjacent to every vertices in V_2. A path on n vertices denoted by P_n, is a connected graph with all but two vertices have degree 2 and $V(P_n)=\{v_i/1\le i\le n\}$ with $v_iv_{i+1}\in E(P_n)$ for $i<n$.
The Complete graph K_p has every pair of p vertices adjacent. A complete bipartite graph is a bipartite graph with disjoint vertex sets V_1 and V_2 in which every pair of vertices in the two sets are adjacent and is denoted by $K_{m,n}$. The Multi-star graph $K_{m(a_1,a_2,\ldots,a_m)}$ is formed by joining a_i end vertices to each vertex x_i of a complete graph $K_m(1\le i\le m)$ where $V(K_m)=\{x_1,x_2,\ldots,x_m\}$. The windmill graph $W_n{}^{(m)}$ is the graph obtained by taking m-copies of the complete graph K_n with a vertex in common. The n-barbell graph B_n is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge. The (m,n)-lollipop graph $L_{m,n}$ is the graph obtained by joining a complete graph K_m to a path graph P_n

with a bridge. The complete m-partite graph $K_{a_1,a_2,\ldots,a_m}$ is a simple graph whose vertices can be partitioned into m disjoint nonempty sets $V_1, V_2, \ldots, V_m$, such that each vertex in one partite set, say, V_i is adjacent to every vertices in other partite sets

$$V_1, V_2, \ldots, V_{i-1}, V_{i+1}, V_{i+2}, \ldots, V_m$$

with $|V_i| = a_i (1 \le i \le n)$. The Fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m} + P_n$ where $\overline{K_m}$ is the complement of K_m with vertex set $\{u_1, u_2, \ldots, u_m\}$ and P_n is a path with vertex set $\{v_1, v_2, \ldots, v_n\}$. The crown graph S_n^0 for an integer $n \ge 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and edge set

$$\{(u_i, v_j)/1 \le i, j \le n, i \ne j\}.$$

The cocktail party graph of order $2n$ is the graph consisting of two rows of paired paths $P_n{}^{(1)}, P_n{}^{(2)}$ such that each vertex in $P_n{}^{(1)}$ is adjacent to all vertices in $P_n{}^{(2)}$ except the corresponding paired vertex and is denoted by T_v.

2. Main Results

Theorem 2.1. *Let G be a connected graph of order p. Then $\gamma_\chi(G) = p$ if and only if $G \cong K_p$, for $p \ge 2$*

Proof. Let G be a non-complete graph with $\delta(G) > 0$. We show that $\gamma_\chi(G) < p$. Let $u_1u_2 \notin E(G)$. We consider the following two cases.

Case(1): Let u_1 and u_2 are adjacent to a same vertex u_3. Then we allot color 1 to u_1 and u_2 and colors $2, 3, \ldots, p$ - 1to the remaining $p-2$ vertices. This is clearly a γ_χ -coloring of G.

Case(2): Let u_1 and u_2 are not adjacent to a same vertex. Then there must be a path connecting u_1 and u_2 and in that path there are two non-adjacent vertices as in case (1). Proceed as in case (1), we show that $\gamma_\chi(G) < p$. The Converse is obvious. □

Proposition 2.2. *For the Complete bipartite* graph $K_{m,n}$, *m*, $n \ge 2$,

$$\gamma_\chi(K_{m,n}) = 2.$$

Theorem 2.3. *The multi-star graph $K_{m(a_1,a_2,\ldots,a_m)}$ has*

$$\gamma_\chi\left(K_{m(a_1,a_2,\ldots,a_m)}\right) = m.$$

Proof. Let $K_{m(a_1,a_2,\ldots,a_m)}$ be the multi-star graph, and let

$$V\left(K_{m(a_1,a_2,\ldots,a_m)}\right) = \left\{v_1^i \quad /1 \le i \le m\right\} \cup \left\{u_i^j / 1 \le i \le m, 1 \le j \le a_i\right\}$$

with each $v_1^i, 1 \le i \le m$, is adjacent to v_1^j for $j \ne i, 1 \le i \le m$ and $u_i^j, 1 \le j \le a_i$. For $1 \le i \le m-1$,assign color i to the vertices $\{v_1^i\} \cup \left\{u_{i+1}^j\right\}$, where $1 \le j \le a_{i+1}$ and color m to the vertices $\{v_1^m\} \cup \left\{u_1^j\right\}$, where $1 \le j \le a_1$ respectively. Then clearly, each color class $\mathscr{C}_i, 1 \le i \le m-1$ is dominated by the vertex v_1^{i+1} and the color class $\mathscr{C}_m$ is dominated by the vertex v_1^1. So

$$\gamma_\chi\left(K_{m(a_1,a_2,\ldots,a_m)}\right) = m.$$

□

Example 2.4.

