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Even and odd strongly multiplicative graphs

M. Joice Punitha[1] and A. Josephine Lissie[2*]

Abstract
A graph $G=(V(G),E(G))$ with p vertices is said to be even strongly multiplicative if the vertices of G can be labeled with p distinct integers $1,2,\ldots,p$ such that the labels induced on the edges by the product of labels of the end vertices are all distinct and even.
If the vertices of G are labeled with distinct integers $1,2,3,\ldots,2p-1$ such that the label induced on the edges by the product of labels of the end vertices should all be odd and distinct then the graph is said to be odd strongly multiplicative.

Keywords
Even multiplicative labeling, odd multiplicative labeling.

AMS Subject Classification
05C10.

[1] *Department of Mathematics, Bharathi Women's College, Affiliated to University of Madras, Chennai, India.*
[2] *Department of Mathematics, Stella Maris College, Affiliated to University of Madras, Chennai, India.*
*__Corresponding author__: [2]lissie84@gmail.com
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1. Introduction

The study of strongly multiplicative graphs and strongly multiplicative labeling methods was introduced and defined by assigning of values, usually represented by integers, to the edges and/or vertices of a graph such that the labels induced on the edges by the product of labels of the end vertices are all distinct [1]. Now strongly multiplicative labeled graphs often serve as models in a wide range of applications. Such applications including coding theory and communication net work addressing. The brief survey on applications of strongly multiplicative labeled graphs is reported in [2]. The present work is targeted to discuss one such labeling known as odd and even strongly multiplicative labeling has been introduced and defined as follow.

A graph G is said to be even strongly multiplicative if the labels induced on the edges by the product of labels of the end vertices are all distinct and even.

A graph G is said to be odd strongly multiplicative if the labels induced on the edges by the product of labels of the end vertices are all distinct and odd.

The strongly multiplicative, even strongly multiplicative and odd strongly multiplicative are different concepts. A graph may posses one or both of these or neither.

Remark 1.1. *For strongly multiplicative labeling the p vertices are labeled with p consecutive integers* $1,2,3,\ldots,p$ *but for odd strongly multiplicative labeling p vertices are labeled with integers* $1,2,3,\ldots,2p-1$. *We denote even strongly multiplicative as ESML and odd strongly multiplicative as OSML.*

Remark 1.2. *Most of graphs can be proved to be odd strongly multiplicative by the choice of the large vertex set but identifying even strongly multiplicative graphs is challenging and interesting. For example most of the complete graph* k_n, *are SML and OSML but not ESML See fig* 1.

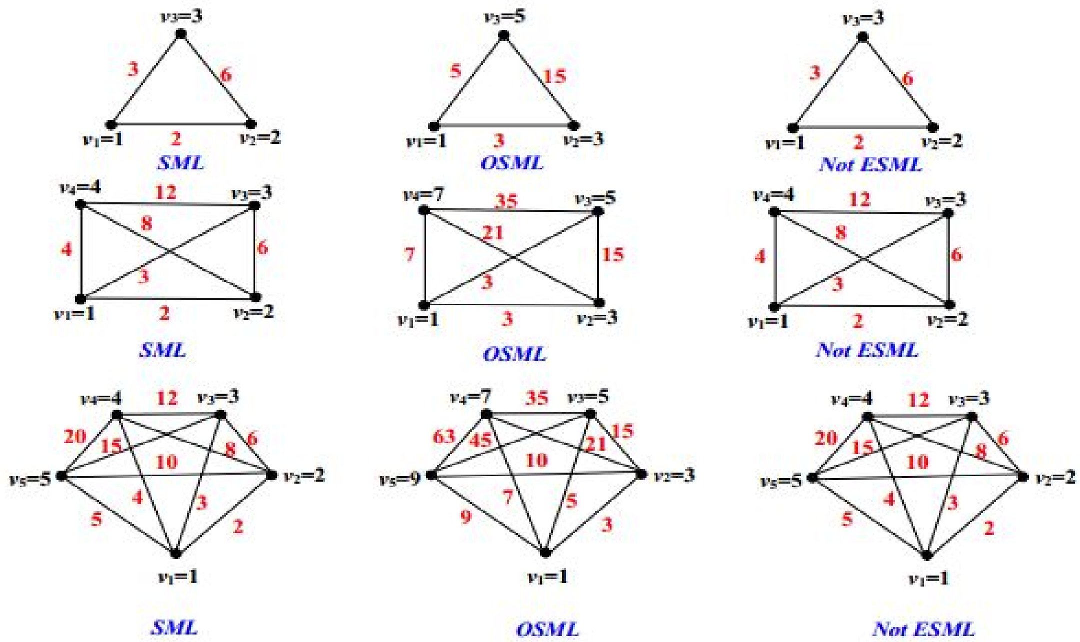

Figure 1. Variation of SML in Complete Graph K_3, K_4, K_5

2. Path related Graph

2.1 Path Graph P_n

Theorem 2.1. *The path graph P_n is odd strongly multiplicative.*

Proof. Let the vertex set of path graph P_n be $V=\{v_i, 1\le i\le n\}$ and the edge set be $E=\{e_i=(v_i,v_{i+1})/1\le i\le n-1\}$. Define the vertex labeling $V(P_n)$ as a bijective map $f:V\to N$ such that $f(v_i)=2i-1,\quad 1\le i\le n$.
Claim: All the edge labelings in the edge set E are odd and distinct. If e_i and e_p are distinct edges in E then to prove $g(e_i)\neq g(e_p)$. Assume that

