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# Pronic Heron Mean labeling on special cases of generalized Peterson graph $P(n,k)$ and disconnected graphs

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**Abstract**
A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. In this paper for a graph $G(V,E)$ each of the vertices $v \in V(G)$ are assigned by pronic numbers. In this paper, we investigate the pronic heron mean labeling on special cases of generalized Petersen graph $P(n,k)$. Also we described an algorithm to label the vertices for the pronic heron mean labeling for certain disconnected graphs.

**Keywords**
Pronic Heron Mean labeling, Generalized Peterson Graph $P(n,k)$, Disconnected Graphs.

**AMS Subject Classification**
05C10.

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## Contents

**1 Introduction** .......... **262**
**2 Main Theorems** .......... **263**
2.1 Special cases of $P(n,k)$ .......... 263
2.2 PHML on Disconnected Graphs .......... 264
2.3 Algorithm for union of Path graphs $P_m \cup P_n$ .......... 264
2.4 Algorithm for union of Path Graph and Cycle Graph $C_m \cup P_n$ .......... 265
2.5 Algorithm for union of $mK_3$, $m \geq 2$ .......... 265
2.6 Algorithm for union of $C_n$, $n \geq 5$ and $mK_3$, $m \geq 2$ 265
2.7 Algorithm for corona product of Comb Graph and Path Graph of $(P_n \odot K_1) \cup P_m$, $m,n \geq 3$ .......... 266
**3 Conclusion** .......... **266**
**References** .......... **266**

## 1. Introduction

All graphs we consider here are finite, undirected, simple and connected. For standard notations and terminology in graph theory, we follow [1]. We refer [2] for a comprehensive survey on graph labeling and [3] for heron mean labeling. A pronic number is a number which is the product of two number of the form $n(n+1)$. Let $G(p,q)$ be a graph with $p \geq 2$. A *pronic heron mean labeling*[4] of a graph $G$ is a bijection $f: V(G) \to \{0,2,6,12,...,p(p+1)\}$ such that the resulting edge labels obtained by $f^*(uv) = \lceil \frac{f(u)+f(v)+\sqrt{f(u)f(v)}}{3} \rceil$ or $f^*(uv) = \lfloor \frac{f(u)+f(v)+\sqrt{f(u)f(v)}}{3} \rfloor$ for every $uv \in E(G)$ are all distinct. The *present work* is aimed to provide pronic heron mean labeling of some special cases of Generalized Peterson graphs namely Durer Graph $P(6,2)$, Mobius Kantor Graph$P(8,3)$, Dodecahedren Graph $P(10,2)$, Desargues Graph $P(10,3)$ and Nauru Graph $P(12,5)$. Also we investigate the existence of the same labeling on certain disconnected graphs.

**Definition 1.1.** ***Generalized Peterson Graph***
*For natural numbers n and k, where $n > 2k$, a Generalized Peterson graph $P(n,k)$ is the graph whose vertex set is $\{u_1,u_2,...,u_n\} \cup \{v_1,v_2,...,v_n\}$ and its edge set is $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k}, 1 \leq i \leq n\}$, where the subscript arithmetic is done modulo n using the residues $0,1,2,...,n-1$.*

**Note:**
⇒ For $n=6$ and $k=2$, the graph $P(6,2)$ is said to be Durer Graph.
⇒ For $n=8$ and $k=3$, the graph $P(8,3)$ is said to be Mobius Kantor Graph.

$\Rightarrow$ For $n = 10$ and $k = 2$, the graph $P(10,2)$ is said to be Dodecaheran or Dodecahedral Graph.
$\Rightarrow$ For $n = 10$ and $k = 3$, the graph $P(10,3)$ is said to be Desargues Graph.
$\Rightarrow$ For $n = 12$ and $k = 5$, the graph $P(12,5)$ is said to be Nauru Graph.

