Malaya Journal of Matematik, Vol. 9, No. 1, 259-261, 2021

https://doi.org/10.26637/MJM0901/0043

# Induced stress of some graph operations

Shiny Joseph [1*]

**Abstract**

The induced vertex stress of any vertex of a graph $G$ measures the contribution of that vertex in the total stress of the graph $G$. This paper investigates the total vertex stress in some merged graphs, cartesian product graphs and the join of two graphs. The induced vertex stress of geodetic graphs and its relation with Wiener index and betweenness centrality is also studied.

**Keywords**

vertex stress, induced vertex stress.

**AMS Subject Classification**

05C12, 05C07.

[1] *Department of Mathematics, Government Polytechnic College, Mananthavady-670645, Wayanad, Kerala, India.*

*Corresponding author: shinyjoseph314@gmail.com;

**Article History**: Received **13** December **2020**; Accepted **26** January **2021**



## Contents



## 1. Introduction

The concept of stress is a shortest path dependent vertex parameter and was introduced by Shimbel(1953).

Let $G(V(G),E(G))$ be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $n$ and $m$ denote the number of its vertices and edges respectively. For any two vertices $u,v\in V(G)$, the distance $d(u,v)$ between $u$ and $v$ is the length of a shortest path between $u$ and $v$ in G.

**Definition 1.1.** *[6] : For a simple connected undirected graph $G(V,E)$ , the stress of a vertex $x\in V$ is defined as*

$$S_G(x) = \sum_{u\neq v\neq x} \sigma_{uv}(x)$$

*where $\sigma_{uv}(x)$ is the number of shortest paths with vertices $u$ and $v$ as their end vertices and include the vertex $x$.*

The total vertex stress of $G$ is given by, $S(G)=\sum_{v\in V(G)} S_G(v)$. The average vertex stress of $G$ of order $n\geq 1$ and denoted by, $\overline{S}(G)$ follows naturally as, $\overline{S}(G)=\frac{1}{n}\sum_{v\in V(G)} S_G(v)$.

**Definition 1.2.** *[5] Let $V(G)=\{v_i : 1\leq i\leq n\}$ and for the ordered vertex pair $(v_i,v_j)$ let there be $k_G(i,j)$ distinct shortest paths of length $l_G(i,j)$ from $v_i$ to $v_j$. Then, $\mathfrak{s}_G(v_i)=\sum_{j=1,j\neq i}^{n} k_G(i,j)(\ell_G(i,j)-1)$ and the total induced vertex stress of $G$ is $\mathfrak{s}(G)=\sum_{i=1}^{n}\mathfrak{s}_G(v_i)$.*

## 2. Total vertex stress of some merged graphs

Merging two cycles :

Consider two cycles $C_n$ and $C_m$ having vertices $v_1,v_2,\ldots,v_n$ and $u_1,u_2,\ldots,u_m$ respectively. Merging any two vertices $v_i$ and $u_j$ we get two cycles with a common vertex.

Case 1. Merging one odd cycle $C_{2t+1}$ and one even cycle $C_{2s}$ denoted by $C^o_{\triangle,\square}(t,s)$.

Let $n=s+t$ then

**Theorem 2.1.** *The total stress of $C^o_{\triangle,\square}(t,s)$ is given by $S(C^o_{\triangle,\square}(t,s))=\frac{(2n-3)t^2-(2n^2-4n+1)t+2n^3-2n^2}{2}$.*

*Proof.* The total stress of $C^o_{\triangle,\square}(s,t)$ consists of the total stress of $C_{2t+1}$ , the total stress of $C_{2(n-t)}$, the stress induced by $2(n-t)-1$ vertices on the cycle $C_{2t+1}$ and the stress induced by $2t$ vertices on the cycle $C_{2(n-t)}$ . Hence $S(C^o_{\triangle,\square}(t,s)) = \frac{(2t+1)t(t-1)}{2} + \frac{2(n-t)(n-t)(n-t-1)}{2} + (2t)2(n-t) + 2(\frac{2(n-t)t(t-1)}{2} + \frac{2t(n-t)(n-t-1)}{2})$ .

