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On β^*-open and β^*-closed sets in fuzzy topological space

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Abstract
A new concept of fuzzy β^*-open and fuzzy β^*-closed sets in fuzzy topological space is presented in this paper. Also some basic concepts and properties of them are investigated. Some theorems and examples for β^*-open and β^*-closed sets are introduced.

Keywords
β^*-open, β^*-closed, β-closure, fuzzy open.

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Contents

1. Introduction and Preliminaries

The method of fuzzy set was introduced by Zedeh [1]. Chang [2] has added fuzzy topological space. Khalik [3] has made a study on certain types of fuzzy separation axioms in fuzzy topological space on fuzzy sets. Singal [4] introduced regularly open sets in fuzzy topological spaces and Lee [5] introduced fuzzy delta separation axioms. Zahran [6] further added the new class of fuzzy open sets: there are fuzzy β-open sets, fuzzy β-closed sets, fuzzy regular open sets and fuzzy regular closed sets. In the present work, some properties and theorems are explored.

β^*-open set and β^*-closed set in fuzzy topological space are introduced based on work of Mubarki [7]. Also some properties and some theorems are investigated.

The present paper (X,τ), (Y,σ) (or simply X,Y) represents non-empty fuzzy topological spaces. Let μ be a fuzzy subset of a space X. The fuzzy closure of μ, fuzzy interior of μ, fuzzy δ-closure of μ and the fuzzy δ-interior of μ are denoted by $\mathrm{cl}(\mu)$, $\mathrm{int}(\mu)$, $\mathrm{cl}_\delta(\mu)$ and $\mathrm{int}_\delta(\mu)$ respectively. The fuzzy δ-interior of fuzzy subset μ of X is the union of all fuzzy regular open sets contained in μ. A fuzzy subset μ is called fuzzy δ-open [8] if $\mu = \mathrm{int}_\delta(\mu)$. The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e, $\mu = \mathrm{cl}_\delta(\mu)$.

2. Fuzzy Topological Space

Definition 2.1. *A family $\tau \subseteq I^X$ of fuzzy subsets is called a fuzzy topology (in the sense of Chang [2]) for X if it satisfies the following conditions:*

(i) $0_X, 1_X \in \tau$.

(ii) $\lambda,\mu \in \tau$, *then* $\wedge\mu \in \tau$.

(iii) $\lambda_i \in \tau$ *for each* $i \in \mathscr{F}$, *then* $\vee_{i\in\mathscr{F}}\lambda_i \in \tau$.

Members of τ are called fuzzy open subsets and the complement of fuzzy open subsets is called fuzzy closed subsets on fuzzy topological space (X,τ).

Definition 2.2. *A fuzzy set μ of (X,τ) is called as follows [9–13]:*

(i) Fuzzy semi open if $\mu \leq \mathrm{cl}(\mathrm{int}(\mu)$.

(ii) Fuzzy α-open if $\mu \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mu))$.

(iii) Fuzzy pre-open (fuzzy pre-closed)) if $\mu \leq \mathrm{int}(\mathrm{cl}(\mu))(\mathrm{cl}(\mathrm{int}(\mu)) \leq \mu)$.

(iv) Fuzzy regular open if $\mu = \mathrm{cl}(\mathrm{int}(\mu)\ (\mu = \mathrm{int}(\mathrm{cl}(\mu))$ closed$)$.

(v) Fuzzy e-open if $\mu \leq \mathrm{cl}(\mathrm{int}_\delta(\mu)) \vee \mathrm{int}(\mathrm{cl}_\delta(\mu))$ $(\mu \geq \mathrm{cl}(\mathrm{int}_\delta(\mu)) \wedge \mathrm{int}(\mathrm{cl}_\delta(\mu))$ closed$)$.

3. Fuzzy β^*-open Sets in Fuzzy Topological Space

In this section a new open set in fuzzy topological space is introduced.

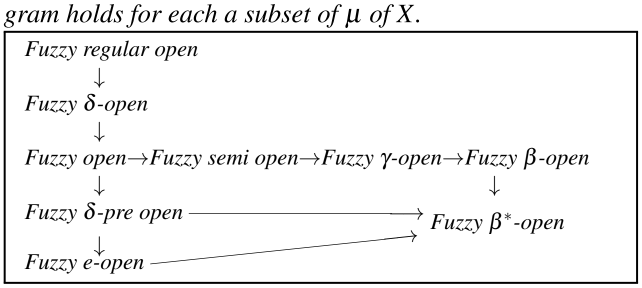
Definition 3.1. *A fuzzy subset μ of a fuzzy topological space (X,τ) is said to be β^*-open set if*
$\mu \leq \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu)\big)$.