## 2. Main Theorems

### 2.1 Special cases of $P(n,k)$

**Theorem 2.1.** *Durer Graph $P(6,2)$ admits pronic heron mean labeling.*

*Proof.* Let $\{v_0,v_1,v_2,v_3,v_4,v_5\}$ be the inner vertices and $\{u_0,u_1,u_2,u_3,u_4,u_5\}$ be the outer vertices of $P(6,2)$.
Define a bijection $f : V(G) \to \{p_1, p_2, ..., p_{2n}\}$ by

$$f(u_i) = p_{i+1}, i = 0,1,...,5; \quad f(v_i) = p_{7+i}, i = 0,1,...,5.$$

For the above vertex labeling, the edge labeling $f^* : E(G) \to \{4,5,6,...,p_{2n}\}$ is defined by

$$f^*(u_iv_{i+1}) = (i+2)^2, i = 0,1,2,...,4;$$
$$f^*(u_0u_{n-1}) = \lceil\frac{n^2+n+2+\sqrt{2(n)^2+2n}}{3}\rceil;$$
$$f^*(v_iv_{i+2}) = p_{8+i}, i = 0,1,2,3;$$
$$f^*(v_iv_{i+4}) = 91+20i, i = 0,1;$$
$$f^*(u_iv_i) = (i+5)^2-(i+2), i = 0,1,2,...5.$$

In the view of the above defined labeling, the Durer graph admits pronic heron mean labeling. $\square$

**Theorem 2.2.** *Mobius Kantor Graph $P(8,3)$ admits pronic heron mean labeling.*

*Proof.* Let $\{v_0,v_1,v_2,...,v_7\}$ be the inner vertices and $\{u_0,u_1,u_2,...,u_7\}$ be the outer vertices of $P(8,3)$.
Define a bijection $f : V(G) \to \{p_1, p_2, ..., p_{2n}\}$ by

$$f(u_i) = p_{i+1}, i = 0,1,2,...,7; f(v_i) = \begin{cases} p_{i+10} & i = 0,1,2,...,6. \\ p_{i+2} & i = 7 \end{cases}$$

For the above vertex labeling, the edge labeling $f^* : E(G) \to \{4,5,6,...,p_{2n}\}$ is defined by

$$f^*(u_iu_{i+1}) = (i+2)^2, i = 0,1,2...,6;$$

$$f^*(u_0u_{n-1}) = \lceil\frac{n^2+n+2+\sqrt{2(n)^2+2n}}{3}\rceil.$$

$$f^*(u_iv_i) = \begin{cases} (n+i-2)^2+6 & i = 0,1,2...,6; \\ (i+2)^2 & i = 7 \end{cases}$$

$$f^*(v_iv_{i+3}) = \begin{cases} (n+i+4)^2 & i = 0,1,2,3; \\ (i+8)^2+2 & i = 4 \end{cases}$$

$$f^*(v_iv_{i+5}) = \begin{cases} (n+5)^2+2 & i = 0 \\ (n+6)^2+2 & i = 1. \\ (n+3)^2 & i = 2 \end{cases}$$

In the view of the above defined labeling, the Mobius Kantor graph admits pronic heron mean labeling. $\square$

**Theorem 2.3.** *Dodecahedral Graph $P(10,2)$ is a pronic heron mean graph.*

*Proof.* Let $\{v_0,v_1,v_2,...,v_9\}$ be the inner vertices and $\{u_0,u_1,u_2,...,u_9\}$ be the outer vertices of $P(10,2)$.
Define a bijection $f : V(G) \to \{p_1, p_2, ..., p_{2n}\}$ by

$$f(u_i) = \begin{cases} p_{i+1} & i = 2,3,...9 \\ p_{i+2} & i = 0; \\ p_i & i = 1 \end{cases} \quad ; \quad f(v_i) = \begin{cases} p_{i+10} & i = 1,2...9. \\ p_{20} & i = 0 \end{cases}$$

