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A study of SIQR model with Holling type–II incidence rate

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Abstract

In this study, we propose an SIQR epidemic model with a Holling type-II incidence rate. In this model, the total population N is divided into five compartments; namely susceptible individual class (S), infective individual class (I), quarantine from susceptible individual class (Q_1), quarantine from infective individual class (Q_2), and recovered individual class (R). The basic reproduction number ($\mathfrak{R}_0$) of the model is found by the next generation method and then disease-free (DF) and endemic equilibrium points of the system are found and their existence conditions are presented. This study concludes that if the basic reproduction number $\mathfrak{R}_0$ is less than one, the disease-free equilibrium is globally asymptotically stable and if the basic reproduction number $\mathfrak{R}_0$ is greater than one, then the endemic equilibrium exists and globally. In this study, we also discuss the behavior of the disease-free equilibrium points by using manifold theory when the basic reproduction number $\mathfrak{R}_0$ is equal to one. This study is very helpful in those pandemic diseases wherein the quarantine process of an infected individual is one of the most effective solutions to get recover from the disease and also to control the spreading of disease from an infected individual to uninfected individuals. The numerical simulation is given, and to analyze the found results, at the last conclusion is also given.

Keywords

SIQR epidemic model, Holling type-II incidence rate, basic reproductive number, Routh-Herwitz criterion, second additive compound matrix, Lyapunov function, Stability.

AMS Subject Classification

34D23, 93A30, 93D20.

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1. Introduction

Looking at the present situation, spreading and controlling infectious diseases is very important for the interest of society. There is a huge role of Mathematical models for making policies, health-economy policy, emergency planning, risk assessment, control program evaluation, and for optimizing various detection. Many authors [3–5, 7, 10] considered the various incidence rates in their literature, and the reproduction numbers and subthreshold endemic equilibrium for compartmental models of disease transmission are discussed by Van den Driessche and Watmough [8].

Treatment is a key to controlling the spread of diseases such as measles and an epidemic model with non-monotonic incidence rate under treatment is considered by Kar and Batabyal [4]. Wang and Ruan [9] discussed on the piecewise treatment function in their research work

$$T(I) = \begin{cases} r & I > 0, \\ 0 & I = 0. \end{cases}$$

where r is a constant removal rate of the infectives

2. Preliminaries

Definition 2.1. Susceptible Individuals (S(t)). *Susceptible*

people are those who are healthy and can be infected under appropriate conditions.

Definition 2.2. Infected Individuals (I(t)). *These are individuals who have already been infected with the virus and can transfer it to individuals who are susceptible.*

Definition 2.3. Quarantine Individuals (Q(t)). *These are individuals who have isolated from susceptible and infected and then return to the recovered class.*

Definition 2.4. Recovered Individuals (R(t)). *These are individuals who have recovered from the infection and are believed to be immune.*

Definition 2.5. Basic reproduction number (Q(t)). *The basic reproductive number or simply the reproductive number $\mathfrak{R}_0$ is the average number of secondary infections produced by one infected individuals during the mean course of infection (infectious period) in a completely susceptible population.*

Definition 2.6. Holling type-II incidence function. *A function $\frac{\beta SI}{1+\alpha I}$ with $\alpha > 0$, is called the Holling type II incidence function, where I is the number of infectious individuals and β is the effective contact rate (the average number of contacts sufficient for transmitting infection).*

Definition 2.7. Routh Hurwitz's criterion. *The Eigen values of a $(m \times m)$ matrix A are the roots of an m^{th} degree polynomial and hence cannot be conveniently evaluated analytically. However, there are some useful mathematical theorems which provide necessary and sufficient conditions for all Eigen values of a matrix to have negative real parts without involving explicit calculations of Eigen values.*

Let the characteristics equation of variation matrix of order $m \times m$ interacting species as

$$\lambda^m + a_1\lambda^{m-1} + a_2\lambda^{m-2} + +a_3\lambda^{m-3} + \ldots + a_m = 0.$$

The stability conditions (known as Routh-Hurwitz criteria) now involve only these coefficients and are stated in the following table (for $m \leq 5$):

S. No.	Number of Species	Stability Criteria
01	2	$a_1 > 0, a_2 > 0$
02	3	$a_1 > 0, a_2 > 0, a_1a_2 - a_3 > 0$
03	4	$a_1 > 0, a_3 > 0, a_4 > 0,$ $a_1a_2a_3 > a_3^2 + a_1^2a_4$
04	5	$a_i > 0$, for all i $a_1a_2a_3 > a_3^2 + a_1^2a_4$ $(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > 0$
05	6	$\cdots$

The Routh-Hurwitz stability criteria for a multi-species model with small numbers of species the inequalities are readily evaluated and they constitute one of the most powerful and widely used theoretical tools in deterministic modeling.

