Integral solutions of fractional neutral mixed type integro-differential systems with non-instantaneous impulses in Banach space

M. Mallika Arjunan[1]

Abstract

The main purpose of this article is to examine the existence and uniqueness of integral solutions for a class of fractional order neutral mixed type integro-differential equations with non-instantaneous impulses and non-densely defined linear operators in Banach spaces. Based on the Banach contraction principle, we develop the main results.

Keywords

Fractional neutral differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification

34K37, 37L05, 47J35, 26A33.

[1] *Department of Mathematics, Vel Tech High Tech Dr. Rangarajan Dr. Sakunthala Engineering College, Avadi-600062, Tamil Nadu, India.*
Corresponding author: [1] arjunphd07@yahoo.co.in

Contents

1. Introduction

Since fractional differential equations characterize the principal properties of the system more precisely, therefore fractional calculus assumes a noteworthy job in the qualitative theory of differential equations, please see the book [9] and the references therein.

There are two types of impulsive processes in the present hypothesis: the instantaneous impulsive system, and the non-instantaneous impulsive system. In the instantaneous impulsive system, the duration of these abrupt shifts has nothing to do with the period of the whole phase of progression. Such abrupt impulses arise in cardiac pulses, vibrations, and natural events, while in non-instantaneous impulses the duration of these rapid changes proceeds over a short time interval. Hernandez & O'Regan [4] developed a new class of differential equations with non-instantaneous impulses and established the existence of mild and classical solutions. For more details,

see for instance [1–3, 8, 10, 11].

Motivated by [1–7], in this paper we consider a class of fractional mixed type integro-differential systems with non-instantaneous impulses of the form

$$
\begin{aligned}
{}^{c}D^{\alpha}&[x(t) + h(t, x(t))] \\
&= A[x(t) + h(t, x(t))] \\
&\quad + f(t, x(t), E_1 x(t), E_2 x(t)), \\
&\qquad t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, m \\
x(t) &= g_i(t, x(t)), \quad t \in (t_i, s_i], i = 1, 2, \dots, m \quad (1.1) \\
x(0) &= x_0,
\end{aligned}
$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $A : D(A) \subset X \to X$ not necessarily a densely defined closed operator on the Banach space $(X, \|\cdot\|)$, $0 = t_0 = s_0 < t_1 \le s_1 < t_2 \le s_2 < \cdots < t_m \le s_m < t_{m+1} = T$ are fixed numbers, $h : [0, T] \times X \to X$; $g_i \in C\left(\left(t_i, s_i\right] \times X; \overline{D(A)}\right)$, $f : [0, T] \times X^3 \to X$ is a nonlinear function and the functions E_1 and E_2 are defined by

$$
E_1 x(t) = \int_0^t k(t, s, x(s)) ds \quad \text{and} \quad E_2 x(t) = \int_0^T \widetilde{k}(t, s, x(s)) ds,
$$

$k, \widetilde{k} : \Delta \times X \to X$, where $\Delta = \{(x, s) : 0 \le s \le x \le \tau\}$ are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence and uniqueness results of problem (1.1).

2. Preliminaries

Let us set $J = [0,T], J_0 = [0,t_1]$, $J_1 = (t_1,t_2], \ldots, J_{m-1} = (t_{m-1},t_m], J_m = (t_m,t_{m+1}]$ and introduce the space $PC(J,X) := \{u : J \to X \mid u \in C(J_k,X), k = 0,1,2,\ldots,m,$ and there exist $u(t_k^+)$ and $x(t_k^-), k = 1,2,\ldots,m,$ with $x(t_k^-) = x(t_k)\}$. It is clear that $PC(J,X)$ is a Banach space with the norm $\|x\|_{PC} = \sup\{\|x(t)\| : t \in J\}$.