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_i,v_{i+1}) &= g(v_p,v_{p+1}) \\ f(v_i)f(v_{i+1}) &= f(v_p)f(v_{p+1}) \\ (2i-1)(2i+1) &= (2p-1)(2p+1) \Rightarrow i=p, \end{aligned}$$

a contradiction for i. Hence $g(e_i)\neq g(e_p), \forall 1\le i,p\le n$. Thus all edge labelings in E are distinct and odd. Hence path is odd strongly multiplicative. □

Theorem 2.2. *The Path P_n is even strongly multiplicative.*

Proof. Let the vertex set of P_n be $V=\{v_i/1\le i\le n\}$ and the edge set of P_n be $E=\{e_i=(v_i,v_{i+1})/1\le i\le n-1\}$. The labeling of vertices of P_n is defined as bijection map $f:V\to N$ such that $f(v_i)=i, 1\le i\le n$. Define an edge induced function $g:E\to N$ such that for all $e_i\in E, g(e_i)=f(v_i)f(v_{i+1}), 1\le i\le n-1$. Thus by this edge labelings of E it is obivous that all labelings in E are even and distinct. Hence the Path P_n is even strongly multiplicative. □

2.2 Comb Graph $P_n\odot K_1$

Definition 2.3 ([5])**.** *Let P_n be a path on n vertices. The comb graph is defined as $P_n\odot K_1$. It has $2n$ vertices and $2n-1$ edges.*

Theorem 2.4. *The Comb graph $P_n\odot K_1$ is odd strongly multiplicative for all n.*

Proof. Let the vertex set of $P_n\odot K_1$ be $V=\{v_i, 1\le i\le n\}$ and the edge set be $E=E_1\cup E_2$, where

$$\begin{aligned} E_1 &= \{e_i=(v_i,v_{i+1}), 1\le i\le n-1\}, \\ E_2 &= \{e_i=(v_i,v_{n+i}), 1\le i\le 2n\} \end{aligned}$$

Define the vertex labeling of $P_n\odot K_1$ as a bijection map $f: V\to N$ such that

$$f(v_i)=2i-1, f(v_{n+i})=2(n+i)-1, 1\le i\le n.$$

To prove all the edge labelings in E are odd and distinct. It is sufficient to prove that the labeling within the edge set E_1 and E_2; among E_1 and E_2 are odd and distinct.
Claim: All the labelings in edge set E_1 are odd and distinct. Define an edge induced function $g:E_1\to N$ such that for all $e_i\in E_1$

$$g(e_i)=f(v_i)f(v_{i+1})=(2i-1)(2(i+1)-1), 1\le i\le n-1.$$

Therefore the edge labelings in E_1 are odd yet it remains to prove that all edge labelings in E_1 are distinct. If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$. Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{i+1}) =& g(v_p, v_{p+1}) \\ f(v_i) f(v_{i+1}) =& f(v_p) f(v_{p+1}) \\ (2i-1)(2i+1) =& (2p-1)(2p+1) \Rightarrow i = p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n-1$. Thus all edge labelings within E_1 are odd and distinct.

Claim: All the labelings within edge set E_2 are odd and distinct. Define an edge induced function $g : E_2 \to N$ such that for all $e_i \in E_2$

$$g(e_i) = f(v_i) f(v_{n+i}) = (2i-1)(2(n+i)-1), 1 \leq i \leq n-1.$$

Therefore the edge labelings in E_2 are odd yet it remains to prove that all edge labelings in E_2 are distinct. If e_i and e_p are distinct edges in E_2 then to prove $g(e_i) \neq g(e_p)$. Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{n+i}) =& g(v_p, v_{n+p}) \\ f(v_i) f(v_{n+i}) =& f(v_p) f(v_{n+p}) \\ (2i-1)(2(n+i)-1) =& (2p-1)(2(n+p)-1) \\ x(2n+x) = y(2n+y),& \text{ where } x = 2i-1, y = 2p-1 \\ x = -2n-y \Rightarrow i =& (1-n-p), \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n$. Thus all edge labelings within E_2 are odd and distinct.

Claim: All edge labelings among E_1 and E_2 are odd and distinct. If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$. For $i \neq p, 1 \leq i \leq n-1$ and $1 \leq p \leq n$. Assume that,

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{i+1}) =& g(v_p, v_{n+p}) \\ f(v_i) f(v_{i+1}) =& f(v_p) f(v_{n+p}) \\ (2i-1)(2i+1) =& (2p-1)(2(n+p)-1) \\ \Rightarrow i =& \frac{1}{2}\sqrt{(2p-1)(2n+2p-1)}, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i \leq n-1$.