In this study, we consider an SIQR epidemic model with Holling type-II incidence rate. In the main results, we present the mathematical model and the basic reproduction number $\mathfrak{R}_0$ is calculated after that equilibrium points of the system are found and their existence conditions are presented and atlast of the main results, we prove some theorems for the global stability of the disease-free and endemic equilibrium points. In section 4, we give an example that demonstrates the validity of the main results. In the last section, we give a conclusion.

3. Main Results

In this section, we shall discuss about the formulation of the model, basic properties of the model, disease free equilibrium, the basic reproduction number, endemic equilibrium of the system and stability of the equilibriums.

Formulation of the model

Here, the total population N is partitioned into five divisions; symbolically in (S), (I), (Q_1), (Q_2) and (R). Susceptible people are those who are healthy and can be infected under appropriate conditions. Quarantine from susceptible people who will never get the infection by separating from susceptible and Quarantine from infected people are those who will never infect the susceptible by separating from the infected class. Let us π be the constant recruitment rate of susceptible i.e. it may be birth rate or immigration rate. Let the susceptible individuals are moving in quarantine from susceptible class at a rate φ_1, hence φ_1 defines the rate of quarantine of susceptible individuals and β is the transmission rate of susceptible to infected individuals, α is the measure of inhibition taken by the infected. In this study, we consider the incidence rate $\frac{\beta SI}{1+\alpha I}$ which is also known as the Holling type–II incidence rate. In this study we consider the following parameters; μ is the natural death rate, d is the disease-related death rate, φ_2 is the rate of quarantine of infected individuals, and θ_1, θ_2 are the recovery rate of infected and quarantine from infected respectively. Therefore the SIR epidemic model with quarantine from susceptible and infected and Holling type-II incidence rate is the system of following non-linear ordinary differential equation:

$$\begin{aligned}
\frac{dS}{dt} &= \pi - \frac{\beta SI}{1+\alpha I} - (\mu + \varphi_1)S, \\
\frac{dI}{dt} &= \frac{\beta SI}{1+\alpha I} - (\mu + d + \varphi_2 + \theta_1)I, \\
\frac{dQ_1}{dt} &= \varphi_1 S - \mu Q_1, \\
\frac{dQ_2}{dt} &= \varphi_2 I - (\mu + d + \theta_2)Q_2, \\
\frac{dR}{dt} &= \theta_1 I - \theta_2 Q_2 - \mu R,
\end{aligned} \tag{1}$$

where $S(0) \geq 0, I(0) \geq 0, Q_1(0) \geq 0, Q_2(0) \geq 0$ and $R(0) \geq 0$.

Basic Properties of the model

In this section, we shall find the feasible region of the model (1), discuss the disease-free equilibrium and investigate the basic reproduction number by the next generation method which is introduced by Diekmann et al. [2]. In last, we shall find an endemic equilibrium of the mathematical model. Now the total population $N(t)$ is the sum of all compartments i.e. $N(t) = S(t)+I(t)+Q_1(t)+Q_2(t)+R(t)$. On adding all the equations of system (1), we have

$$\frac{dN(t)}{dt} = \pi - \mu N - d(I+Q_2) \leq \pi - \mu N.$$

Hence $N(t) \leq N(0)e^{-\mu t} + \frac{\pi}{\mu}(1-e^{-\mu t})$. Thus, we find $N(t) \to \pi/\mu$ as $t \to \infty$. Therefore the feasible region for the system (1) is $W = \left\{ (S,I,Q_1,Q_2,R) \in \mathrm{IR}^5_+ : 0 < S+I+Q_1+Q_2+R < \pi/\mu \right\}$ which is bounded and positively invariant. Thus we have the following lemma:

Lemma 3.1. *The set* $W = \left\{ (S,I,Q_1,Q_2,R) \in \mathrm{IR}^5_+ : 0 < S+I+Q_1+Q_2+R < \pi/\mu \right\}$ *is a positively invariant region of the model (1).*

As because the first four equations of the model do not contain $R(t)$*. So without loss of generality, the fifth equation can be left out for theoretical analysis. Thus this study considers the following diminished system of equations for the theoretical study:*

$$\begin{aligned}
\frac{dS}{dt} &= \pi - \frac{\beta SI}{1+\alpha I} - (\mu+\varphi_1)S,\\
\frac{dI}{dt} &= \frac{\beta SI}{1+\alpha I} - (\mu+d+\varphi_2+\theta_1)I,\\
\frac{dQ_1}{dt} &= \varphi_1 S - \mu Q_1,\\
\frac{dQ_2}{dt} &= \varphi_2 I - (\mu+d+\theta_2)Q_2.
\end{aligned} \tag{2}$$

Disease free equilibrium On keeping all the equations of system (2) equal to zero and then on solving, we can find out the equilibrium points. The disease-free equilibrium (DFE) for the system (2) is $E^0 = (S^0, 0, Q_1^0, 0)$ where $S^0 = \frac{\pi}{\mu+\varphi_1}$, $Q_1^0 = \frac{\pi\varphi_1}{(\mu+\varphi_1)\mu}$.

