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On fractional neutral Volterra-Fredholm integro-differential systems with non-dense domain and non-instantaneous impulses

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Abstract

The key purpose of this manuscript is to examine the existence and uniqueness of integral solutions for a class of fractional neutral Volterra-Fredholm integro-differential systems with non-instantaneous impulses and non-densely defined linear operators in Banach spaces. We are constructing the main findings on the basis of the Banach contraction theory. An example is given to support the validation of the theoretical results achieved.

Keywords

Fractional neutral differential equations, Integral solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification

34K30, 35R12, 26A33.

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1. Introduction

The principle of fractional or non-integer calculus is not a new subject. This is the generalization of the traditional integer-order calculus. Its appearance dates back to the argument between the Leibniz mathematicians and the Hospital. It has currently become a common area of inquiry in the light of its practical application, following its theoretical creation a few hundred years ago. It is a reasonable way to describe memory for specific aspects, so there are diverse disciplines of study in science and engineering, such as astronomy, biology, chemistry, economics, management of complex processes, etc. Various books and articles in this domain, see the monographs and papers [4, 8, 9].

There are various genuine issues that have some unexpected changes in their states, such abrupt changes are known as impulsive effects in the issues. In the current theory, there are two sorts of impulsive frameworks, one is the instantaneous impulsive framework and another is known as the non-instantaneous impulsive framework. In the instantaneous impulsive framework, the length of these unexpected changes is minimal in connection with the range of an entire progression measure. These quick driving forces occur in heart throbs, stuns and catastrophic events, while in the non-instantaneous impulsive system, the length of these abrupt changes continues throughout a restricted time span. Hernandez and O'Regan [3] gave the new class of differential conditions non- instantaneous impulses and set up the existence of mild and classical solutions.

Motivated by [1–3, 5–7], in this paper we consider a class of fractional neutral Volterra-Fredholm integro-differential systems with non-instantaneous impulses of the form

$$\begin{aligned} {}^cD^{\alpha}[x(t)+h(t,x(t))] &= A[x(t)+h(t,x(t))] \\ &\quad + f\left(t,x(t),\int_0^t k(t,s,x(s))ds,\int_0^T \widetilde{k}(t,s,x(s))ds\right), \\ &\qquad t\in(s_i,t_{i+1}],\ i=0,1,2,\dots,m \\ x(t) &= g_i(t,x(t)),\quad t\in(t_i,s_i],\ i=1,2,\dots,m \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in$

$(0,1)$ with the lower limit zero, $A: D(A) \subset X \to X$ not necessarily a densely defined closed operator on the Banach space $(X, \|\cdot\|), x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $h: [0,T] \times X \to X$; $g_i \in C\left((t_i, s_i] \times X; \overline{D(A)}\right)$, $f: [0,T] \times X^3 \to X$ is a nonlinear function and $k, \widetilde{k}: \Delta \times X \to X$, where $\Delta = \{(x,s): 0 \leq s \leq x \leq \tau\}$ are given functions which satisfies assumptions to be specified later on. For our convenience, we denote $E_1 x(t) = \int_0^t k(t,s,x(s))ds$ and $E_2 x(t) = \int_0^T \widetilde{k}(t,s,x(s))ds$.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence and uniqueness result of problem (1.1). An example is given in Section 4 to illustrate the results.

2. Preliminaries

In this section, we recall basic definitions of fractional calculus and integral solutions which are very useful to prove our main results.

Definition 2.1. *[4] The Riemann-Liouville fractional integral of order q with the lower limit zero for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad q > 0$$

provided the integral exists, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *[4] The Riemann-Liouville derivative of order q with the lower limit zero for a function $f: [0,\infty) \to \mathbb{R}$ can be written as*

$${}^{\mathrm{L}}D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-q-1} f(s) ds,\ n-1 < q < n, t > 0$$

Let us set $J = [0,T], J_0 = [0,t_1], J_1 = (t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, t_{m+1}]$ and introduce the space $PC(J,X) := \{u: J \to X \mid u \in C(J_k, X), k = 0,1,2,\ldots,m,$ and there exist $u\left(t_k^+\right)$ and $u\left(t_k^-\right), k = 1,2,\ldots,m,$ with $u\left(t_k^-\right) = u(t_k)\}$. It is clear that $PC(J,X)$ is a Banach space with the norm $\|u\|_{PC} = \sup\{\|u(t)\|: t \in J\}$.