Example 3.2. *$X = \{x,y,z\}$ and the fuzzy topology*

$$\begin{aligned}\tau = \big\{&0,1,\{x_{0.2},y_{0.1},z_{0.1}\},\{x_{0.4},y_{0.1},z_{0.1}\},\\ &\{x_{0.6},y_{0.9},z_{0.9}\},\{x_{0.6},y_{0.5},z_{0.7}\},\\ &\{x_{0.8},y_{0.9},z_{0.9}\}\big\} \quad \textit{and}\\ \tau^c = \big\{&0,1,\{x_{0.8},y_{0.9},z_{0.9}\},\{x_{0.6},y_{0.9},z_{0.9}\},\\ &\{x_{0.4},y_{0.1},z_{0.1}\},\{x_{0.4},y_{0.5},z_{0.3}\},\\ &\{x_{0.2},y_{0.1},z_{0.1}\}\big\}.\\ \textit{Let } \mu = &\{x_{0.6},y_{0.9},z_{0.9}\},\mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}(\mu)\big)\big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu)\big)\\ =&\{x_{0.6},y_{0.9},z_{0.9}\}.\end{aligned}$$

i.e., μ is β^*-open set.

Remark 3.3. *From the definitions I obtain the following diagram holds for each a subset of μ of X.*



Result 3.4.

(i) Fuzzy β^*-open is fuzzy δ-preopen if
$\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu)\big)\Big) = 0.$

(ii) Fuzzy β^*-open is β-open if
$\mathrm{int}\big(\mathrm{cl}_\delta(\mu)\big) = 0.$

Proposition 3.5. *If μ is fuzzy δ-pre open and λ is fuzzy β-open then $\mu \vee \lambda$ is fuzzy β^*-open.*

Proof. Obvious from Definition 3.1. □

Proposition 3.6. *Let (X,τ) be a fuzzy topological space. Then the union of any two fuzzy β^*-open sets is an β^*-open set.*

Proof. Let μ_1,μ_2 be two fuzzy β^*-open sets,
$\mu_1 \leq \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_1)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_1)\big)$ and
$\mu_2 \leq \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_2)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_2)\big)$ (by Definition 3.1)

Then we have,

$$\begin{aligned}\mu_1 \vee \mu_2 \leq\ &\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_1)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_1)\big)\\ &\vee \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_2)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_2)\big).\\ \mu_1 \vee \mu_2 \leq\ &\mathrm{cl}(\mathrm{int}\big(\mathrm{cl}(\mu_1 \vee \mu_2)\big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_1 \vee \mu_2)\big).\end{aligned}$$

Since, the arbitrary union of fuzzy β^*-open sets is fuzzy β^*-open set. □

Theorem 3.7. *Let (X,τ) be a fuzzy topological space and let $\{\mu_\alpha\}_{\alpha\in\mathscr{F}}$ be the collection of fuzzy β^*-open sets in fuzzy topological space X, then $\vee_{\alpha\in\mathscr{F}}(\mu_\alpha)$ is fuzzy β^*-open set.*

Proof. Let $\mathscr{F}$ be the collection of fuzzy β^*-open sets in fuzzy topological space (X,τ).

For each $\alpha \in \mathscr{F}$, $\mu_\alpha \leq \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_\alpha)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_\alpha)\big)$. Thus,

$$\begin{aligned}\vee_{\alpha\in\mathscr{F}}(\mu_\alpha) &\leq \vee_{\alpha\in\mathscr{F}}\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_\alpha)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_\alpha)\big).\\ \vee_{\alpha\in\mathscr{F}}(\mu_\alpha) &\leq \mathrm{cl}\Big(\mathrm{int}\Big(\mathrm{cl}\big(\vee_{\alpha\in\mathscr{F}}\big)(\mu_\alpha)\Big)\\ &\qquad \vee \mathrm{int}\Big(\mathrm{cl}_\delta\big(\vee_{\alpha\in\mathscr{F}}(\mu_\alpha)\big)\Big)\end{aligned}$$