The basic reproduction number $\mathfrak{R}_0$

The basic reproduction number $\mathfrak{R}_0$ is the average number of secondary infections generated by a single infective when it is introduced into a purely susceptible population. To find the basic reproduction number $\mathfrak{R}_0$ by the next-generation method [2].

Let $X = (I, Q_1, Q_2, S)^T$. System (2) becomes

$$\frac{dX}{dt} = F(X) - V(X)$$

where

$$F(X) = \begin{pmatrix} \frac{\beta SI}{1+\alpha I} \\ 0 \\ 0 \\ 0 \end{pmatrix}, V(X) = \begin{pmatrix} (\mu+d+\theta_1+\varphi_2)I \\ -\varphi_1 S + \mu Q_1 \\ -\varphi_2 I + (\mu+d+\theta_2)Q_2 \\ -\pi + \frac{\beta SI}{1+\alpha I} + (\mu+\varphi_1)S \end{pmatrix}.$$

The Jacobian matrix of $F(X)$ and $V(X)$ at the disease-free equilibrium E^0 are,

$$DF(E^0) = \begin{pmatrix} F_1 & 0 \\ 0 & 0 \end{pmatrix}, DV(E^0) = \begin{pmatrix} V_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ respectively,}$$

where

$$F_1 = \begin{pmatrix} 0 & \frac{\beta\pi}{\mu+\varphi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$V_1 = \begin{pmatrix} 0 & \mu+d+\theta_1+\varphi_2 & 0 & 0 \\ -\varphi_1 & 0 & \mu & 0 \\ 0 & -\varphi_2 & 0 & \mu+d+\theta_2 \\ \mu+\varphi_1 & \frac{\beta\pi}{\mu+\varphi_1} & 0 & 0 \end{pmatrix}.$$

The next-generation matrix of the system is

$$F_1 V_1^{-1} = \begin{pmatrix} \frac{\beta\pi}{(\mu+\varphi_1)(\mu+d+\theta_1+\varphi_2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The spectral radius of $F_1V_1^{-1}$ is called the basic reproduction number, therefore

$$\mathfrak{R}_0 = \frac{\beta\pi}{(\mu+\varphi_1)(\mu+d+\theta_1+\varphi_2)}.$$

Endemic equilibrium of the system

The endemic equilibrium of the system (2) is given by $E^1 = (S^*, I^*, Q_1^*, Q_2^*)$, where $S^* = \frac{\pi(1+\alpha I^*)}{(\mu+\varphi_1)(1+\alpha I^*)+\beta}$, $Q_1^* = \frac{\varphi_1 S^*}{\mu}$, $Q_2^* = \frac{\varphi_2 I^*}{\mu+d+\theta_2}$ and I^* is given by $I^* = \frac{\beta\pi-(\mu+\varphi_1)(\mu+d+\theta_1+\varphi_2)}{\alpha(\mu+\varphi_1)+\beta}$. Thus we have the following lemma

Lemma 3.2. *system (2) will have a unique positive equilibrium disease free* $E^1 = (S^*, I^*, Q_1^*, Q_2^*)$*, which is called endemic equilibrium, where* $S^* = \frac{\pi(1+\alpha I^*)}{(\mu+\varphi_1)(1+\alpha I^*)+\beta}$, $Q_1^* = \frac{\varphi_1 S^*}{\mu}$, $Q_2^* = \frac{\varphi_2 I^*}{\mu+d+\theta_2}$, $I^* = \frac{\beta\pi-(\mu+\varphi_1)(\mu+d+\theta_1+\varphi_2)}{\alpha(\mu+\varphi_1)+\beta}$.

Stability analysis

To investigates the local and global stability of the system (2) at equilibriums. The linearized matrix of the system (2) is given by

$$J = \begin{pmatrix} -\frac{\beta I}{1+\alpha I} - (\mu+\varphi_1) & -\frac{\beta S}{(1+\alpha I)^2} & 0 & 0 \\ \frac{\beta I}{1+\alpha I} & \frac{\beta S}{(1+\alpha I)^2} - (\mu+d+\varphi_2+\theta_1) & 0 & 0 \\ \varphi_1 & 0 & -\mu & 0 \\ 0 & \varphi_2 & 0 & -(\mu+d+\theta_2) \end{pmatrix}.$$