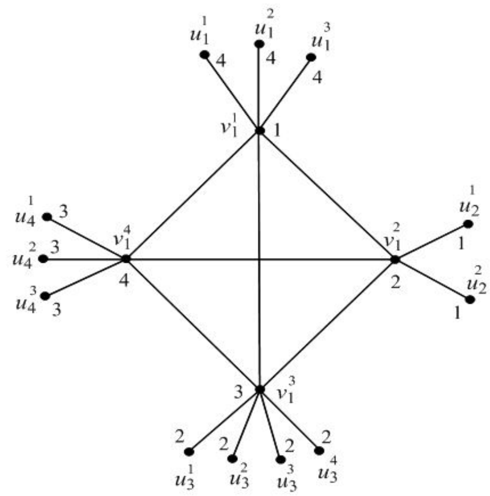

Figure 1

Theorem 2.5. *For the windmill graph $G = W_n{}^{(m)}, \gamma_\chi\left(W_n{}^{(m)}\right) = n$.*

Proof. Let $G = W_n{}^{(m)}$ be the windmill graph with $n, m \ge 3$ formed by m-copies of the complete graph K_n with $V\left(W_n^{(m)}\right) = \left\{v_{ij}/1 \le i \le m, 1 \le j \le n\right\}$ with $v_{11} = v_{21} = \cdots = v_{n1}$ is a common vertex, say, v_{11} and assign distinct colors $1, 2, 3, \ldots, n$ to the vertices $v_{11}, \{v_{i2}/1 \le i \le m\}, \{v_{i3}/1 \le i \le m\}, \ldots, \{v_{in}$ $/1 \le i \le m\}$ respectively, we get a $\gamma_\chi-$ coloring. Thus $\gamma_\chi\left(W_n^{(m)}\right) = n$. □

Example 2.6.

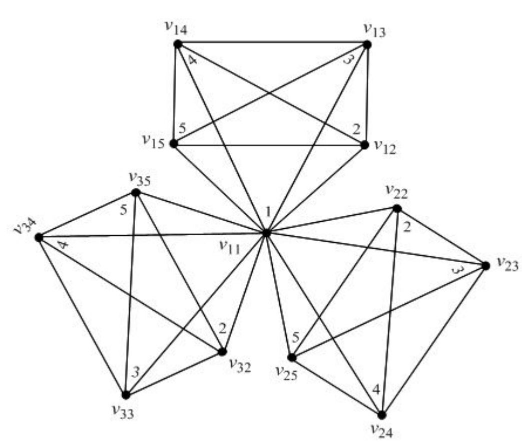

Figure 2

Theorem 2.7. *The Barbell graph B_n with $n \geq 3$ has $\gamma_\chi(B_n) = 2n-2$.*

Proof. Let B_n be the n-Barbell graph with

$$V(B_n) = \{v_1, v_2, \dots, v_n\} \cup \{v_{n+1}, v_{n+2} \dots, v_{2n}\}.$$

Assign color i $(1 \leq i \leq 2n-2)$ to the vertices $\{(v_1, v_{n+2}), v_2, v_3, \dots, v_{n-1}, (v_n, v_{n+1})\ v_{n+3}, v_{n+4}, \dots, v_{2n}\}$ respectively, we get a γ_χ -coloring of B_n. Hence $\gamma_\chi(B_n) = 2n-2$. □

Example 2.8.

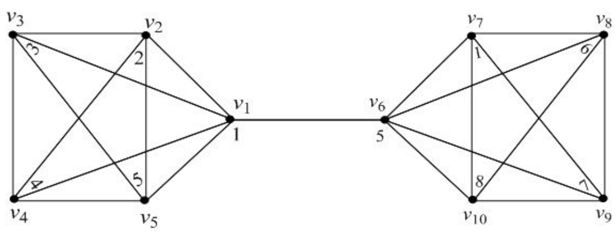

Figure 3

Theorem 2.9. *The Lollipop graph $L_{m,n}, m \geq 3$ has*

$$\gamma_\chi(L_{m,n}) = \begin{cases} m + \left(\frac{n-2}{2}\right) & \text{if } n \equiv 2(\bmod 4) \\ m + \frac{n}{2} & \text{if } n \equiv 0(\bmod 4) \\ m + \left\lfloor \frac{n-2}{2} \right\rfloor & \text{if } n \equiv 1,3(\bmod 4) \end{cases}$$