Hence $S(C^o_{\triangle,\square}(t,s)) = \frac{(2n-3)t^2-(2n^2-4n+1)t+2n^3-2n^2}{2}$. □

Case 2. Merging two even cycles $C_{2t}$ and $C_{2s}$ denoted by $C^o_{\square,\square}(t,s)$. Let $n = t+s$ then

**Proposition 2.2.** *The total vertex stress of $C^o_{\square,\square}(t,s)$ is given by*
$S(C^o_{\square,\square}(t,s)) = (n-2)t^2 - (n^2-2n)t + n^3 - n^2$

*Proof.* The total stress of $C^o_{\square,\square}(t,s)$ consists of
(a)The total stress of $C_{2t}$
(b)The total stress of $C_{2(n-t)}$
(c)The stress induced by $2t-1$ vertices on the cycle $C_{2(n-t)}$
(d)The stress induced by $2(n-t)-1$ vertices on the cycle $C_{2t}$
(a)+(b)+(c)+(d) gives the result
□

Case 3. Merging two odd cycles $C_{2t+1}$ and $C_{2s+1}$ denoted by $C^o_{\triangle,\triangle}(t,s)$. Let $n = t+s+1$, then

**Proposition 2.3.** *The total vertex stress of $C^o_{\triangle,\triangle}(t,s)$ is*
$S(C^o_{\triangle,\triangle}(t,s)) = \frac{(2n-4)t^2-(2n^2-6n+4)t+2n^3-7n^2+7n-2}{2}$

*Proof.* Similar to the proof of Proposition 2.1 and Proposition 2.2 □

Merging two paths:
Consider two paths $P_t$ having vertices $v_1, v_2, ..., v_t$ and $P_s$ having vertices $u_1, u_2, ..., u_s$. Merging any two vertices $v_i$ and $u_j$ gives a tree, denoted by $P^o_{ij}(t,s)$.
Case 1. If two pendent vertices are merged, we get the path $P_{t+s-1}$. Hence $S(P^o_{ij}(t,s)) = S(P_{t+s-1})$ , for $i = 1,t$ and $j = 1,s$
Merging a path with a cycle :
Consider the path graph $P_t$ and the cycle graph $C_n$. By merging the pendant vertex of $P_t$ with any vertex of $C_n$ we get a tadpole graph $T(n,t)$.

**Theorem 2.4.** *The total vertex stress of $T(2r+1,t)$ is*
$S(T(2r+1,t)) = S(P_t) + S(C_{2r+1}) + r(t-1)(t+r-1)$.

*Proof.* The total vertex stress of $T(2r+1,t)$ consists of , the total stress of the cycle $C_{2r+1}$, the total stress of the path $P_t$, the stress induced by the $2r+1$ vertices on the path $p_{t-1}$ and the stress induced by the $t-1$ vertices on the cycle $C_{2r+1}$. Thus total stress of $T(2r+1,t)$ is $S(T(2r+1,t)) = S(P_t) + S(C_{2r+1}) + \frac{2rt(t-1)}{2} + \frac{2(t-1)r(r-1)}{2}$. □

**Corollary 2.5.** *The total vertex stress of $T(2r,t)$ is given by*
$S(T(2r,t)) = S(P_t) + S(C_{2r}) + r(t-1)(t+r-1)$.

## 3. Total vertex stress of some cartesian product graphs

**Theorem 3.1.** *The total vertex stress of the ladder graph $K_2\square P_n$ is , $S(K_2\square P_n) = \frac{n(n-1)[(n+1)(n+7)-3(n+3)]}{6}$*