Let $X_0 = \overline{D(A)}$ and A_0 be the part of A in $\overline{D(A)}$ defined by

$$D(A_0) = \left\{x_1 \in D(A): Ax_1 \in \overline{D(A)}\right\}, A_0(x_1) = A(x_1)$$

Throughout our analysis, the following hypotheses will be considered:

(H1) $A: D(A) \subset X \to X$ satisfies the Hille-Yosida condition, that is, there exist two constants $\omega \in \mathbb{R}$ and $\Lambda_0 \geq 0$ such that $(\omega, \infty) \subset \rho(A)$ and

$$\left\|(\lambda I - A)^{-n}\right\|_{L(X)} \leq \frac{\Lambda_0}{(\lambda - \omega)^n}, \text{ for all } \lambda > \omega, n \geq 1$$

(H2) The part A_0 of A generates a compact C_0 -semigroup $\{T(t)\}_{t \geq 0}$ in X_0 which is uniformly bounded, that is, there exists $\Lambda_A \geq 1$ such that $\sup_{t \in [0,\infty)} \|T(t)\| < \Lambda_A$.

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then for all $x_1 \in X_0, B_\lambda x_1 \to x_1$ as $\lambda \to \infty$. Also from Hille-Yosida condition, it is clear that $\lim_{\lambda \to \infty} \|B_\lambda\| \leq \Lambda_0$.

Definition 2.3. *[8, 9] A function $x \in C(J,X)$ is said to be a mild solution of the following problem:*

$$\begin{cases} {}^{\mathrm{c}}D^\alpha x(t) = Ax(t) + y(t), & t \in (0,T] \\ x(0) = x_0 \end{cases}$$

if it satisfies the integral equation

$$x(t) = P_\alpha(t) x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) y(s) ds.$$

Here

$$P_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T\left(t^\alpha \theta\right) d\theta,\ Q_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T\left(t^\alpha \theta\right) d\theta$$
$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \sigma_\alpha\left(\theta^{-\frac{1}{\alpha}}\right) \geq 0$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha),\ \theta \in (0,\infty)$$

and ξ_α is a probability density function defined on $(0,\infty)$, that is,

$$\xi_\alpha(\theta) \geq 0, \theta \in (0,\infty), \quad \int_0^\infty \xi_\alpha(\theta) d\theta = 1$$

It is not difficult to verify that

$$\int_0^\infty \theta \xi_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}.$$

We make the following assumption on A in the whole paper.

H(A) : The operator A generators a strongly continuous semigroup $\{T(t): t \geq 0\}$ in X, and there is a constant $\Lambda_A \geq 1$ such that $\sup_{t \in [0,\infty)} \|T(t)\|_{L(X)} \leq \Lambda_A$. For any $t > 0, T(t)$ is compact.

Lemma 2.4. *[8, 9] Let* H(A) *hold, then the operators P_α and Q_α have the following properties:*

(1) For any fixed $t \geq 0, P_\alpha(t)$ and $Q_\alpha(t)$ are linear and bounded operators, and for any $x \in X$,

$$\|P_\alpha(t)x\| \leq \Lambda_A \|x\|, \quad \|Q_\alpha(t)x\| \leq \tfrac{\alpha\Lambda_A}{\Gamma(1+\alpha)} \|x\|$$

(2) $\{P_\alpha(t), t \geq 0\}$ and $\{Q_\alpha(t), t \geq 0\}$ are strongly continuous;

(3) for every $t > 0, P_\alpha(t)$ and $Q_\alpha(t)$ are compact operators.

Next, by using the concept discussed in [2], we define the following definition of the mild solution for problem (1.1).

Definition 2.5. *A function $x \in PC(J,X)$ is said to be a PC -mild solution of problem (1.1) if it satisfies the following relation:*

$$x(t) = \begin{cases} P_\alpha(t)[x_0 + h(0,x(0))] - h(t,x(t)) + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\alpha-1} \\ Q_\alpha(t-s)B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds,\ t\in[0,t_1] \\ g_1(t,x(t)),\ t\in(t_1,s_1] \\ P_\alpha(t-s_1)\,d_1 - h(t,x(t)) + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s) \\ B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds,\times\in[s_1,t_2] \\ \ldots, \\ g_i(t,x(t)),\ t\in(t_i,s_i],i=1,2,\ldots,m, \\ P_\alpha(t-s_i)\,d_i - h(t,x(t)) + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s) \\ B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds,\ t\in[s_i,t_{i+1}] \end{cases}$$

where, for $i=1,2,\ldots,m$

$$\begin{aligned} d_i = & g_i(s_i,x(s_i)) + h(s_i,x(s_i)) - \lim_{\lambda\to\infty}\int_0^{s_i}(s_i-s)^{\alpha-1} \\ & Q_\alpha(s_i-s)B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds \end{aligned} \tag{2.1}$$

and $x_0 \in X_0$.