Since, the arbitrary union of fuzzy β^*-open sets is fuzzy β^*-open set. □

Theorem 3.8. *Let (X,τ) and (X,σ) be any two fuzzy topological spaces such that X is product related to Y. Then the product $\mu_1 \times \mu_2$ of a fuzzy β^*-open set μ_1 of X and fuzzy β^*-open set μ_2 of Y is fuzzy β^*-open set of the fuzzy product space $X \times Y$.*

Proof. Let μ_1, μ_2 are two fuzzy β^*-open sets of X and Y respectively, From Definition 3.1,

$$\begin{aligned}\mu_1 &\leq \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_1)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_1)\big) \text{ and}\\ \mu_2 &\leq \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_2)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_2)\big).\end{aligned}$$

Then we have,

$$\begin{aligned}\mu_1 \times \mu_2 \leq\ &\mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_1)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_1)\big)\\ &\times \mathrm{cl}\Big(\mathrm{int}\big(\mathrm{cl}(\mu_2)\big)\Big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_2)\big).\\ \mu_1 \times \mu_2 \leq\ &\mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(\mu_1 \times \mu_2)\big) \vee \mathrm{int}\big(\mathrm{cl}_\delta(\mu_1 \times \mu_2)\big).\end{aligned}$$

$\mu_1 \times \mu_2$ is fuzzy β^*-open in the fuzzy product space $X \times Y$. □

4. Fuzzy β^*-Closed Sets in Fuzzy Topological Space

In this section a new closed set in fuzzy topological space is introduced.

Definition 4.1. *A fuzzy subset μ of a fuzzy topological space (X,τ) is said to be β^*-closed set if*
$\mu \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu)\big)$.

Example 4.2. $X = \{x,y,z\}$. *And the fuzzy topology*

$$\tau = \big\{0, 1, \{x_{0.2}, y_{0.1}, z_{0.1}\}, \{x_{0.4}, y_{0.1}, z_{0.1}\}, \{x_{0.6}, y_{0.9}, z_{0.9}\}, \{x_{0.6}, y_{0.5}, z_{0.7}\}, \{x_{0.8}, y_{0.9}, z_{0.9}\}\big\}$$

and

$$\tau^c = \big\{0, 1, \{x_{0.8}, y_{0.9}, z_{0.9}\}, \{x_{0.6}, y_{0.9}, z_{0.9}\}, \{x_{0.4}, y_{0.1}, z_{0.1}\}, \{x_{0.4}, y_{0.5}, z_{0.3}\}, \{x_{0.2}, y_{0.1}, z_{0.1}\}\big\}.$$

Let $\mu = \{x_{0.6}, y_{0.9}, z_{0.9}\}$,
$\operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu)\big) = \{x_{0.6}, y_{0.9}, z_{0.9}\}$.
μ is β^*-closed set.

Result 4.3.

(i) Fuzzy β^*-closed is fuzzy δ-semi open if $\operatorname{cl}\Big(\operatorname{int}\big(\operatorname{cl}(\mu)\big)\Big) = 0$.

(ii) Fuzzy β^*-closed is α-open if $\operatorname{cl}\big(\operatorname{int}_\delta(\mu)\big) = 0$.

Proposition 4.4. *If μ is fuzzy δ-semi open and λ is fuzzy α-open then $\mu \wedge \lambda$ is fuzzy β^*-closed.*

Proof. Obvious from Definition 4.1. □

Proposition 4.5. *Let (X,τ) be a fuzzy topological space. Then the intersection of two fuzzy β^*-closed sets is a β^*-closed set in the fuzzy topological space (X,τ).*

Proof. Let μ_1, μ_2 be two fuzzy β^*-closed sets.
$\mu_1 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_1)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_1)\big)$. And
$\mu_2 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_2)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_2)\big)$ by Definition 4.1.
Then we have,

$$\mu_1 \wedge \mu_2 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_1)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_1)\big) \wedge \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_2)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_2)\big).$$

$$\mu_1 \wedge \mu_2 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_1 \wedge \mu_2)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_1 \wedge \mu_2)\big).$$

Therefore, $\mu_1 \wedge \mu_2$ is fuzzy β^*-closed set. □

Theorem 4.6. *Let (X,τ) be a fuzzy topological space and let $\{\mu_\alpha\}_{\alpha \in \mathscr{F}}$ be the collection of fuzzy β^*-closed sets in fuzzy topological space X, then $\wedge_{\alpha \in \mathscr{F}}(\mu_\alpha)$ is fuzzy β^*-closed set.*