Let $X_0 = \overline{D(A)}$ and A_0 be the part of A in $\overline{D(A)}$ defined by

$$D(A_0) = \left\{x_1 \in D(A) : Ax_1 \in \overline{D(A)}\right\}, A_0(x_1) = A(x_1)$$

Throughout our analysis, the following hypotheses will be considered:

(H1) $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, that is, there exist two constants $\omega \in \mathbb{R}$ and $\Lambda_0 \geq 0$ such that $(\omega,\infty) \subset \rho(A)$ and

$$\left\|(\lambda I - A)^{-n}\right\|_{L(X)} \leq \frac{\Lambda_0}{(\lambda - \omega)^n}, \text{ for all } \lambda > \omega, n \geq 1$$

(H2) The part A_0 of A generates a compact C_0-semigroup $\{Q(t)\}_{t \geq 0}$ in X_0 which is uniformly bounded, that is, there exists $\Lambda \geq 1$ such that $\sup_{t \in [0,\infty)} \|Q(t)\| < \Lambda$.

Let $(U(t))_{t \geq 0}$ be the integrated semigroup generated by A. It is to be noted that $(U'(t))_{t \geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by A_0 and $\|U'(t)\| \leq \Lambda e^{\omega t}, t \geq 0, \Lambda$ and ω are the constants used in the Hille-Yosida condition.

Let $B_\lambda = \lambda R(\lambda,A) := \lambda(\lambda I - A)^{-1}$. Then for all $x_1 \in X_0, B_\lambda x_1 \to x_1$ as $\lambda \to \infty$. Also from Hille-Yosida condition, it is clear that $\lim_{\lambda \to \infty} \|B_\lambda\| \leq \Lambda_0$.

Based on the above discussion along with [2], we define the integral solution of the given system (1.1).

Definition 2.1. *[2] A function $x \in PC(J,X)$ is said to be an integral solution of the Cauchy problem (1.1) if it satisfies $x(0) = x_0 \in X_0$, $x(t) = g_i(t,x(t))$ for all $t \in (t_i,s_i], i = 1,2,\ldots,m$:*

$$x(t) = U_\alpha(t)[x_0 + h(0,x(0))] - h(t,x(t))$$

$$+ \lim_{\lambda \to \infty} \int_0^t V_\alpha(t-s)B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds,$$

$$t \in [0,t_1] \text{ and}$$

$$x(t) = U_\alpha(t-s_i)[g_i(s_i,x(s_i)) + h(s_i,x(s_i))] - h(t,x(t))$$

$$+ \lim_{\lambda \to \infty} \int_{s_i}^t V_\alpha(t-s)B_\lambda f(s,x(s),E_1x(s),E_2x(s))ds,$$

$$t \in [s_i,t_{i+1}], i = 1,2,\ldots,m,$$

where

$$U_\alpha(t) = I_{0+}^{1-\alpha}V_\alpha(t), V_\alpha(t) = t^{\alpha-1}W_\alpha(t), W_\alpha(t)$$

$$= \int_0^\infty \alpha\theta\Lambda_\alpha(\theta)Q(t^\alpha\theta)d\theta.$$

Remark 2.2. *For any fixed $t > 0, V_\alpha(t)$ and $U_\alpha(t)$ are linear operators, and for any $x_1 \in X_0$,*

$$\|V_\alpha(t)x_1\| \leq \frac{\Lambda t^{\alpha-1}}{\Gamma(\alpha)}\|x_1\| \text{ and } \|U_\alpha(t)x_1\| \leq \Lambda\|x_1\|.$$

3. Existence and Uniqueness Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under Banach contraction principle fixed point theorem.

To establish our results on the existence of solutions, we consider the following hypotheses:

Here, we take $a = \frac{\alpha-1}{1-\alpha_1} \in (-1,0)$.

(H(f)) The function $f \in C(J \times X^3;X)$ and there exists a constant $\alpha \in (0,\alpha)$ and a function $L_f \in L^{\frac{1}{\alpha_1}}(J,\mathbb{R}^+)$ such that

$$\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\|$$
$$\leq L_f(t)[\|x_1 - y_1\| + \|x_2 - y_2\| + \|x_3 - y_3\|]$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3) \in X$ and every $t \in J$.