3. Existence Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under Banach contraction principle fixed point theorem.

From Definition 3.1, we define an operator $S: PC(J,X) \to PC(J,X)$ as

$$(Sx)(t) = \begin{cases} P_\alpha(t)[x_0 + h(0,x(0))] - h(t,x(t)) + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\alpha-1} \\ Q_\alpha(t-s)B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds,\ t\in[0,t_1] \\ g_i(t,x(t)),\ t\in(t_i,s_i] \\ P_\alpha(t-s_i)\,d_i + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s) \\ B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds,\ t\in[s_i,t_{i+1}] \end{cases}$$

with $d_i, i=1,2,\ldots,m$, defined by (2.1) and $x_0 \in X_0$..

To prove our first existence result we introduce the following assumptions:

Here, we take $a = \frac{\alpha-1}{1-\alpha_1} \in (-1,0)$.

(H(f)) The function $f \in C(J\times X^3;X)$ and there exists $L_f \in L^{\frac{1}{\alpha_1}}(J,\mathbb{R}^+)$ with $\alpha_1 \in (0,\alpha)$ such that

$$\begin{aligned} & \|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\| \\ & \quad \le L_f(t)[\|x_1-y_1\| + \|x_2-y_2\| + \|x_3-y_3\|] \end{aligned}$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3) \in X$ and every $t\in J$.

(H(h)) The function $h \in C([0,T]\times X;X)$ and there exists a positive constant $L_h > 0$ in a ways that

$$\|h(t,x) - h(t,y)\| \le L_h\|x-y\|, \quad x,y\in X \text{ and } t\in J.$$

$(H(k,\ \widetilde{k}))$ The functions $k,\widetilde{k}: \Delta\times X \to X$ are continuous and there exist constants $L_k, L_{\widetilde{k}} > 0$ such that

$$\left\| \int_0^t [k(t,s,x(s)) - k(t,s,y(s))]ds \right\| \le L_k\|x-y\|,$$

for all, $x,y\in X$; and

$$\left\| \int_0^T [\widetilde{k}(t,s,x(s)) - \widetilde{k}(t,s,y(s))]ds \right\| \le L_{\widetilde{k}}\|x-y\|,$$

for all, $x,y\in X$;

(H(g)) For $i=1,2,\ldots,m$, the functions $g_i \in C((t_i,s_i]\times X;X_0)$ and there are positive constants L_{g_i} such that

$$\|g_i(t,x) - g_i(t,y)\| \le L_{g_i}\|x-y\|$$

for all $x,y\in X$ and $t\in(t_i,s_i]$.

Theorem 3.1. *Assume* H(f)$,H(h),H(k,\widetilde{k})$ *and* H(g) *are satisfied and*

$$\begin{aligned} C = \max\Bigg\{ & \max_{1\le i\le m}\Bigg\{\Lambda_A\Bigg(L_{g_i} + L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)}\frac{s_i^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \\ & \|L_f\|_{L^{\frac{1}{\alpha_1}}([0,s_i],\mathbb{R}^+)}[1+L_k+L_{\widetilde{k}}]\Bigg) \\ & + L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)}\frac{t_{i+1}^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}\|L_f\|_{L^{\frac{1}{\alpha_1}}([0,t_{i+1}],\mathbb{R}^+)}[1+L_k+L_{\widetilde{k}}]\Bigg\}, \\ & L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)}\frac{t_1^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}\|L_f\|_{L^{\frac{1}{\alpha_1}}([0,t_1],\mathbb{R}^+)}[1+L_k+L_{\widetilde{k}}]\Bigg\} < 1. \end{aligned} \tag{3.1}$$

Then there exists a unique integral solution in $PC(J,X)$ of the system (1.1) provided $x_0\in X_0$.