Proof. Let $\mathscr{F}$ be the collection of fuzzy β^*-closed sets in fuzzy topological space (X,τ).
For each $\alpha \in \mathscr{F}$, $\mu_\alpha \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_\alpha)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_\alpha)\big)$. Thus,

$$\wedge_{\alpha \in \mathscr{F}}(\mu_\alpha) \geq \wedge_{\alpha \in \mathscr{F}} \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_\alpha)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_\alpha)\big).$$

$$\wedge_{\alpha \in \mathscr{F}}(\mu_\alpha) \geq \operatorname{int}\Big(\operatorname{cl}\Big(\operatorname{int}\big(\wedge_{\alpha \in \mathscr{F}}(\mu_\alpha)\big)\Big)\Big) \wedge \operatorname{cl}\Big(\operatorname{int}_\delta\big(\wedge_{\alpha \in \mathscr{F}}(\mu_\alpha)\big)\Big).$$

Since, the arbitrary intersection of fuzzy β^*-closed sets is fuzzy β^*-closed set. □

Theorem 4.7. *Let (X,τ) and (X,σ) be any two fuzzy topological spaces such that X is product related to Y. Then the product $\mu_1 \times \mu_2$ of a fuzzy β^*-closed set μ_1 of X and fuzzy β^*-closed set μ_2 of Y is fuzzy β^*-closed set of the fuzzy product space $X \times Y$.*

Proof. Let μ_1, μ_2 are two fuzzy β^*-closed sets of X and Y respectively. From Definition 4.1,
$\mu_1 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_1)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_1)\big)$. And
$\mu_2 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_2)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_2)\big)$.
Then we have,

$$\mu_1 \times \mu_2 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_1)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_1)\big) \times \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_2)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_2)\big).$$

$$\mu_1 \times \mu_2 \geq \operatorname{int}\Big(\operatorname{cl}\big(\operatorname{int}(\mu_1 \times \mu_2)\big)\Big) \wedge \operatorname{cl}\big(\operatorname{int}_\delta(\mu_1 \times \mu_2)\big).$$

$\mu_1 \times \mu_2$ is fuzzy β^*-closed in the fuzzy product space $X \times Y$. □

5. Conclusion

In this paper, a new class of open and closed sets in fuzzy topological space, namely β^*-open and β^*-closed sets is introduced. Then some new examples and theorems in separation axioms on fuzzy topological space are developed.

References

[1] Zadeh, L. A. Fuzzy sets. Information and control, 8(3)(1965), 338–353.

[2] Chang, C. L. Fuzzy topological spaces. Journal of mathematical Analysis and Applications, 24(1)(1968), 182–190.

[3] Khalik, A. P. D. M. A., and Hussein, A. K. G. A. On Some Types of Fuzzy Separation Axioms in Fuzzy Topological Space on Fuzzy Sets.

[4] Singal, M. K., and Rajvanshi, N. Regularly open sets in fuzzy topological spaces. Fuzzy sets and systems, 50(3)(1992), 343–353.

[5] Lee, S. J., and Yun, S. M. Fuzzy delta separation axioms. In 2011 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2011)(2011, June) 595–602.

[6] Zahran, A. M. Fuzzy δ-continuous, Fuzzy almost Regularity (Normality) on Fuzzy Topology No Fuzzy Sets. Fuzzy Mathematics, 3(1)(1995), 89–96.

[7] Mubarki, A. M., Al-Rshudi, M. M., and Al-Juhani, M. A. β^*-Open sets and β^*-continuity in topological spaces. Journal of Taibah University for Science, 8(2)(2014), 142–148.

[8] Velicko, N. V. H-closed topological spaces. Amer. Math. Soc. Transl, 78(1968), 103–118.

[9] Al-Hawary, T. Fuzzy M-open sets. Theory and Applications of Mathematics & Computer Science, 7(1)(2017), 72–77.

[10] Azad, K. K. On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity. Journal of mathematical analysis and applications, 82(1)(1981), 14–32.

[11] El-Shafei, M. E., and Zakari, A. Theta-Generalized closed sets in fuzzy topological spaces. Arabian Journal for Science and Engineering, 31(2)(2006), 197.

[12] Hanafy, I. M. Fuzzy γ-open sets and fuzzy γ-continuity. J. Fuzzy Math, 7(2)(1999), 419–430.

[13] Othmana, H. A. On fuzzy θ-generalized-semi-closed sets. Journal of Advanced Studies in Topology, 7(2)(2016), 84–92.