(H(h)) The function $h \in C([0,T] \times X;X)$ and there exists a positive constant $L_h > 0$ in a ways that

$$\|h(t,x) - h(t,y)\| \leq L_h\|x-y\|, \quad x,y \in X \text{ and } t \in J.$$

$(H(k,\widetilde{k}))$ The functions $k,\widetilde{k} : \Delta \times X \to X$ are continuous and there exist constants $L_k, L_{\widetilde{k}} > 0$ such that

$$\left\|\int_0^t [k(t,s,x(s)) - k(t,s,y(s))]ds\right\| \leq L_k\|x-y\|,$$

for all, $x,y \in X$; and

$$\left\|\int_0^T [\widetilde{k}(t,s,x(s)) - \widetilde{k}(t,s,y(s))]ds\right\| \leq L_{\widetilde{k}}\|x-y\|,$$

for all, $x,y \in X$;

(H(g)) For $i = 1,2,\ldots,m,$ the functions $g_i \in C((t_i,s_i] \times X;X_0)$ and there exists $L_{g_i} \in C(J,\mathbb{R}^+)$ such that

$$\|g_i(t,x) - g_i(t,y)\| \leq L_{g_i}\|x-y\|$$

for all $x,y \in X$ and $t \in (t_i,s_i]$.

Theorem 3.1. *Assume the hypotheses* $H(f), H(h), H(k, \widetilde{k})$, $H(g)$ *to hold and*

$$C = \max \left\{ \max_{1 \le i \le m} \left\{ \Lambda(L_{g_i} + L_h) + L_h + \frac{\Lambda\Lambda_0}{\Gamma(\alpha)} \frac{(t_{i+1} - s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \right. \right.$$

$$\left. \|L_f\|_{L^{\alpha_1}([s_i, t_{i+1}])} [1 + L_k + L_{\widetilde{k}}] \right\},$$

$$\left. L_h + \frac{\Lambda\Lambda_0}{\Gamma(\alpha)} \frac{t_1^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\alpha_1}([0,t_1])} [1 + L_k + L_{\widetilde{k}}] \right\} < 1.$$

Then there exists a unique integral solution in $PC(J, X)$ *of the system (1.1) provided* $x_0 \in X_0$.

Proof. Define the operator $\Upsilon : PC(J, X) \to PC(J, X)$ by $\Upsilon x(0) = x_0$, $\Upsilon x(t) = g_i(t, x(t))$ for $t \in (t_i, s_i]$ and

$$\Upsilon x(t) = U_\alpha (t - s_i) [g_i(s_i, x(s_i)) + h(s_i, x(s_i))] - h(t, x(t))$$

$$+ \lim_{\lambda \to \infty} \int_{s_i}^t V_\alpha(t - s) B_\lambda f(s, x(s), E_1 x(s), E_2 x(s)) ds,$$

$$t \in [s_i, t_{i+1}], i \ge 0.$$

Let $x, y \in PC(J, X)$. For $t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m$, we obtain

$$\|\Upsilon x(t) - \Upsilon y(t)\|$$
$$\le \|U_\alpha(t - s_i) g_i(s_i, x(s_i)) - U_\alpha(t - s_i) g_i(s_i, y(s_i))\|$$
$$+ \|U_\alpha(t - s_i) h(s_i, x(s_i)) - h(s_i, y(s_i))\|$$

$$\|h(t, x(t)) - h(t, y(t))\| + \lim_{\lambda \to \infty} \int_{s_i}^t \|V_\alpha(t - s)$$

$$B_\lambda f(s, x(s), E_1 x(s), E_2 x(s)) - V_\alpha(t - s)$$

$$B_\lambda f(s, y(s), E_1 y(s), E_2 y(s))\| ds$$

$$\le \Lambda(L_{g_i} + L_h)\|x - y\|_{PC} + \left\{ L_h + \frac{\Lambda\Lambda_0}{\Gamma(\alpha)} \int_{s_i}^t (t - s)^{\alpha - 1} L_f(s) \right.$$