Thus all edge labelings in E_1 and E_2 are odd and distinct. Hence all labelings induced on the edge set E of comb graph are odd and distinct. Hence comb graph $P_n \odot K_1$ is odd strongly multiplicative for even $n \geq 2$. □

Theorem 2.5. *The comb graph $P_n \odot K_1$, is even strongly multiplicative for all $n \geq 2$.*

Proof. Let the vertex set of comb graph $P_n \odot K_1$ be $V = \{v_i / 1 \leq i \leq 2n\}$ and the edge set be $E = E_1 \cup E_2$, where

$$\begin{aligned} E_1 =& \{e_i = (v_i, v_{i+1}) / 1 \leq i \leq n-1\}, \\ E_2 =& \{e_i = (v_i, v_{n+i}) / 1 \leq i \leq n\} \end{aligned}$$

Define the vertex labeling of L_n as $f : V \to N$ such that for $1 \leq i \leq n$

$$f(v_i) = \begin{cases} 2i-1, i-odd \\ 2i, \quad i- \text{ even} \end{cases}$$

$$f(v_{n+i}) = \begin{cases} 2i, i-o\bar{d}d \\ 2i-1, \quad i- \text{ even} \end{cases}$$

We claim that all the edge labelings in E are even and distinct.

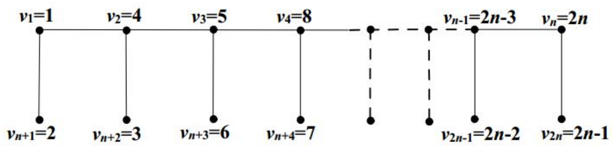

Figure 2. ESML of Comb Graph $P_n \odot K_1$

Claim: All edge labelings are distinct and even in E_1. Define an edge induced function $g : E_1 \to N$ such that for all $e_i \in E_1$,

$$g(e_i) = f(v_i) f(v_{i+1}) = (2i-1)2i, 1 \leq i \leq n-1.$$

Therefore the edge labeling in E_1 is even yet it remains to prove that the edge labelings are distinct. If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$. Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{i+1}) =& g(v_p, v_{p+1}) \\ f(v_i) f(v_{i+1}) =& f(v_p) f(v_{p+1}) \\ (2i-1)2i =& (2p-1)2p \Rightarrow i = \frac{1}{2} - p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n-1$. Thus all edge labelings in E_1 are distinct and even.

Claim: All edge labelings in E_2 are distinct and even. Define an edge induced function $g : E_2 \to N$ such that for all $e_i \in E_2$,

$$g(e_i) = f(v_i) f(v_{n+i}) = (2i-1)2i, 1 \leq i \leq n.$$

Therefore the edge labelings in E_2 is even yet it remains to prove that the edge labelings are distinct. If e_i and e_p are distinct edges in E_2 then to prove $g(e_i) \neq g(e_p)$. Assume that,

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{n+i}) =& g(v_p, v_{n+p}) \\ f(v_i) f(v_{n+i}) =& f(v_p) f(v_{n+p}) \\ (2i-1)2i =& (2p-1)2p \Rightarrow i = \frac{1}{2} - p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n$. Thus all edge labelings in E_2 are distinct and even.

Claim: All edge labelings are distinct and even in E_1 and E_2. If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$. For $i \neq p, 1 \leq i \leq n-1$ and $1 \leq p \leq n$. Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{i+1}) =& g(v_p, v_{n+p}) \\ f(v_i) f(v_{i+1}) =& f(v_p) f(v_{n+p}) \\ (2i-1)2i =& (2p-1)2p \Rightarrow i = \frac{1}{2} - p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i \leq n-1$. Thus all edge labelings in E_1 and E_2 are distinct and even. Hence all edge labeling induced on the edge set E of comb graph are distinct and even. Therefore Comb graph is even strongly multiplicative. □

3. Ladder graph L_n

Definition 3.1 ([6]). *The ladder graph L_n is defined by $L_n = P_n \times K_2$ where P_n is a path on n vertices and K_2 is a path on two-vertices.*

Theorem 3.2. *The Ladder graphs L_n is odd strongly multiplicative for all $n \geq 2$.*

Proof. Let the vertex set of L_n be $V = \{v_i, 1 \leq i \leq n\}$ and the edge set be $E = E_1 \cup E_2$, where

$$E_1 = \{e_i = (v_i, v_{i+1}), 1 \leq i \leq n-1, n+1 \leq i \leq 2n-1\}$$
$$E_2 = \{e_i = (v_i, v_{n+i}), 1 \leq i \leq n\}$$