*Proof.* Let the vertices of the ladder graph $K_2\square P_n$ be $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$. Here stress of $v_1, v_n, u_1, u_n$ are same and by considering the path $u_1v_1v_2...v_n$ we get $S_{K_2\square P_n}(v_1) = n-1$. Now consider the vertex $v_i$. By considering paths $v_1v_2...v_nu_n$, $v_1v_2...v_{n-1}u_{n-1}u_n$ and so on, we get the stress of $v_i$ as $S_{K_2\square P_n}(v_i) = (i-1)(n+1-i) + \frac{(n-i)[(n+5)i-(n+3)]}{2}$ for $i = 2,3,...,n-1$. By a similar reasoning we get the stress of $u_i$ as $S_{K_2\square P_n}(u_i) = (i-1)(n+1-i) + \frac{(n-i)[(n+5)i-(n+3)]}{2}$ for $i = 2,3,...,n-1$. Hence total vertex stress of $K_2\square P_n$ is ,
$S(K_2\square P_n) = 2[\sum_{i=2}^{n-1}(S_{K_2\square P_n}(v_i)) + 2(n-1)]$. That is
$S(K_2\square P_n) = 2\{\sum_{i=1}^{n-1}[(i-1)(n+1-i) + \frac{(n-i)[(n+5)i-(n+3)]}{2}]$
$+2(n-1)\}$.
□

**Theorem 3.2.** *The total vertex stress of the book graph $K_2\square S_{n+1}$ is , $S(K_2\square S_{n+1}) = n(7n-3)$.*

*Proof.* Consider the book graph $K_2\square S_{n+1}$ with vertex set $v, v_1, v_2, ..., v_n$ for one copy of $s_{n+1}$ and $u, u_1, u_2, ..., u_n$ for the other copy of $s_{n+1}$. Consider the vertex $u$. For the vertex stress of $u$, $uv_i$ paths contributes $n$, $v_iv_j$ paths contributes $\frac{n(n-1)}{2}$ and $u_iv_j$ paths contributes $2n(n-1)$. Thus total stress of $u$ is $n + \frac{5n(n-1)}{2}$. Now consider $v_i$, for the stress of $v_i$, $vu_i$ paths contributes 1 and $u_iv_j$ paths contributes $(n-1)$. Hence stress of $v_i$ is $n$ for $i = 1,2,...,n$. Thus total stress of book graph is $2[n.n + n + \frac{5n(n-1)}{2}] = n(7n-3)$. □

## 4. Induced vertex stress of join of two graphs

Consider two graphs $G$ and $H$ with order $n$ and $m$ respectively. In this section we discuss the total induced vertex stress of $G+H$, the join of $G$ and $H$. We can define two subsets of $V(G)$ as , $G_2(u_t) = \{u_i \in V(G) : \ell_G(t,i) = 2\}$ and $G_{>2}(u_t) = \{u_i \in V(G) : \ell_G(t,i) > 2\}$ . Let $r_{(G,t)} = | G_2(u_t) |$ and $p_{(G,t)} = | G_{>2}(u_t) |$.

**Theorem 4.1.** *The total induced vertex stress of join of two graphs is given by*
$$\mathfrak{s}(G+H) = \sum_{t=1}^{n}(\sum_{i=1,i\neq t}^{r_{(G,t)}}(k_{G_2}(t,i)+m) + p_{(G,t)}\cdot m)$$
$$+\sum_{s=1}^{m}(\sum_{j=1,j\neq s}^{r_{(H,s)}}(k_{H_2}(s,j)+n) + p_{(H,s)}\cdot n).$$

*Proof.* Let $G$ and $H$ be two graphs with vertex set $\{u_1, u_2, ..., u_n\}$ and $\{v_1, v_2, ..., v_m\}$ respectively.
$\mathfrak{s}(G+H) = \sum_{x\in V(G+H)} \mathfrak{s}_{G+H}(x) = \sum_{t=1}^{n}\mathfrak{s}_{G+H}(u_t) + \sum_{s=1}^{m}\mathfrak{s}_{G+H}(v_s)$.