At disease-free equilibrium $E^0 = (S^0, 0, Q_1^0, 0)$

The Jacobian matrix of the system (2) at disease-free equilibrium $E^0 = (S^0, 0, Q_1^0, 0)$ is

$$J(E^0) = \begin{pmatrix} -(\mu+\varphi_1) & -\frac{\beta\pi}{\mu+\varphi_1} & 0 & 0 \\ 0 & \frac{\beta\pi}{\mu+\varphi_1} - (\mu+d+\varphi_2+\theta_1) & 0 & 0 \\ \varphi_1 & 0 & -\mu & 0 \\ 0 & \varphi_2 & 0 & -(\mu+d+\theta_2) \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix of the system (2) is given by

$$(-\mu-\varphi_1-\lambda)\left(\frac{\beta\pi}{\mu+\varphi_1} - \mu - d - \varphi_2 - \theta_1 - \lambda\right)$$
$$(-\mu-\lambda)(-\mu-d-\theta_2-\lambda) = 0.$$

The eigen values of the Jacobian matrix are

$$-(\mu+\varphi_1), -\mu, -(\mu+d+\theta_2) \text{ and } \frac{1}{(\mu+d+\varphi_2+\theta_1)}(\Re_0 - 1).$$

The first three eigen values of the Jacobian matrix are negative and the fourth eigen value will be negative when $\Re_0 < 1$. By the Routh Hurwitz criterion the disease-free equilibrium is locally asymptotically stable.

From the above discussion, we get the following theorem:

Theorem 3.3. *The disease-free equilibrium $E^0 = (S^0, 0, Q_1^0, 0)$ is locally asymptotically stable if $\Re_0 < 1$ and unstable if $\Re_0 > 1$.*

Now using the center manifold theory [1, 6], we will study the stability of DFE at $\Re_0 = 1$. Let $\varphi = \beta = \beta^* = \frac{(\mu+\varphi_1)(\mu+d+\theta_1+\varphi_2)}{\pi}$ be the transmission parameter.

Let $S = x_1, I = x_2, Q_1 = x_3$ and $Q_4 = x_4$ then the system (2) can be written as

$$\begin{aligned} \frac{dx_1}{dt} &= \pi - \frac{\beta x_1 x_2}{1+\alpha x_2} - (\mu+\varphi_1)x_1 \equiv f_1, \\ \frac{dx_2}{dt} &= \frac{\beta x_1 x_2}{1+\alpha x_2} - (\mu+d+\varphi_2+\theta_1)x_2 \equiv f_2, \\ \frac{dx_3}{dt} &= \varphi_1 x_1 - \mu x_3 \equiv f_3, \\ \frac{dx_4}{dt} &= \varphi_2 x_2 - (\mu+d+\theta_2)x_4 \equiv f_4, \end{aligned} \tag{3}$$

The Jacobian matrix J^* at $\Re_0 = 1$ i.e. $\beta = \beta^* = \frac{(\mu+\varphi_1)(\mu+d+\theta_1+\varphi_2)}{\pi}$ is

$$J^* = \begin{pmatrix} -(\mu+\varphi_1) & -\frac{\beta\pi}{\mu+\varphi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varphi_1 & 0 & -\mu & 0 \\ 0 & \varphi_2 & 0 & -(\mu+d+\theta_2) \end{pmatrix}.$$

Here J^* has one simple eigen value at zero. Let $u = [u_1, u_2, u_3, u_4]$ and $w = [w_1, w_2, w_3, w_4]^T$ be the left and a right eigen vector of J^* corresponding to the zero eigen value respectively. Then we have $u_1 = 0, u_2 = 1, u_3 = 0, u_4 = 0$ and $w_1 = -\frac{\mu}{\varphi_1}$, $w_2 = \frac{\mu(\mu+\varphi_1)^2}{\beta^*\varphi_1\pi}$, $w_3 = 1$, $w_4 = -\frac{\varphi_2\mu(\mu+\varphi_1)^2}{\beta^*\varphi_1\pi(\mu+d+\theta_2)}$.

To find the non zero partial derivatives of f_2 which allied with the first equation of the system (3) at $\Re_0 = 1$ i.e. $\beta = \beta^* = \frac{(\mu+\varphi_1)(\mu+d+\theta_1+\varphi_2)}{\pi}$

$$\left(\frac{\partial^2 f_2}{\partial x_2 \partial x_1}\right)_{E^0} = \beta^*, \left(\frac{\partial^2 f_2}{\partial x_2 \partial \beta^*}\right)_{E^0} = \frac{\pi}{\mu+\varphi_1},$$
$$\left(\frac{\partial^2 f_2}{\partial x_2^2}\right)_{E^0} = -\frac{2\beta^*\alpha\pi}{\mu+\varphi_1}.$$

In pursuance of bifurcation theory which is based on center manifold theory, to obtain the bifurcation constants a_1 and a_2 with the formula:

$$\begin{aligned} a_1 &= \sum_{k,i,j=1}^{4} u_k w_i w_j \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j}\right)_{E^0} \\ &= -\frac{2\mu^2(\mu+\varphi_1)^2}{\varphi_1^2\pi} - \frac{2\mu^2(\mu+\varphi_1)^3}{\beta^*\varphi_1^2\pi} < 0, \\ a_2 &= \sum_{k,i=1}^{4} u_k w_i \left(\frac{\partial^2 f_k}{\partial x_i \partial \beta^*}\right)_{E^0} = \frac{\mu(\mu+\varphi_1)}{\beta^*\varphi_1} > 0. \end{aligned}$$