For the above vertex labeling, the edge labeling $f^* : E(G) \to \{4,5,6,...,p_{2n}\}$ is defined by

$$f^*(u_iv_{i+1}) = \begin{cases} (i+2)^2 & i = 0,2,3,...,8; \\ (i+1)(i+2) & i = 1 \end{cases}$$

$$f^*(u_0u_{n-1}) = \lceil\frac{n^2+n+6+\sqrt{6(n)^2+6n}}{3}\rceil;$$

$$f^*(v_iv_{i+1}) = \begin{cases} p_{11+i}, i = 1,2,...,7; \\ p_{16}+5, i = 0 \end{cases}$$

$$f^*(v_iv_{i+8}) = 380-135i, i = 0,1;$$

$$f^*(u_iv_i) = \begin{cases} (n+i-4)^2+6, i = 2,3,...9 \\ 5(i+9), i = 1. \\ 15n+9, i = 0 \end{cases}$$

In the view of the above defined labeling, the Dodecahedral graph admits pronic heron mean labeling. $\square$

**Theorem 2.4.** *Desargues Graph $P(10,3)$ admits pronic heron mean labeling.*

*Proof.* Let $\{v_0,v_1,v_2,...,v_9\}$ be the inner vertices and $\{u_0,u_1,u_2,...,u_9\}$ be the outer vertices of $P(10,3)$.
Define a bijection $f : V(G) \to \{p_0, p_1, p_2, ..., p_{2n}\}$ by

$$f(u_i) = \begin{cases} p_{i+1} & i = 2,3,...9 \\ p_{i+2} & i = 0; \\ p_i & i = 1 \end{cases}$$

$$f(v_i) = \begin{cases} p_{i+10} & i = 1,2...9. \\ p_{20} & i = 0 \end{cases}$$

For the above vertex labeling, the edge labeling $f^* : E(G) \to \{4,5,6,...,p_{2n}\}$ is defined by

$$f^*(u_iv_{i+1}) = \begin{cases} (i+2)^2 & i = 0,2,3,...,8; \\ (i+1)(i+2) & i = 1 \end{cases}$$

$$f^*(u_0u_{n-1}) = \lceil \frac{n^2+n+6+\sqrt{6(n)^2+6n}}{3} \rceil;$$

$$f^*(v_iv_{i+3}) = \begin{cases} (n+2+i)^2 & i=1,...,6; \\ (n+7)^2+4 & i=0 \end{cases}$$

$$f^*(v_iv_{i+7}) = \begin{cases} (n+9)^2 & i=0 \\ (n+4+i)^2+4 & i=1,2; \end{cases}$$

$$f^*(u_iv_i) = \begin{cases} (n+i-4)^2+6, i=2,3,...9 \\ 5(i+9), i=1. \\ 15n+9, i=0 \end{cases}$$

In the view of the above defined labeling, the Desargues graph admits pronic heron mean labeling. □

**Theorem 2.5.** *Nauru Graph $P(12,5)$ admits pronic heron mean labeling.*

*Proof.* Let $\{v_0,v_1,v_2,...,v_{11}\}$ be the inner vertices and $\{u_0,u_1,u_2,...,u_{11}\}$ be the outer vertices of $P(12,5)$.
Define a bijection $f:V(G)\to\{p_0,p_1,p_2,...,p_{2n}\}$ by

$$f(u_i) = p_{i+1}, i=0,1,2...11;$$

$$f(v_i) = \begin{cases} p_{i+12} & i=1,2...11. \\ p_{2n} & i=0 \end{cases}$$

For the above vertex labeling, the edge labeling $f^*:E(G)\to\{4,5,6,...,p_{2n}\}$ is defined by