Consider $u_t \in V(G+H)$. The total induced vertex stress of $u_t$ is, $\mathfrak{s}_{G+H}(u_t) = \sum_{j=1, j\neq t}^{n+m} k_{G+H}(t,j)(\ell_{G+H}(t,j)-1)$.

Since $u_t v_j \in E(G+H)$, for induced vertex stress of $u_t$ we consider only $(u_t, u_j), j = 1,2,...,n$. Hence

$\mathfrak{s}_{G+H}(u_t) = \sum_{j=1, j\neq t}^{n} k_{G+H}(t,j)(\ell_{G+H}(t,j)-1)$.

If $\ell_G(t,j) = 2$, then $\ell_{G+H}(t,j) = 2$ and $k_{G+H}(t,j) = k_G(t,j) + m$. If $\ell_G(t,j) > 2$, then $\ell_{G+H}(t,j) = 2$ and $k_{G+H}(t,j) = m$.Therefore

$$\mathfrak{s}_{G+H}(u_t) = \sum_{j=1, j\neq t}^{r_{(G,t)}} (k_{G_2}(t,j)+m) + \sum_{j=1, j\neq t}^{p_{(G,t)}} m.$$

By a similar argument,

$$\mathfrak{s}_{G+H}(v_s) = \sum_{j=1, j\neq s}^{r_{(H,s)}} (k_{H_2}(s,j)+n) + \sum_{j=1, j\neq s}^{p_{(H,s)}} n\ .$$

Hence the result.

□

## 5. Induced vertex stress of geodetic graphs

A geodetic graph is an undirected graph such that there exists a unique shortest path between each two vertices.

The Wiener index or Wiener number $W(G)$ of $G$ is defined as $W(G) = \frac{1}{2} \sum_{u\in V(G)} \sum_{v\in V(G)} d_G(u,v)$

**Theorem 5.1.** *If $G$ is a geodetic graph of order $n$, then $\mathfrak{s}(G) = 2W(G) - n(n-1)$.*

*Proof.* consider $u_t \in V(G)$. Then induced vertex stress of $u_t$ is given by $\mathfrak{s}_G(u_t) = \sum_{i=1, i\neq t}^{n} (\ell_G(t,i)-1)$

Hence $\mathfrak{s}(G) = \sum_{t=1}^{n} \sum_{i=1, i\neq t}^{n} (\ell_G(t,i)-1) = 2W(G) - n(n-1)$

□

Hence the total vertex stress of a geodetic graph $G$ is given by $S(G) = W(G) - \frac{n(n-1)}{2}$. Since the total betweenness centrality $\sum_{v\in V(G)} B(v) = W(G) - \frac{n(n-1)}{2}$ the total betweenness centrality is same as total vertex stress in a geodetic graph.

## 6. Conclusion

In this paper total vertex stress of some graph classes and some graph operations are computed and calculate the induced vertex stress of join of two graphs. Also establish a relation between the total induced vertex stress and Wiener index in geodetic graphs and proved that in geodetic graphs total betweenness centrality and total vertex stress are the same.

## References


[1] Hammack, Richard and Imrich, Wilfried and Klavžar, Sandi, *Handbook of Product Graphs*, CRC Press, 2011.

[2] Harary, F., *Graph Theory*, Addison-Wesley, Reading MA, 1969.

[3] Freeman, Linton C, A set of measures of centrality based on betweenness, *Sociometry*, 1977, 35–41.

[4] Shimbel, Alfonso, Structural parameters of communication networks, *The Bulletin of Mathematical Biophysics*, 15(4)(1993), 501–507.

[5] Shiny, J., Kok, J. and Ajitha, V., *Total induced vertex stress in barbell-like graph*, Communicated.

[6] Shiny, J. and Ajitha, V., Stress regular graphs, *Malaya Journal of Matematik*, 8(3)(2020) 1152–1154.