Since $a_1 < 0$ and $a_2 > 0$, then the disease-free equilibrium is unstable and there exists a positive equilibrium as $\Re_0$ crosses 1. As per discussion for $\Re_0 = 1$ the following theorem can be given:

Theorem 3.4. *The disease-free equilibrium is unstable at $\Re_0 = 1$, hence we get a positive equilibrium as $\Re_0$ crosses 1.*

Now we discuss the global stability of the disease-free equilibrium of the system (2).

Consider the Lyapunov function

$$L = I$$

Now

$$\frac{dL}{dt} = \left(\frac{\beta S}{1+\alpha I} - (\mu+d+\varphi_2+\theta_1)\right) I \leq 0.$$

Here

$$\frac{dL}{dt} = 0 \Leftrightarrow I = 0.$$

If $\Re_0 < 1$, the largest compact invariant set is the singleton set $\{(S^0, 0, Q_1^0, 0)\}$ in $\left\{(S, I, Q_1, Q_2) : \frac{dL}{dt} = 0\right\}$. Thus by the Lasalle invariance principle, the disease-free equilibrium is globally asymptotically stable.

The following theorem can be given:

Theorem 3.5. *If $\Re_0 < 1$, then the disease-free equilibrium is globally asymptotically stable.*

At endemic equilibrium

Now we will discuss the local and global asymptotically stability of the endemic equilibrium $E^1 = \left(S^*, I^*, Q_1^*, Q_2^*\right)$.

The Jacobian matrix $J(E^1)E^1$ *is given by*

$$J(E^1)=\begin{pmatrix} -\frac{\beta I^*}{1+\alpha I^*}-(\mu+\varphi_1) & -\frac{\beta S^*}{(1+\alpha I^*)^2} & 0 & 0 \\ \frac{\beta I^*}{1+\alpha I^*} & \frac{\beta S^*}{(1+\alpha I^*)^2}-(\mu+d+\varphi_2+\theta_1) & 0 & 0 \\ \varphi_1 & 0 & -\mu & 0 \\ 0 & \varphi_2 & 0 & -(\mu+d+\theta_2) \end{pmatrix}.$$

The characteristic equation $J(E^1)$ *is*

$$(-\mu-d-\theta_2-\lambda)(-\mu-\lambda)\left\{\left(-\mu-\varphi_1-\frac{\beta I^*}{1+\alpha I^*}-\lambda\right)\left(\frac{\beta S^*}{(1+\alpha I^*)^2}-(\mu+d+\varphi_2+\theta_1)-\lambda\right)+\frac{\beta^2 S^* I^*}{(1+\alpha I^*)^3}\right\}=0.$$

Implies that

$$(-\mu-\lambda)(-\mu-d-\theta_2-\lambda)(\lambda^2+b_1\lambda+b_2)=0,$$

where

$$b_1 = \mu+\varphi_1+\frac{\beta I^*}{1+\alpha I^*}-\frac{\beta S^*}{(1+\alpha I^*)^2} + (\mu+d+\varphi_2+\theta_1)$$

$$b_1 = (\mu+\varphi_1)\left\{-\frac{\beta S^*}{(1+\alpha I^*)^2}+(\mu+d+\varphi_2+\theta_1)\right\} + \frac{\beta(\mu+d+\varphi_2+\theta_1)I^*}{1+\alpha I^*}$$

The first two eigen values of the Jacobian matrix are negative and the remaining two eigen values will be the root of the equation $\lambda^2+b_1\lambda+b_2=0$.

It can be easily seen that b_1 *and* b_2 *are positive, when* $-\frac{\beta S^*}{(1+\alpha I^*)^2}+(\mu+d+\varphi_2+\theta_1)\geq 0$ *i.e.* $\frac{\beta S^*}{(1+\alpha I^*)^2}\leq(\mu+d+\varphi_2+\theta_1)$. *Therefore the equation* $\lambda^2+b_1\lambda+b_2=0$ *has two roots whose real parts are negative. Thus all the eigen values of the Jacobian matrix at endemic equilibria are either negative or the real part is negative under the condition* $\frac{\beta S^*}{(1+\alpha I^*)^2}\leq(\mu+d+\varphi_2+\theta_1)$. *Therefore the endemic equilibrium is locally asymptotically stable under the condition* $\frac{\beta S^*}{(1+\alpha I^*)^2}\leq(\mu+d+\varphi_2+\theta_1)$.

Thus we have the following theorem:

Theorem 3.6. *The endemic equilibrium* $E^1=\left(S^*,I^*,Q_1^*,Q_2^*\right)$ *is locally asymptotically stable when* $\frac{\beta S^*}{(1+\alpha I^*)^2}\leq(\mu+d+\varphi_2+\theta_1)$.