$$f^*(u_iu_{i+1}) = (i+2)^2, i=0,1,2...,10;$$

$$f^*(u_0u_{n-1}) = \lceil \frac{n^2+n+2+\sqrt{2(n)^2+2n}}{3} \rceil;$$

$$f^*(u_iv_i) = \begin{cases} (i+7)^2+10 & i=1,2...,11; \\ (n+3)^2-13 & i=0 \end{cases}$$

$$f^*(v_iv_{i+5}) = \begin{cases} (15+i)^2+2 & i=1,2,...6; \\ (2n-3)^2+4 & i=0 \end{cases}$$

$$f^*(v_iv_{i+7}) = \begin{cases} (16+i)^2+4 & i=1,2,3,4. \\ (2n-2)^2+2 & i=0 \end{cases}$$

In the view of the above defined labeling, the Nauru graph admits pronic heron mean labeling. □

## 2.2 PHML on Disconnected Graphs

**Note 2.6.** *In this section the assignment of the labels are in both anticlockwise and clockwise directions. Denote the anticlockwise direction by AC and the clockwise direction by C.*

### 2.3 Algorithm for union of Path graphs $P_m \cup P_n$

The union graph $P_m \cup P_n$ where $m,n \geq 2$has the vertex set $V(P_m \cup P_n)$= $V(P_m)\cup V(P_n)$ with the cardinality $m+n$ and edge set $E(P_m \cup P_n)$= $E(P_m)\cup E(P_n)$ with cardinality $q=m+n-2$.
Let the path $P_m$ is demonstrated by listing the vertices and the edges in order
$u_1,e_1,u_2,e_2,...u_{m-1},e_{m-1},u_m$. We name the vertex $u_1$, the active vertex and the vertex $u_m$ is the end vertex of the edge $e_{m-1}$. Now the path $P_n$ is demonstrated by listing the vertices and the edges in order $v_1,e_1^{'},v_2,e_2^{'},...v_{m-1},e_{m-1}^{'},v_m$. We name the vertex $v_1$, the first vertex and the vertex $v_m$ is the end vertex of the edge $e_{m-1}^{'}$.
Label the vertices in cloclwise direction $C$. The algorithm has single pass: it labels the vertices of both the paths $P_m$ and $P_n$. At the end of the algorithm, we compute the labels of the edges which results the graph, a pronic heron mean graph.

**Algorithm**

The parameters of the algorithm are described as follows:

- i, the index of the vertices ranging from 1 to $max\{m,n\}$.
- $\{u_1,u_2...,u_m\}$, the vertices of $P_m$, $m\geq 2$.
- $\{v_1,v_2...,v_n\}$, the vertices of $P_n$, $n\geq 2$.
- $f(u_i)$, the "i"th value of the vertex $u_i$.
- $f(v_i)$, the "i"th value of the vertex $v_i$.
- $p_i$, the "i"th pronic number: $p_i = i(i+1)$ for $i = 0,1,2...m+n$.

```
    if(m < 2 || n < 2)
{
        return;
}
let pointersize;
if(m is greater than n)
{
        pointersize = m;
}
elseif(n is greater than m)
{
        pointersize = n;
}
else
{
        pointersize = m;
}
for(i=1; i ≤ pointersize; i++)
{
        if(i ≤ m)
        {
                f(u_i) = p_i
        }
        if(i ≤ n)
        {
                f(v_i) = p_{m+i}
        }
}
```