The labeling of vertices of L_n is defined as bijection $f : V \to N$ such that for all $1 \leq i \leq n$ $f(v_i) = 2i-1$ and $f(v_{n+i}) = 2(n+i)-1$. To prove Ladder graphs L_n is odd strongly multiplicative, the labelings in the edge set E should be odd and distinct.
Claim: All the labelings in edge set E_1 are odd and distinct. Define an edge induced function $g : E_1 \to N$ such that for all $e_i \in E_1$,

$$g(e_i) = f(v_i) f(v_{i+1}) = (2i-1)(2i+1), 1 \leq i \leq n-1$$

Therefore the edge labelings in E_1 are odd. If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$. For $i \neq p, 1 \leq i, p \leq n-1, n+1 \leq i, p \leq 2n-1$. Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{i+1}) =& g(v_p, v_{p+1}) \\ f(v_i) f(v_{i+1}) =& f(v_p) f(v_{p+1}) \\ (2i-1)(2i+1) =& (2p-1)(2p+1) \Rightarrow i = p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n-1$. Thus all edge labelings in E_1 are odd and distinct.
Claim: All the labelings in edge set E_2 are odd and distinct. Define an edge induced function $g : E_2 \to N$ such that for all $e_i \in E_2$,

$$g(e_i) = f(v_i) f(v_{n+i}) = (2i-1)(2n+2i-1), 1 \leq i \leq n$$

Therefore the edge labelings in E_2 are odd. If e_i and e_p are distinct edges in E_2 then to prove $g(e_i) \neq g(e_p)$ Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{n+i}) =& g(v_p, v_{n+p}) \\ f(v_i) f(v_{n+i}) =& f(v_p) f(v_{n+p}) \\ (2i-1)(2n+2i-1) =& (2p-1)(2n+2p-1) \\ x = -2n - y \Rightarrow& i = 1-n-p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n$. Thus all edge labelings in E_2 are odd and distinct.
Claim: All the edge labelings among E_1 and E_2 are odd and distinct. If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$. For $i \neq p, 1 \leq i \leq n-1, n+1 \leq i \leq 2n-1$ and $1 \leq p \leq n$. Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{i+1}) =& g(v_p, v_{n+p}) \\ f(v_i) f(v_{i+1}) =& f(v_p) f(v_{n+p}) \\ (2i-1)(2i+1) =& (2p-1)(2n+2p-1) \\ \Rightarrow i =& \frac{1}{2}\sqrt{(2p-1)(2n+2p-1)}, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i \leq n-1$. Thus all edge labelings in E_1 and E_2 are odd and distinct. Hence all edge labelings in ladder graph are odd and distinct. □

Theorem 3.3. *The Ladder graph L_n is even strongly multiplicative for all $n \geq 2$.*

Proof. Let the vertex set of L_n be $V = \{v_i, 1 \leq i \leq 2n\}$ and the edge set $E = E_1 \cup E_2$, where

$$E_1 = \{e_i = (v_i, v_{i+1}), 1 \leq i \leq n-1 \text{ and } n+1 \leq i \leq 2n-1\}$$
$$E_2 = \{e_i = (v_i, v_{n+i}), 1 \leq i \leq n\}.$$

Define the vertex labeling of L_n as a bijection map $f : V \to N$ such that for $1 < i < n$,

$$f(v_i) = \begin{cases} 2i-1, i-o\bar{d}d \\ 2i, \quad i- \text{ even} \end{cases}$$
$$f(v_{n+i}) = \begin{cases} 2i, i- \text{ odd} \\ 2i-1, \quad i- \text{ even} \end{cases}$$

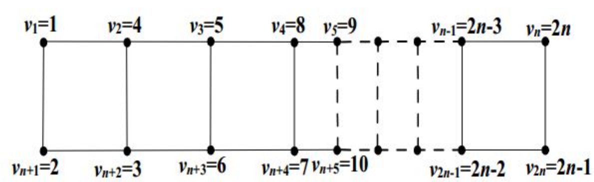

Figure 3. ESML of Ladder L_n

To prove even strongly multiplicative, we claim that all the edge labelings in E are distinct and even. Claim: All edge labelings are distinct and even in E_1 Define an edge induced function $g : E_1 \to N$ such that for all $e_i \in E_1$,

$$g(e_i) = f(v_{n+i}) f(v_{n+i+1}) = 2i(2i-1), n+1 \leq i \leq 2n-1.$$

Therefore the edge labelings in E_1 is always be even yet it remains to prove that the edge labelings are distinct. If e_i

and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$. Assume that,

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_{n+i}, v_{n+i+1}) &= g(v_{n+p}, v_{n+p+1}) \\ f(v_{n+i}) f(v_{n+i+1}) &= f(v_{n+p}) f(v_{n+p+1}) \\ (2i-1)2i &= (2p-1)2p \Rightarrow i = \frac{1}{2} - p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall n+1 \leq i, p \leq 2n-1$. Thus all edge labelings in E_1 are distinct and even. For the cases in proving the labeling of edges in E_1, E_2, and $E_1 \& E_2$ are distinct and even, the proof is similar to that of the proof of comb graph. Therefore all edge labeling in ladder graph are distinct and even. Hence ladder graph is even strongly multiplicative for all $n \geq 2$. □

Thus all Path related graphs like path, comb and ladder graphs are both odd and even strongly multiplicative.