To study the global stability of an endemic equilibrium $E^1=\left(S^*,I^*,Q_1^*,Q_2^*\right)$

System (2) is divided into two parts.

The first subsystem as:

$$\begin{aligned}\frac{dS}{dt}&=\pi-\frac{\beta SI}{1+\alpha I}-(\mu+\varphi_1)S\\ \frac{dI}{dt}&=\frac{\beta SI}{1+\alpha I}-(\mu+d+\varphi_2+\theta_1)I,\\ \frac{dQ_2}{dt}&=\varphi_2 I-(\mu+d+\theta_2)Q_2,\end{aligned}\tag{4}$$

and the limit system is

$$\frac{dQ_1}{dt}=\varphi_1 S-\mu Q_1\tag{5}$$

The Jacobian matrix of the system is

$$J=\begin{bmatrix} -\frac{\beta I}{1+\alpha I}-(\mu+\varphi_1) & -\frac{\beta S}{(1+\alpha I)^2} & 0 \\ \frac{\beta I}{1+\alpha I} & \frac{\beta S}{(1+\alpha I)^2}-(\mu+d+\varphi_2+\theta_1) & 0 \\ 0 & \varphi_2 & -(\mu+d+\theta_2) \end{bmatrix}.$$

The second additive compound matrix is given by

$$J^{[2]}=\begin{bmatrix} -\frac{\beta I}{1+\alpha I}+\frac{\beta S}{(1+\alpha I)^2}-m_1 & 0 & 0 \\ \varphi_2 & -\frac{\beta I}{1+\alpha I}-m_2 & -\frac{\beta S}{(1+\alpha I)^2} \\ 0 & \frac{\beta I}{1+\alpha I} & \frac{\beta S}{(1+\alpha I)^2}-m_2 \end{bmatrix},$$

where

$$\begin{aligned}m_1&=2\mu+d+\theta_1+\varphi_1+\varphi_2,\\ m_2&=2\mu+d+\theta_2+\varphi_1,\\ m_3&=2\mu+2d+\theta_1+\theta_2+\varphi_2.\end{aligned}$$

Let us choose the function

$$P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{I}{Q_2} & 0 \\ 0 & 0 & \frac{I}{Q_2} \end{bmatrix},$$

and

$$P^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{Q_2}{I} & 0 \\ 0 & 0 & \frac{Q_2}{I} \end{bmatrix},$$

then

$$P_f=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\dot{I}Q_2-I\dot{Q}_2}{Q_2^2} & 0 \\ 0 & 0 & \frac{\dot{I}Q_2-I\dot{Q}_2}{Q_2^2} \end{bmatrix}.$$

From P_f *and* P^{-1}, *it follows that*

$$P_fP^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\dot{I}}{I}-\frac{\dot{Q}_2}{Q_2} & 0 \\ 0 & 0 & \frac{\dot{I}}{I}-\frac{\dot{Q}_2}{Q_2} \end{bmatrix}.$$

Now we have

$${}_PJ^{[2]}{}_{P^{-1}}=\begin{bmatrix} -\frac{\beta I}{1+\alpha I}+\frac{\beta S}{(1+\alpha I)^2}-m_1 & 0 & 0 \\ \varphi_2 & -\frac{\beta I}{1+\alpha I}-m_2 & -\frac{\beta S}{(1+\alpha I)^2} \\ 0 & \frac{\beta I}{1+\alpha I} & \frac{\beta S}{(1+\alpha I)^2}-m_2 \end{bmatrix},$$

where $m=\min\{m_1,m_2\}$.

Now

$$B=P_fP^{-1}+PJ^{[2]}P^{-1}$$

$$B=\begin{bmatrix} -\frac{\beta I}{1+\alpha I}+\frac{\beta S}{(1+\alpha I)^2}-m_1+1 & 0 & 0 \\ \varphi_2 & \frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}-\frac{\beta I}{1+\alpha I}-m_2 & -\frac{\beta S}{(1+\alpha I)^2} \\ 0 & \frac{\beta I}{1+\alpha I} & \frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}+\frac{\beta S}{(1+\alpha I)^2}-m_2 \end{bmatrix}.$$

In the Block form, the matrix B can be written as

$$B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$B_{11}=-\frac{\beta I}{1+\alpha I}+\frac{\beta S}{(1+\alpha I)^2}-m_1+1,$$

$$B_{12}=\begin{bmatrix} 0 & 0 \end{bmatrix}, B_{21}=\begin{bmatrix} \varphi_2 \\ 0 \end{bmatrix},$$

$$B_{22}=\begin{bmatrix} \frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}-\frac{\beta I}{1+\alpha I}-m_2 & -\frac{\beta S}{(1+\alpha I)^2} \\ \frac{\beta I}{1+\alpha I} & \frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}+\frac{\beta S}{(1+\alpha I)^2}-m_3 \end{bmatrix}$$

$$B_{22}=\begin{bmatrix} \frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}-\frac{\beta I}{1+\alpha I}-m_2 & -\frac{\beta S}{(1+\alpha I)^2} \\ \frac{\beta I}{1+\alpha I} & \frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}+\frac{\beta S}{(1+\alpha I)^2}-m_3 \end{bmatrix}.$$