Proof. Proof From the assumptions it is easy to show that the operator S is well defined on $PC(J,X)$ Let $x,y\in PC(J,X)$. For $t\in[0,t_1]$, from Lemma 2.1, we have

$$\begin{aligned} & \|(Sx)(t) - (Sy)(t)\| \\ & \le \|h(t,x(t)) - h(t,y(t))\| + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\alpha-1}\|Q_\alpha(t-s) \\ & B_\lambda(f(s,x(s),E_1x(s),E_2x(s)) - f(s,y(s),E_1y(s),E_2y(s)))\|ds \\ & \le \left[L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L_f(s)[1+L_k+L_{\widetilde{k}}]ds\right]\|x-y\|_{PC} \\ & \le \Bigg[L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)}\left[\int_0^t\left((t-s)^{\alpha-1}\right)^{\frac{1}{1-\alpha_1}}ds\right]^{1-\alpha_1}\|L_f\|_{L^{\frac{1}{\alpha_1}}([0,t_1],\mathbb{R}^+)} \\ & [1+L_k+L_{\widetilde{k}}]\Bigg]\|x-y\|_{PC} \\ & \le \left[L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)}\frac{t_1^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}\|L_f\|_{L^{\frac{1}{\alpha_1}}([0,t_1],\mathbb{R}^+)}[1+L_k+L_{\widetilde{k}}]\right] \\ & \quad \|x-y\|_{PC}. \end{aligned}$$

Similarly, we have, for $t\in(t_i,s_i], i=1,2,\ldots,m$

$$\begin{aligned} \|(Sx)(t) - (Sy)(t)\| & \le \|g_i(t,x(t)) - g_i(t,y(t))\| \\ & \le L_{g_i}\|x-y\|_{PC} \le \Lambda L_{g_i}\|x-y\|_{PC} \end{aligned}$$

and, for $t \in [s_i, t_{i+1}], i = 1, 2, \ldots, m$

$$\begin{aligned}
&\|(Sx)(t) - (Sy)(t)\| \\
&\leq \Bigg\| P_\alpha(t - s_i)\Big[g_i(s_i, x(s_i)) - g_i(s_i, y(s_i)) \\
&+ h(s_i, x(s_i)) - h(s_i, y(s_i)) \\
&- \lim_{\lambda\to\infty} \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) \\
&\quad B_\lambda(f(s, x(s), E_1x(s), E_2x(s)) \\
&- f(s, y(s), E_1y(s), E_2y(s)))ds \Big] \Bigg\| \\
&+ \|h(t, x(t)) - h(t, y(t))\| + \lim_{\lambda\to\infty} \int_0^t (t - s)^{\alpha-1} \\
&\quad \|Q_\alpha(t - s)B_\lambda(f(s, x(s), E_1x(s), E_2x(s)) \\
&- f(s, y(s), E_1y(s), E_2y(s)))\| ds \\
&\leq \Lambda_A \Bigg(L_{g_i} + L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)} \frac{s_i^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\frac{1}{\alpha_1}}([0,s_i],\mathbb{R}^+)} \\
&[1 + L_k + L_{\tilde{k}}]\Bigg) \|x - y\|_{PC} \\
&+ \Bigg[L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)} \frac{t_{i+1}^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\frac{1}{\alpha_1}}([0,t_{i+1}],\mathbb{R}^+)} \\
&[1 + L_k + L_{\tilde{k}}]\Bigg] \|x - y\|_{PC} \\
&\leq \Bigg[\Lambda_A \Bigg(L_{g_i} + L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)} \frac{s_i^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\frac{1}{\alpha_1}}([0,s_i],\mathbb{R}^+)} \\
&[1 + L_k + L_{\tilde{k}}]\Bigg) \\
&+ \Bigg[L_h + \frac{\Lambda_0\Lambda_A}{\Gamma(\alpha)} \frac{t_{i+1}^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\frac{1}{\alpha_1}}([0,t_{i+1}],\mathbb{R}^+)} \\
&[1 + L_k + L_{\tilde{k}}]\Bigg]\Bigg] \|x - y\|_{PC}.
\end{aligned}$$

From the above we can deduce that

$$\|(Sx)(t) - (Sy)(t)\|_{PC} \leq C\|x - y\|_{PC}.$$

Then it follows from condition (3.1) that S is a contraction on the space $PC(J, X)$. Hence by the Banach contraction mapping principle, S has a unique fixed point $x \in PC(J, X)$ which is just the unique integral solution of problem (1.1). The proof is now complete. □

4. Application

A simple example is given in this section to illustrate the result.

Let $X = C^2([0, \pi], \mathbb{R})$. Define an operator $A : D(A) \subseteq X \to X$ by $Ax = x''$ with $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$.