## 2.4 Algorithm for union of Path Graph and Cycle Graph $C_m \cup P_n$

The union graph $C_m \cup P_n$ where $m,n \geq 2$has the vertex set $V(C_m \cup P_n)= V(C_m) \cup V(P_n)$ with the cardinality $m+n$ and edge set $E(C_m \cup P_n)= E(C_m) \cup E(P_n)$ with cardinality $q = m+n-1$. Let the cycle $C_m$ is demonstrated by listing the vertices and the edges in order $u_1, e_1, u_2, e_2, ...u_{m-1}, e_{m-1}, u_m, e_m, u_1$. We name the vertex $u_1$, the active vertex and the vertex $u_m$ is the end vertex of the edge $e_{m-1}$. Now the path $P_n$ is demonstrated by listing the vertices and the edges in order $v_1, e_1^{'}, v_2, e_2^{'}, ...v_{m-1}, e_{m-1}^{'}, v_m$. We name the vertex $v_1$, the first vertex and the vertex $v_m$ is the end vertex of the edge $e_{m-1}^{'}$. Label the vertices in cloclwise direction $C$. The algorithm has single pass: it labels the vertices of both the paths $C_m$ and $P_n$. At the end of the algorithm, we compute the labels of the edges which results the graph, a pronic heron mean graph.

**Algorithm**

The parameters of the algorithm are described as follows:

- i, the index of the vertices ranging from 1 to $max\{m,n\}$.
- $\{u_1, u_2, u_3, ..., u_m\}$ be the vertices of $C_m$, $m \geq 3$.
- $\{v_1, v_2 ..., v_n\}$ be the pendant edges attached to the corresponding vertices of $P_n$
- $f(u_i)$, the "i"th value of the vertex $u_i$.
- $f(v_i)$, the "i"th value of the vertex $v_i$.
- $p_i$, the "i"th pronic number: $p_i = i(i+1)$ for $i = 1,2...m+n$.

```
if(m < 3 || n < 2)
{
        return;
}
let pointersize;
if(m is greater than n)
{
        pointersize = m;
}
elseif(n is greater than m)
{
        pointersize = n;
}
else
{
        pointersize = m;
}
for(i=1; i ≤ pointersize; i++)
{
        if(m==4)
        {
        if(i=1 || i = 2)
        {
                f(u_i) = p_i
        }
        elseif(i=3)
        {
                f(u_i) = p_{i+1}
        }
        elseif(i=4)
        {
                f(u_i) = p_{i-1}
        }
        }
        else
        {
                f(u_i) = p_i
        }
                f(v_i) = p_{m+i}
        }
}
```

## 2.5 Algorithm for union of $mK_3$, $m \geq 2$

Given a graph $mK_3$ where $m$ denotes the number of copies of $K_3$. Let the vertex set of $mK_3$ be $V = \{V_1 \cup V_2 \cup ... \cup V_m\}$ where $V_i = \{v_1^i, v_2^i, v_3^i\}$ where $i = 1,2,3...,m$ and the edge set of $mK_3$ be $E = \{(v_1v_2)^i, (v_2v_3)^i, (v_3v_1)^i\}$ where $i = 1,2,3...,m$.
Label the vertices in cloclwise direction $C$. The algorithm has single pass: it labels the vertices of all the $mK_3$ . At the end of the algorithm, we compute the labels of the edges which results the graph, a pronic heron mean graph.

**Algorithm**

The parameters of the algorithm are described as follows:

- i, the index of the vertices ranging from 1 to m.
- $\{v_1^i, v_2^i, v_3^i\}$, the vertices of $mK_3$, $m \geq 2$.
- $f(v_i^{'})$, the "i"th value of the vertex $v_i^{'}$.
- $p_i$, the "i"th pronic number: $p_i = i(i+1)$ for $i = 1,2...3m$.

```
for(i = 1; i ≤ m; i++)
{
        f(v_1^i) = p_{3i-2}
        f(v_2^i) = p_{3i-1}
        f(v_3^i) = p_{3i}
}
```