Suppose $v=(v_1,v_2,v_3)\in \mathrm{IR}^3$ *and its norm* $B_{12}=[0\ 0]$, *defined as*

$$PvP=\max\{|v_1|,|v_2|,|v_3|\}.$$

Let the Lozinski measure concerning this norm is denoted by $F(B)$.

It follows from [5]

$$F(B)\le \sup\{g_1,g_2\},$$

where

$$g_1=F(B_{11})+|B_{12}|, g_2=|B_{21}|+F(B_{22})$$

$$|B_{12}|=\max[0,0]=0, |B_{21}|=|\varphi_2|=\varphi_2$$

$$F(B_{11})=\lim_{h\to 0}\frac{|I+hB_{11}|-1}{h}$$
$$=\frac{\beta I}{1+\alpha I}-\frac{\beta S}{(1+\alpha I)^2}+m_1,$$
$$F(B_{22})=\lim_{h\to 0}\frac{|I+hB_{22}|-1}{h}$$
$$=\frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}-\frac{\beta I}{1+\alpha I}-\min\{m_2,m_3\}.$$

Therefore, we have

$$g_1=\frac{\beta I}{1+\alpha I}-\frac{\beta S}{(1+\alpha I)^2}+m_1$$
$$=\frac{\beta I}{1+\alpha I}-\frac{\beta S}{(1+\alpha I)^2}+2\mu+d+\theta_1+\varphi_1+\varphi_2$$
$$=\frac{\beta I}{1+\alpha I}-\frac{\beta S}{(1+\alpha I)^2}+2\mu+d+\theta_2+\varphi_1\le\frac{\dot I}{I}-\mu,$$
$$g_2=\varphi_2+\frac{\dot I}{I}-\frac{\dot Q_2}{Q_2}-\frac{\beta I}{1+\alpha I}-\min\{m_2,m_3\}\le\frac{\dot I}{I}-\mu.$$

$$\therefore F(B)\le\sup\{g_1,g_2\}\le\frac{\dot I}{I}-\mu.$$

Then

$$q=\frac{1}{t}\int_0^t F(B)dS\le\frac{1}{t}\int_0^t\left(\frac{\dot I}{I}-\mu\right)dS=\frac{1}{t}\ln\frac{I(t)}{I(0)}-\mu$$

which implies $q\le-\frac{\mu}{2}<0$. *Thus the sub-system (4) is globally asymptotically stable. Consider the limit system*

$$\frac{dQ_1}{dt}=\varphi_1 S-\mu Q_1 \tag{6}$$

Equation (6) can be written as

$$\frac{dQ_1}{dt}=\varphi_1 S^*-\mu Q_1$$
$$\therefore Q_1=e^{-\mu t}\left[Q_1(0)-\frac{\varphi_1 S^*}{\mu}\right]+\frac{\varphi_1 S^*}{\mu},$$

which implies $Q_1\to Q_1^*$ *when* $t\to\infty$, *thus the endemic equilibrium* $E^1=\left(S^*,I^*,Q_1^*,Q_2^*\right)$ *is globally asymptotically stable.*

Thus we have the following theorem

Theorem 3.7. *The disease-free and endemic equilibriums of the system (2) are globally asymptotically stable, when* $R_0<1$ *and* $R_0>1$ *respectively.*

4. Examples

For numerical simulation, when $\Re_0<1$, we take the following parameters $\pi=2,\alpha=0.5,\beta=1,\mu=0.007,d=0.05,\theta_1=0.002,\theta_2=0.003,\varphi_1=1,\varphi_2=2$. With these values of the parameters, we conclude that $\Re_0=0.96<1$, then by theorem 3.5 trajectories of S,I,Q_1,Q_2 and R with initial values $S(0)=200,I(0)=0,Q_1(0)=100,Q_2(0)=0,R(0)=0$ approach to the disease-free equilibrium $E^0=(1.99,0,284.286,0)$ as shown in the Fig. 4.1. Therefore, the disease-free equilibrium E^0 is a global asymptotically stable when $\Re_0<1$.