$$[1 + L_k + L_{\widetilde{k}}] ds \bigg\} \|x - y\|_{PC}$$

$$\le \Lambda(L_{g_i} + L_h)\|x - y\|_{PC} + \left\{ L_h + \frac{\Lambda\Lambda_0}{\Gamma(\alpha)} \left[\int_{s_i}^t \left((t - s)^{\alpha - 1} \right)^{\frac{1}{1 - \alpha_1}} ds \right]^{1 - \alpha_1} \right.$$

$$\|L_f\|_{L^{\alpha_1}([s_i, t_{i+1}])} [1 + L_k + L_{\widetilde{k}}] \bigg\} \|x - y\|_{PC}$$

$$\le \left[\Lambda(L_{g_i} + L_h) + L_h + \frac{\Lambda\Lambda_0}{\Gamma(\alpha)} \frac{(t_{i+1} - s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\alpha_1}([s_i, t_{i+1}])} \right.$$

$$[1 + L_k + L_{\widetilde{k}}] \bigg] \|x - y\|_{PC}.$$

For $t \in [0, t_1]$

$$\|\Upsilon x(t) - \Upsilon y(t)\|$$

$$\le \left[L_h + \frac{\Lambda\Lambda_0}{\Gamma(\alpha)} \frac{t_1^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\alpha_1}([0,t_1], \mathbb{R}^+)} [1 + L_k + L_{\widetilde{k}}] \right]$$

$$\|x - y\|_{PC}.$$

For $t \in (t_i, s_i], i = 1, 2, \ldots, m$, we have

$$\|\Upsilon x(t) - \Upsilon y(t)\| \le L_{g_i}\|x - y\|_{PC} \le \Lambda L_{g_i}\|x - y\|_{PC}.$$

From above, we observe that

$$\|\Upsilon(x) - \Upsilon(y)\|_{PC} \le C\|x - y\|_{PC}$$

which implies that $\Upsilon(\cdot)$ is a contraction and there exists a unique integral solution of the system (1.1).

□

References

[1] J. Bora and S.N. Bora, Sufficient conditions for existence of integral solution for non-instantaneous impulsive fractional evolution equations, Indian J. Pure Appl. Math., 51(3)(2020), 1065-1082.

[2] H. Gu, Y. Zhou, B. Ahmad and A. Alsaedi, Integral solutions of fractional evolution equations with nondense domain, *Electron. J. Differential Equations*, Vol. 2017 (2017), No. 145, 1–15.

[3] X. Fu, X. Liu and B. Lu, On a new class of impulsive fractional evolution equations, *Advances in Difference Equations*, (2015) 2015:227.

[4] E. Hernández and D. O'Regan, On a new class of abstract impulsive differential equations, *Proc.Amer. Math. Soc.*, 141 (2013), 1641–1649.

[5] M. Mallika Arjunan, Existence results for nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space, *Malaya Journal of Matematik*, 7(4)(2019), 837–840.

[6] M. Mallika Arjunan, Integral solutions of fractional order mixed type integro-differential equations with non-instantaneous impulses in Banach space, *Malaya Journal of Matematik*, 8(4)(2020), 2212–2214.

[7] M. Mallika Arjunan, On fractional Volterra-Fredholm integro-differential systems with non-dense domain and non-instantaneous impulses, *Malaya Journal of Matematik*, 8(4)(2020), 2228–2232.

[8] B. Zhu, B. Han, L. Liu and W. Yu, On the fractional partial integro-differential equations of mixed type with non-instantaneous impulses, *Boundary Value Problems*, (2020), 2020:154

[9] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam, 2006.

[10] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal., Real World Appl.* 11(2010), 4465–4475.

[11] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, 59(2010), 1063–1077.