4. Star Related Graphs

In this section star related graphs such as star and bistar are confirmed to be odd and even strongly multiplicative graphs.

Definition 4.1 ([3]). *A star S_n is the complete bipartite graph $K_{1,n}$. It is also defined as a tree with one internal node and n leaves.*

4.1 Star Graphs

Theorem 4.2. *The star graphs are odd strongly multiplicative.*

Proof. let the vertex set of star graph be $V = \{v_i, 1 \leq i \leq n\}$ and the edge set be $E = \{e_i = (v_1, v_i), 2 \leq i \leq n\}$. Define the vertex labeling of S_n as a bijection map $f : V \to N$ such that

$$f(v_i) = \begin{cases} 1, & i = 1 \\ 2i-1, & i = 2,3\ldots n \end{cases}$$

To prove Star graphs are odd strongly multiplicative, the labelings in the edge set E should be odd and distinct. If e_i and e_p are distinct edges in E then to prove $g(e_i) \neq g(e_p)$. Assume that,

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_1, v_i) &= g(v_1, v_p) \\ f(v_1) f(v_i) &= f(v_1) f(v_p) \\ 1(2i-1) &= 1(2p-1) \Rightarrow i = p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 2 \leq i, p \leq n$. Thus all edge labelings in the star graphs are distinct and odd. Thus the Star graph S_n is odd strongly multiplicative. □

Theorem 4.3. *The star graph S_n is even strongly multiplicative.*

Proof. Let the vertex set of star graph S_n be $V = \{v_i / 1 \leq i \leq n\}$ and the edge set be

$$E = \{e_i = (v_i, v_n), 1 \leq i \leq n-1\}.$$

Define the vertex labeling of star graph as a bijection map $f : V \to N$ such that

$$f(v_i) = i+1, 2 \leq i \leq n-1, f(v_1) = 1$$

and $f(v_n) = 2$. Define an edge induced function $g : E \to N$

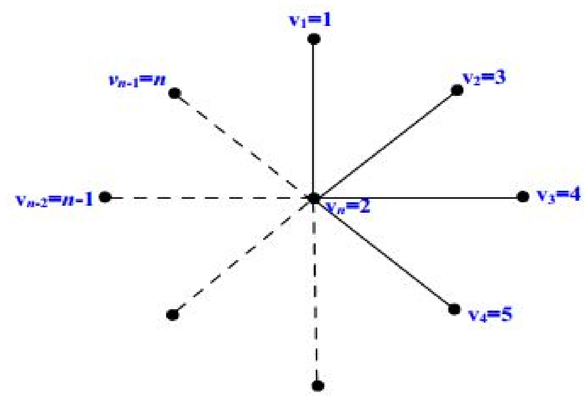

Figure 4. Star Graph

such that for all $e_i \in E$,

$$g(e_i) = f(v_i) f(v_n) = 2(i+1), 2 \leq i \leq n-1.$$

Thus all edge labelings of E are even and distinct. Hence star graph S_n is even strongly multiplicative for all $n \geq 2$. □

4.2 Bi-star Graph $B(m,n)$

Definition 4.4 ([4]). *The Bi-star $B(m,n)$ is a two star graph, one with $m+1$ and other with $n+1$ vertices along with an edge joining the apex of the two star graphs.The Bi-star $B(m,n)$ has $m+n+2$ vertices and $m+n+1$ edges.*

Theorem 4.5. *The Bi-star $B(m,n)$ is even strongly multiplicative for all $m < n$.*

Proof. Let the vertex set of bi-star graph $B(m,n)$ be $V = \{v_i / 1 \leq i \leq m+n+2\}$ and the edge set be $E = E_1 \cup E_2 \cup \{e\}$, where $e = (v_{m+n+1}, v_{m+n+2})$,

$$\begin{aligned} E_1 &= \{e_i = (v_i, v_{m+n+2}), 1 \leq i \leq m\} \\ E_2 &= \{e_i = (v_{m+i}, v_{m+n+1}) / 1 \leq i \leq n\}. \end{aligned}$$

Define the vertex labeling of $B(m,n)$ as a bijective map $f : V \to N$ such that,

$$\begin{aligned} f(v_i) &= 2i, 1 \leq i \leq m \\ f(v_{m+i}) &= 2i-1, 1 \leq i \leq m \\ f(v_{m+i}) &= 2m+i, m+1 \leq i \leq n \end{aligned}$$

If $m+n$ is odd then,

$$f(v_{m+n+1}) = m+n+1, f(v_{m+n+2}) = m+n+2.$$

If $m+n$ is even then

$$f(v_{m+n+1}) = m+n+2, f(v_{m+n+2}) = m+n+1.$$

To prove bi-star is even strongly multiplicative we claim that all the edge labelings in E are distinct and even.