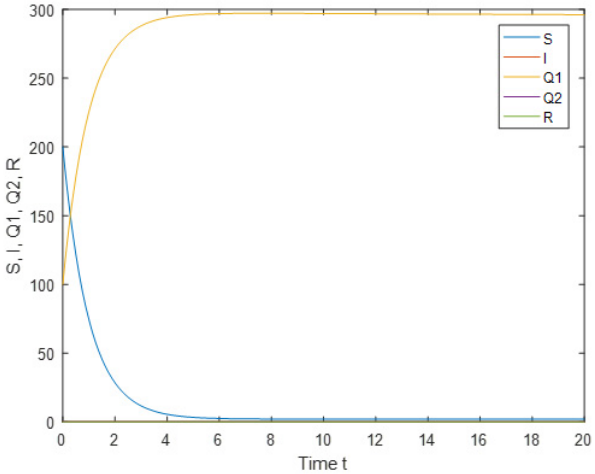

Figure 4.1. When $\Re_0<1$.

For numerical simulation, when $\Re_0 > 1$, we take the following parameters $\pi = 2, \alpha = 0.5, \beta = 2.1, \mu = 0.007, d = 0.05, \theta_1 = 0.002, \theta_2 = 0.003, \varphi_1 = 1, \varphi_2 = 2$. With these values of the parameters, we conclude that $\Re_0 = 2.0256 > 1$, $\frac{\beta S^*}{(1+\alpha I^*)^2} = 1.282 < 2.059 = (\mu + d + \varphi_2 + \theta_1)$, then by theorem 3.6 and 3.7 trajectories of S, I, Q_1, Q_2 and R with initial values $S(0) = 200, I(0) = 2, Q_1(0) = 100, Q_2(0) = 0, R(0) = 1$ approach to the endemic equilibrium $E^1 = (0.66, 0.82, 141, 66)$ as shown in the Fig. 4.2. Therefore, the endemic equilibrium E^1 is a global asymptotically stable under the given conditions,

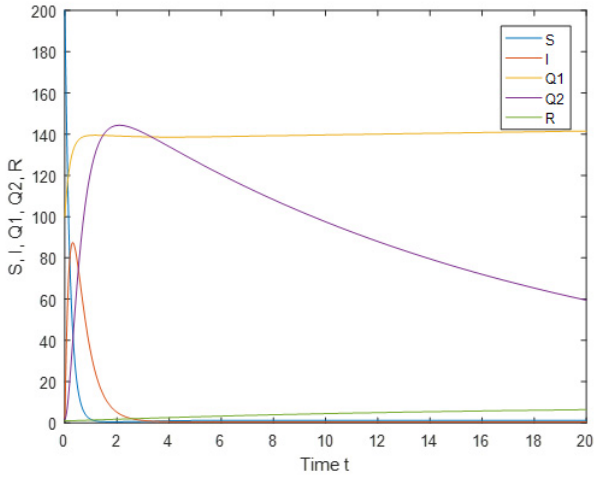

Figure 4.2. When $\Re_0 > 1$.

5. Conclusion

In this paper, we have considered an SIQR epidemic model with Holling type-II incidence rate. We have discussed on the global stability of the equilibriums. Our main results show that the disease-free equilibrium E^0 and the endemic equilibrium $E^* = \left(S^*, I^*, Q_1^*, Q_2^*\right)$ are globally asymptotically stable, when $\Re_0 < 1, \Re_0 > 1$ respectively under certain conditions. Looking at the present situation of this pandemic, spreading and controlling infectious disease is very important for the interest of society; hence our obtained main results are very useful in this situation.

References

[1] C. Castillo-Chavez, B. Song, Dynamical models of tuberculosis and their applications, *Math. Biosci. Eng.,* 1(2004), 361-404.

[2] O. Diekmann, J.A.P. Heesterbeek and J.A.J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, *Journal of Mathematical Biology,* 28(4) (1990), 365–382.

[3] H.W. Hethcote, The mathematics of infectious disease, *SIAM Rev.,* 42(2000), 599-653.

[4] T.K. Kar and A. Batabyal, Modeling, and analysis of an epidemic model with non-monotonic incidence rate under treatment, *Journal of mathematics research,* 2(1) (2010), 103-115.

[5] J. Mena-Lorca and H.W. Hethcote, Dynamic models of infectious diseases as a regulator of population sizes, *J. Math. Biol.,* 30(1992), 693-716.

[6] S. Sastry, Analysis, Stability and Control, *Springer, New York* (1999).

[7] H. Shu, D. Fan and J. Wei, Global Stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission, Nonlinear Analysis: *Real World Applications,* 13(2012), 1581-1592.

[8] P. Van den Driessche and J. Watmough, Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission, *Mathematical biosciences,* 180, (2002), 29-48.

[9] W. Wang and S. Ruan, Bifurcations in an epidemic model with a constant removal rate of the infective, *Journal of mathematical analysis and application,* 219(2) (2004), 775-793.

[10] N. Yi, Q. Zhang, K. Mao, D. Yang and Q. Li, Analysis and control of an SEIR epidemic system with nonlinear transmission rate, *Mathematical and computer modeling,* 50, (2009), 1498-1513.

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