Proof. Let

$$V(L_{m,n}) = \{v_1, v_2, \dots, v_m, v_{m+1}, v_{m+2}, \dots, v_{m+n}\}$$

with $\deg(v_i) = m-1(1 \leq i \leq m-1), \deg(v_m) = m, \deg(v_i) = 2(m+1 \leq i \leq m+n-1)$ and $\deg(v_{m+n}) = 1$. Assign the colors $i(1 \leq i \leq m)$ to the vertices $\{v_i/1 \leq i \leq m-2\}, (v_{m-1}, v_{m+1})$ and $(v_m, v_{m+2}\}$ respectively. Also the induced subgraph $< v_{m+3}, v_{m+4}, \dots, v_{m+n} > \cong P_{n-2}$. So

$$\gamma_\chi(L_{m,n}) = m + \gamma_\chi(P_{n-2}) = \begin{cases} m + \left(\frac{n-2}{2}\right) & \text{if } n \equiv 2(\bmod 4) \\ m + \frac{n}{2} & \text{if } n \equiv 0(\bmod 4) \\ m + \left\lfloor \frac{n-2}{2} \right\rfloor & \text{if } n \equiv 1,3(\bmod 4) \end{cases}$$

□

Example 2.10.

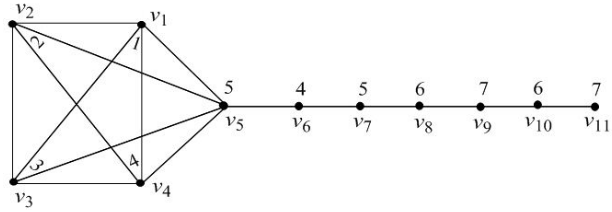

Figure 4

Theorem 2.11. *The complete m-partite graph $K_{a_1,a_2,\dots,a_m}$ has $\gamma_\chi(K_{a_1,a_2,\dots,a_m}) = m$.*

Proof. Let

$$V(K_{a_1,a_2,\dots,a_m}) = \left\{v_i^j / 1 \leq i \leq m, 1 \leq j \leq a_i\right\}.$$

Assign distinct colors $i(1 \leq i \leq m)$ to the vertices

$$\left\{v_i^j / 1 \leq i \leq m, 1 \leq j \leq a_i\right\},$$

we get a γ_χ-coloring of $K_{a_1,a_2,\dots,a_m}$. Hence $\gamma_\chi(K_{a_1,a_2,\dots,a_m}) = m$. □

Example 2.12.

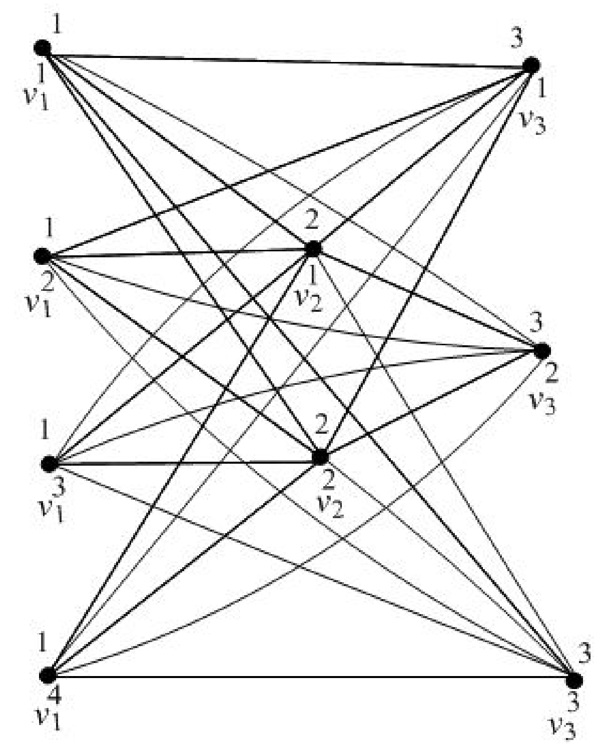

Figure 5

Theorem 2.13. *For the Fan graph $F_{m,n}$, $m \geq 1$, $n \geq 2$,*

$$\gamma_\chi(F_{m,n}) = 3.$$

Proof. Let,

$$F_{m,n} = \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_m\},$$

where $v_1, v_2, \dots, v_n$ be the vertices of the path P_n and $u_1, u_2, \dots, u_m$ be the vertices of $\overline{K_m}$ Consider a proper coloring $\mathscr{C} = \{\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_m\}$ in which $\mathscr{C}_1 = \{u_1, u_2, \dots, u_m\}$ and when n is odd, $\mathscr{C}_2 = \{v_1, v_3, \dots, v_n\}, \mathscr{C}_3 = \{v_2, v_4, \dots, v_{n-1}\}$, when n is even $\mathscr{C}_2 = \{v_1, v_3, \dots, v_{n-1}\}$ and $\mathscr{C}_3 = \{v_2, v_4, \dots, v_n\}$. Then the color class $\mathscr{C}_1$ is dominated by each vertex in the path P_n and the color classes $\mathscr{C}_2$ and $\mathscr{C}_3$ are dominated by the vertices $u_1, u_2, \ldots, u_m$. Therefore $\mathscr{C}$ is a γ_χ -coloring of $F_{m,n}$ with 3 colors and so $\gamma_\chi(F_{m,n}) = 3$. □

Example 2.14.