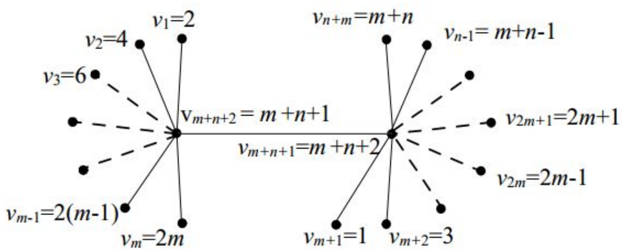

Figure 5. ESML of bi-star $B(m,n)$, $m+n$-even

Consider the case when $m<n$ and $m+n$ is even.

Claim: All the edge labelings in E_1 are distinct and even. Define an edge induced function $g: E_1 \to N$ such that for all $e_i \in E_1$,

$$g(e_i) = f(v_i) f(v_{m+n+2}) = 2i(m+n+1), 1 \leq i \leq m.$$

Therefore all the edge labeling in E_1 is even, it remains to prove that the edge labeling are distinct. If e_i and e_p are distinct edges in E_1 to prove $g(e_i) \neq g(e_p)$. Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{m+n+2}) =& g(v_p, v_{m+n+2}) \\ f(v_i) f(v_{m+n+2}) =& f(v_p) f(v_{m+n+2}) \\ 2i(m+n+1) =& 2p(m+n+1) \Rightarrow i = p, \end{aligned}$$

a contradiction for i Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq n$. Thus all edge labelings in E_1 are distinct and even.

Claim: All edge labelings within E_2 are distinct and even. Define an edge induced function $g: E_2 \to N$ such that for all $e_i \in E_2$,

$$\begin{aligned} g(e_i) =& f(v_{m+i}) f(v_{m+n+1}) = (2i-1)(m+n+2), 1 \leq i \leq m \\ g(e_i) =& f(v_{m+i}) f(v_{m+n+1}) = (2m+i)(m+n+2), m+1 \leq i \leq n \end{aligned}$$

Therefore all the edge labelings in E_2 are even yet it remains to prove that the edge labelings are distinct. If e_i and e_p are distinct edges in E_2 then to prove $g(e_i) \neq g(e_p)$.

Case 1: For $i \neq p, 1 \leq i \leq m$ and $m+1 \leq p \leq n$ Assume that

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_{m+i}, v_{m+n+1}) =& g(v_{m+p}, v_{m+n+1}) \\ f(v_{m+i}) f(v_{m+n+1}) =& f(v_{m+p}) f(v_{m+n+1}) \\ (2i-1)(m+n+2) =& (2m+p)(m+n+2) \\ \Rightarrow i =& \frac{2m+1+p}{2}, \end{aligned}$$

a contradiction for i Hence $g(e_i) \neq g(e_p), \forall 1 \leq i \leq m$.

Case 2: For $i \neq p, 1 \leq i, p \leq m$. Assume that

$$\begin{aligned} (e_i) =& g(e_p) \\ g(v_{m+i}, v_{m+n+1}) =& g(v_{m+p}, v_{m+n+1}) \\ f(v_{m+i}) f(v_{m+n+1}) =& f(v_{m+p}) f(v_{m+n+1}) \\ (2i-1)(m+n+2) =& (2p-1)(m+n+2) \end{aligned}$$

$\Rightarrow i = p$, a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq m$.

Case 3: For $i \neq p, m+1 \leq i, p \leq n$. Assume that,

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_{m+i}, v_{m+n+1}) =& g(v_{m+p}, v_{m+n+1}) \\ f(v_{m+i}) f(v_{m+n+1}) =& f(v_{m+p}) f(v_{m+n+1}) \\ (2m+i)(m+n+2) =& (2m+p)(m+n+2) \end{aligned}$$

$\Rightarrow i = p$, a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq m$. Thus all edge labelings in E_2 are distinct and even.

Claim: All the edge labeling in E_1 and E_2 are distinct and even If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.

Case 1: For $i \neq p, 1 \leq i, p \leq m$. Assume that,

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{m+n+2}) =& g(v_{m+p}, v_{m+n+1}) \\ f(v_i) f(v_{m+n+2}) =& f(v_{m+p}) f(v_{m+n+1}) \\ 2i(m+n+1) =& (2p-1)(m+n+2) \\ \Rightarrow i =& \frac{(2p-1)(m+n+2)}{2(m+n+1)}, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i, p \leq m$.

Case 2: For $i \neq p, 1 \leq i \leq m$ and $m+1 \leq p \leq n$. Assume that,

$$\begin{aligned} g(e_i) =& g(e_p) \\ g(v_i, v_{m+n+2}) =& g(v_{m+p}, v_{m+n+1}) \\ f(v_i) f(v_{m+n+2}) =& f(v_{m+p}) f(v_{2n+1}) \\ 2i(m+n+1) =& (2m+p)(m+n+2) \\ i =& \frac{(2m+p)(m+n+2)}{2(m+n+1)}, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i \leq m$ and $m+1 \leq p \leq n$. Thus all edge labelings in E_1 and E_2 are distinct and even.