Consider the following impulsive problem:

$$\begin{aligned}
&{}^cD_t^\alpha[u(t, y) + H(t, u(t, y))] \\
&\quad = \frac{\partial^2}{\partial y^2}[u(t, y) + H(t, u(t, y))] \\
&\quad + F(t, u(t, y), E_1u(t, y), E_2u(t, y)), \\
&\quad t \in \cup_{i=1}^m (s_i, t_{i+1}], y \in [0, \pi] \\
&u(t, y) = G_i(t, u(t, y)), \quad t \in (t_i, s_i], i = 1, 2, \ldots, m; \\
&\quad y \in [0, \pi] \\
&u(t, y) = u_0(y), \quad y \in [0, \pi] \\
&u(t, 0) = u(t, \pi) = 0, \quad t \in [0, T],
\end{aligned} \tag{4.1}$$

where ${}^cD_t^\alpha$ means that the Caputo fractional derivative is taken for the time variable t with the lower limit zero; $x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C\left((t_i, s_i] \times X; \overline{D(A)}\right), i = 1, 2, \ldots, m$.

Define $x(t)(y) = u(t, y), (t, y) \in [0, T] \times [0, \pi]$. Then F, H and G_i can be rewritten as

$$\begin{aligned}
f(t, x, E_1x, E_2x)(y) &= F(t, u(t, y), E_1u(t, y), E_2u(t, y)), \\
h(t, x)(y) &= H(t, u(t, y)), \\
g_i(t, x)(y) &= G_i(t, u(t, y)), y \in [0, \pi], t \in (t_i, s_i], \\
&i = 1, 2, \ldots, m,
\end{aligned}$$

where E_1 and E_2 are same as defined in (1.1).

This shows that the problem (1.1) is an abstract formulation of the problem (4.1).

It is well known that the operator A satisfies the Hille-Yosida condition with $(0, +\infty) \subset \rho(A), \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ for $\lambda > 0$ and

$$\overline{D(A)} = \{x \in X : x(0) = x(\pi) = 0\} \neq X.$$

This implies that A satisfies $(H1)$ with $|lambda_0 = 1$. Since it is well known that A generates a compact C_0-semigroup $\{(T(t))\}_{t>0}$ on X_0 such that $\|T(t)\| \leq 1$, hence $(H2)$ is satisfied with $\Lambda_A = 1$.

For the validation of Theorem 3.1, let us consider

$$\begin{aligned}
&f(t, x(t), E_1x(t), E_2x(t)) \\
&\quad = \frac{e^{-t}x(t)}{(9 + e^t)(1 + |x(t)|)} + \frac{1}{10}\int_0^t e^{-\frac{1}{2}x(s)}ds \\
&\quad + \frac{1}{10}\int_0^t e^{-\frac{1}{4}x(s)}ds;
\end{aligned}$$

$$h(t, x(t)) = \frac{1}{3}x(t);$$

$$g_i(t, x(t)) = \frac{1}{3}\sin x(t) + e^t, \quad t \in (t_i, s_i], i = 1, 2, \ldots, m.$$

Then clearly $f : [0, T] \times E^3 \to E$ is a continuous function and

$$\begin{aligned}
&\|f(t, x, E_1x, E_2x) - f(t, y, E_1y, E_2y)\| \\
&\leq \frac{e^{-t}}{(9 + e^t)}[1 + L_k + L_{\tilde{k}}]\|x - y\|,
\end{aligned}$$

for all, $x, y \in X$, with $L_k = \frac{1}{2}$, $L_{\widetilde{k}} = \frac{1}{4}$, $L_f(t) = \frac{e^{-t}}{(9+e^t)}$ and it follows that $L_f \in L^{\frac{1}{\alpha_1}}([0,T], \mathbb{R}^+)$.

Further,

$$\|h(t,x) - h(t,y)\| \leq L_h\|x-y\|$$

with $L_h = \frac{1}{3}$. Also $g_i : (t_i, s_i] \times X \to X_0$ are continuous function and

$$\|g_i(t,x) - g_i(t,y)\| \leq L_{g_i}\|x-y\|$$

with $L_{g_i} = \frac{1}{3}$. Thus f, h, g, L_k and $L_{\widetilde{k}}$ are satisfied the hypotheses (Hf), H(h), (Hg) and $(H(k,\widetilde{k}))$ respectively. Therefore, we deduce that the model (4.1) has a unique integral solution.

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