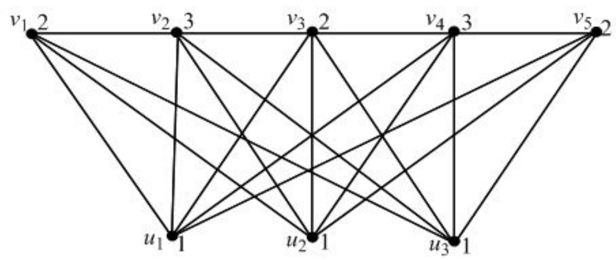

Figure 6

Theorem 2.15. *For the Crown Graph $S_n^0, n \geq 2, \gamma_\chi\left(S_n^0\right) = 4$.*

Proof. Let G be a Crown graph. Let

$$\left(S_n^0\right) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}.$$

Let $\mathscr{C}_1 = \{u_1, u_{2,} \ldots, u_{n-1}\}$, $c_2 = \{v_1, v_2, \ldots,\ v_{n-1}\}$, $c_3 = \{u_n\}$ and $\mathscr{C}_4 = \{v_n\}$ be the γ_χ-coloring of S_n^0. Because $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$ are dominated by the vertices v_n, u_n, u_n, v_n respectively. Hence $\gamma_\chi\left(S_n^0\right) = 4$. □

Example 2.16.

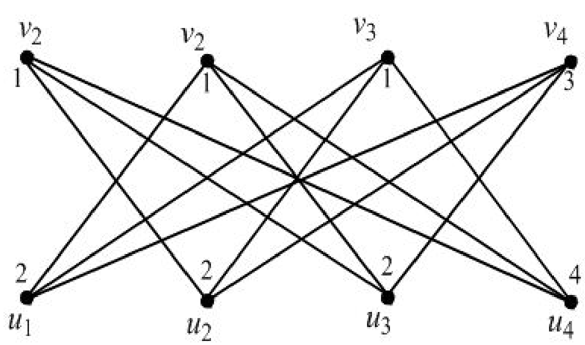

Figure 7

Theorem 2.17. *For the Cocktail party Graph T_v, $\gamma_\chi(T_v) = 4$.*

Proof. Let $V(T_v) = \left\{v_i^{(1)}, v_i^{(2)} / 1 \leq i \leq n\right\}$. We assign two distinct colors, say, 1 and 2 to the vertices $\left\{v_i^{(1)} / i = 1, 3, \ldots, n\right.$ if n is odd $\left.\right\}$ or $\left\{v_i^{(1)} / i = 1, 3, \ldots, n-1\right.$ if n is even $\left.\right\}$ and $\left\{v_i^{(1)} / i = 2, 4, \ldots, n\right.$ if n is even $\left.\right\}$ or $\left\{v_i^{(1)} / i = 2, 4, \ldots, n-1\right.$ if n is odd $\left.\right\}$ respectively. Also we assign colors 3 and 4 to the vertices $\left\{v_i^{(2)} / i = 1, 3, \ldots,\ n\right.$ if n is odd $\left.\right\}$ or $\left\{v_i^{(2)} / i = 1, 3, \ldots, n-1\right.$ if n is even $\left.\right\}$ and $\left\{v_i^{(2)} / i = 2, 4, \ldots, n\right.$ if n is even $\left.\right\}$ or $\left\{v_i^{(2)} / i = 2, 4, \ldots, n-\ 1\right.$ if n is odd $\left.\right\}$ respectively. So, $\gamma_\chi(T_v) = 4$. □

Example 2.18.

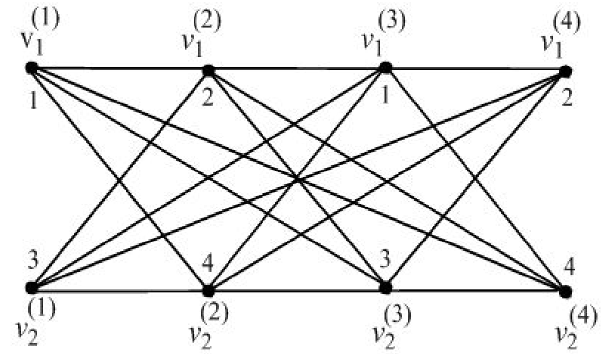

Figure 8

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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