The labelings in $E_1 \cup (e)$ and $E_2 \cup (e)$ can be proved to be distinct and even. Similar proof holds in the case when $m+n$ is odd. Thus $B(m,n), m<n$ is even strongly multiplicative.

Case 2: When $m>n$. Interchanging the role of m and n in above proof, we can prove that all edge labelings in $B(m,n)$ are distinct and even.Thus $B(m,n), m>n$ is even strongly multiplicative.

Case 3: When $m=n$. Define the vertex labeling of $B(m,n)$ as $f: V \to N$ such that

$$f(v_i) = 2i, 1 \leq i \leq m, f(v_{m+i}) = 2i-1, 1 \leq i \leq n$$

and for $m+n$ is even

$$f(v_{m+n+1}) = m+n+2,\ f(v_{m+n+2}) = m+n+1.$$

Thus the edge labelings in $B(m,n)$ are even strongly multiplicative for all m and n. □

Theorem 4.6. *The bi-Star is odd strongly multiplicative for all $m < n$.*

Proof. Let the vertex set of $B(m,n)$ be

$$V = \{v_i, 1 \le i \le m+n+2\}$$

and the edge set be $E = E_1 \cup E_2 \cup \{e\}$, where

$$\begin{aligned} e &= (v_{m+n+1}, v_{m+n+2}), \\ E_1 &= \{e_i = (v_i, v_{m+n+2}), 1 \le i \le m\} \\ E_2 &= \{e_i = (v_{m+i}, v_{m+n+1}), 1 \le i \le n\} \end{aligned}$$

Define the vertex labeling of $B(m,n)$ as a bijection map $f : V \to N$ such that $f(v_i) = 2i-1\ 1 \le i \le m, \quad f(v_{m+i}) = 2(m+i)-1, 1 \le i \le n+2$.

To prove Bi-star is odd strongly multiplicative, the labelings in the edge set E should be odd and distinct.
Claim : All edge labelings in E_1 are odd and distinct Define an edge induced function $g : E_1 \to N$ such that for all $e_i \in E_1$,

$$g(e_i) = f(v_i) f(v_{m+n+2}) = (2i-1)(2(m+n+2)-1), 1 \le i \le m.$$

Therefore the edge labelings in E_1 is odd. If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \ne g(e_p)$. Assume that,

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_i, v_{m+n+2}) &= g(v_p, v_{m+n+2}) \\ f(v_i) f(v_{m+n+2}) &= f(v_p) f(v_{m+n+2}) \\ (2i-1)(2(m+n)+3) &= (2p-1)(2(m+n)+3) \end{aligned}$$

$\Rightarrow i = p$, a contradiction for i. Hence $g(e_i) \ne g(e_p), \forall 1 \le i, p \le n$. Thus all edge labelings in E_1 are odd and distinct.
Claim: All labelings in edge set E_2 are odd and distinct. Define an edge induced function $g : E_2 \to N$ such that for all $e_i \in E_2$,

$$g(e_i) = f(v_{m+i}) f(v_{m+n+1}) = (2i-1)(2(m+n+1)-1), 1 \le i \le n.$$

Therefore the edge labelings in E_2 are odd. If e_i and e_p are distinct edges in E_2 then to prove $g(e_i) \ne g(e_p)$. Assume that,

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_{m+i}, v_{m+n+1}) &= g(v_{m+p}, v_{m+n+1}) \\ f(v_{m+i}) f(v_{m+n+1}) &= f(v_{m+p}) f(v_{m+n+1}) \\ (2i-1)[2(m+n+1)-1] &= (2p-1)[2(m+n+1)-1] \end{aligned}$$

$\Rightarrow i = p$, a contradiction for i. Hence $g(e_i) \ne g(e_p), \forall 1 \le i, p \le n$. Thus all edge labelings in E_2 are odd and distinct.
Claim: All edge labelings among E_1 and E_2 are odd and distinct. If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \ne g(e_p)$. For $i \ne p, 1 \le i \le m, 1 \le p \le n$. Assume that,

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_i, v_{m+n+2}) &= g(v_{m+p}, v_{m+n+1}) \\ f(v_i) f(v_{m+n+2}) &= f(v_{n+p}) f(v_{m+n+1}) \\ (2i-1)(2(m+n+2)-1) &= (2p-1)(2(m+n+1)-1) \\ \Rightarrow i &= \frac{(2p-1)(2(m+n)+1)}{2(m+n)+3}, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \ne g(e_p), \forall 1 \le i \le m$. Thus all edge labelings of Bi-Star graph are odd and distinct. Thus Bi-Star graph is odd strongly multiplicative for $n \ge 2$.