## 2.6 Algorithm for union of $C_n$, $n \geq 5$ and $mK_3$, $m \geq 2$

Given a graph $C_n \cup mK_3$ where $m$ denotes the number of copies of $K_3$ and $n$ denote the number of vertices of $C_n$. Let the cycle $C_m$ is demonstrated by listing the vertices and the edges in order $u_1, e_1, u_2, e_2, ...u_{m-1}, e_{n-1}, u_n, e_n, u_1$. We name the vertex $u_1$, the active vertex and the vertex $u_n$ is the end vertex of the edge $e_{n-1}$. Let the vertex set of $mK_3$ be $V = \{V_1 \cup V_2 \cup ... \cup V_m\}$ where $V_j = \{v_1^j, v_2^j, v_3^j\}$ where $j = 1,2,3...,m$ and the edge set of $mK_3$ be $E = \{(v_1v_2)^j, (v_2v_j)^i, (v_3v_1)^j\}$ where $j = 1,2,3...,m$.
Label the vertices in clockwise direction $C$. The algorithm

has single pass: it labels the vertices of all the $C_n$ and the it labels all the evrtices of $mK_3$ . At the end of the algorithm, we compute the labels of the edges which results the graph, a pronic heron mean graph.

**Algorithm**

The parameters of the algorithm are described as follows:

- i, the index of the vertices ranging from 1 to n.
- j, the index of the vertices ranging from 1 to m.
- $\{u_1, u_2, u_3, ..., u_n\}$ be the vertices of $C_n$, $n \geq 3$.
- $\{v_1^j, v_2^j, v_3^j\}$, the vertices of $mK_3$, $m \geq 2$.
- $p_i$, the "i"th pronic number: $p_i = i(i+1)$ for $i = 1, 2...n+m$.

$if(m < 2 \,||\, n < 5)$
{
$return;$
}
for(i=1; i $\leq n; i++)$
{
$f(u_i) = p_i$
}
for(i=1; j $\leq m; j++)$
{
$f(v_1^j) = p_{n+3j-2}$
$f(v_2^j) = p_{n+3j-1}$
$f(v_3^j) = p_{n+3j}$
}

### 2.7 Algorithm for corona product of Comb Graph and Path Graph of $(P_n \odot K_1) \cup P_m$, $m, n \geq 3$

The **comb graph**, denoted by $P_n \odot K_1$ is defined as the corona product of the path $P_n$ and $K_1$ . It has $2n$ vertices and $2n-1$ edges. Let $\{u_0, u_1, ..., u_{n-1}\}$ be the path $P_n$ with $n$ vertices and$\{v_1', v_2', ..., v_n'\}$ be the pendant edges attached to the corresponding vertices of $P_n$. Let $\{u_1, u_2, u_3, ..., u_m\}$ be the vertices of $P_m$, $m \geq 3$. Algorithm for Comb Graph:
The parameters of the algorithm are described as follows:

- i, the index of the vertices ranging from 1 to 1 to $max\{m, n\}$.
- $\{v_1, v_2 ..., v_n\}$, the vertices of $P_n$, $n \geq 3$.
- $\{v_1', v_2', ..., v_n'\}$, the vertices attached to the corresponding vertices of $P_n$.
- $\{u_1, u_2, u_3, ..., u_m\}$, the vertices of $P_m$, $m \geq 3$.
- $f(v_i)$, the "i"th value of the vertex v$_i$.
- $f(v_i')$, the "i"th value of the vertex $v_i'$.
- $f(u_i)$, the "i"th value of the vertex u$_i$.
- $p_i$, the "i"th pronic number: $p_i = i(i+1)$ where $i = 1, 2...2n+m$.

$if(m < 3 \,||\, n < 2)$
{
$return;$
}
$let\ pointersize;$
$if$(m is greater than n)
{
$pointersize = m;$
}
$elseif$(n is greater than m)
{
$pointersize = n;$
}
$else$
{
$pointersize = m;$
}
for(i=1; i $\leq pointersize; i++)$
{
$f(v_i) = p_{2i-1}$
$f(v_i') = p_{2i}$
$f(u_i) = p_{2n+i}$
}

## 3. Conclusion

In this paper, the results for few special graphs are proved that they admit pronic heron mean labeling. It is possible to investigate similar results for other families of graphs. The authors are highly thankful to the anonymous referees for their valuable suggestions.

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