Remark 4.7. *When $m > n$, bi-star graph can be proved to be OSML and ESML by interchanging the role of m and n in above proof, we can prove that all edge labelings in $B(m,n)$ are distinct and even. Thus $B(m,n), m > n$ is even strongly multiplicative.*

Remark 4.8. *When $m = n$. bi-star graph can be proved to be OSML and ESML by defining the vertex labeling of $B(m,n)$ as $f : V \to N$ such that $f(v_i) = 2i, 1 \le i \le m, f(v_{m+i}) = 2i-1\ 1 \le i \le n$ and for $m+n$ is even $f(v_{m+n+1}) = m+n+2, f(v_{m+n+2}) = m+n+1$.*

Thus all star related grpahs like star and bi-star graphs are both odd and even strongly multiplicative. □

5. Cycle graph C_n

Theorem 5.1. *The cycle C_n is odd strongly multiplicative.*

Proof. To prove C_n, is odd strongly multiplicative when $n \ne 2r^2, r = 2, 3, \ldots\ldots$.

Let the vertex set of C_n be $V = \{v_i, 1 \le i \le n\}$ and the edge set be for $1 \le i \le n-1$, $E = \{e_i = (v_i, v_{i+1}) \cup (v_n, v_1)\}$.

Define the vertex labeling of C_n be defined as bijection map $f : V \to N$ such that $f(v_i) = 2i-1, 1 \le i \le n$.
Claim : All the labelings in edge set E are odd and distinct. Define an edge induced function $g : E \to N$ such that for all $e_i \in E$ $g(e_i) = f(v_i) f(v_{i+1}) = (2i-1)(2i+1), 1 \le i \le n-1$.

Therefore the edge labelings in E is odd yet it remains to prove the labelings are distinct. If e_i and e_p are distinct edges in E then to prove $g(e_i) \ne g(e_p)$. Assume that,

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_i, v_{i+1}) &= g(v_p, v_{p+1}) \\ f(v_i) f(v_{i+1}) &= f(v_p) f(v_{p+1}) \\ (2i-1)(2i+1) &= (2p-1)(2p+1) \Rightarrow i = p, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \ne g(e_p), \forall 1 \le i, p \le n-1$.

For $i \neq p, 1 \leq i \leq n-1, p = n$. Assume that,

$$\begin{aligned} g(e_i) &= g(e_p) \\ g(v_i, v_{i+1}) &= g(v_n, v_1) \\ f(v_i) f(v_{i+1}) &= f(v_n) f(v_1) \\ (2i-1)(2i+1) &= (2n-1)1 \Rightarrow i = \sqrt{\frac{n}{2}}, \end{aligned}$$

a contradiction for i. Hence $g(e_i) \neq g(e_p), \forall 1 \leq i \leq n-1$. Thus all edge labelings in E are odd and distinct. To prove C_n is odd strongly multiplicative when $n = 2r^2, r = 2, 3, \ldots$. Let the vertex set of C_n be $V = \{v_i / 1 \leq i \leq n\}$ and the edge set be

$$E = \{e_i = (v_i, v_{i+1}) \cup (v_n, v_1), 1 \leq i \leq n-1\}.$$

Define the vertex labeling of C_n as follows as $f : V \to N$ such that

$$f(v_i) = \begin{cases} 3, & i = 1 \\ 1, & i = 2 \\ 2i-1, & 3 \leq i \leq n \end{cases}$$

it can be proved similarly that all edge labeling in E are odd distinct. Hence Cycle C_n is odd strongly multiplicative $n \geq 2$. Similar it can analysed that cycle C_n is ESML.

□

6. Conclusion

As every graph does not admit odd and even strongly multiplicative labeling it is very interesting to investigate graphs or graph families which admit odd and even strongly multiplicative labeling. In the present work we investigate three new families of odd and even strongly multiplicative graphs.

Finding odd and even strongly multiplicative labeling for other interconnection networks are quite challenging.

References

[1] B. D. Acharya, K. A. Germina, V. Ajitha, *Multiplicatively Indexable Graphs in Labeling Discrete Structures and Applications*, Narosa Publishing House, New Delhi, pp. 29 - 40, 2008.

[2] L. W. Beineke, S. M. Hegde, Strong Multiplicative Graphs, Discuss. *Math Graph Theory*, 21(2001), 63-75.

[3] J. A. Gallian, A Dynamic Survey of Graph Labeling, *Electronic Journal of Combinatorics*, 2015.

[4] S. K. Vaidya and N. H. Shah, Some Star and Bistar Related Divisor Cordial Graphs, *Annals of Pure and Applied Mathematics,* 3(1)(2013), 67-77.

[5] E. Esakkiammal, B. Deepa, K. Thirusangu, *Some Labelings on Square Graph of Comb*, International Journal of Mathematics Trends and Technology (IJMTT) — Special Issue NCCFQET ISSN: 2231-5373 Page 27, 2018.

[6] Solairaju, S. Subbulakshmi and R. Kokila, Various Labelings for Ladder, Cycle Merging with Fan, and Open Staircase Graph A, *Global Journal of Pure and Applied Mathematics,* 13(5)(2017), 1347-1355.

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