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A note on t-Cayley hypergraphs

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Abstract
In this paper we study some properties of t-Cayley hypergraph in terms of algebraic properties. This did not attract much attention in the literature.

Keywords
Hypergraph, t- hypergraph, k-transitive, mn-transitive.

AMS Subject Classification
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Contents

1. Introduction

A hypergraph H is a pair $(V(H);E(H))$, where $V(H)$ is a finite nonempty set and $E(H)$ is a finite family of nonempty subsets of $V(H)$. The elements of $V(H)$ are called vertices and the elements of $E(H)$ are called edges. Two vertices in a hypergraph are adjacent if there is a hyperedge which contains both vertices [1].

A *path* of length k in a hypergraph $(V(H);E(H))$ is an alternating sequence $(v_1,e_1,v_2,\dots,v_k,e_k,v_{k+1})$ in which $v_i \in V(H)$ for each $i=1,2,\dots,k+1, e_i \in E(H), \{v_i,v_{i+1}\} \subseteq e_i$ for $i=1,2,\dots,k$ and $v_i \neq v_j$ and $e_i \neq e_j$ for $i \neq j$.

A hypergraph is connected if for any pair of vertices, there is a path which connects these vertices [1]

Let G be a group, Ω a subset of $G\setminus\{1\}$ and t an integer satisfying $2 \leq t \leq \max\{o(\omega) : \omega \in \Omega\}$. In [8], M. Buratti introduced t-Cayley Hypergraph $H = t-\mathrm{Cay}[G:\Omega]$ as follows:

$$V(H) := G \text{ and } E(H) = \{\{g, g\omega, \dots, g\omega^{t-1}\} : g \in G, \omega \in \Omega\}.$$

He proved that t-Cayley Hypergraphs are vertex transitive and regular. Moreover, he obtained a necessary and sufficient condition for t-Cayley Hypergraphs to be connected.

In [6] H. Galeana Sanchez and Cesar Hernandez-Cruz introduced the concepts of k- transitivity and k-path transitivity in Cayley digraphs. A digraph G is $k-transitive$ if the existence of a path $(x_0,x_1,\dots,x_k)$ of length k in G implies that x_0 and x_k are adjacent. A digraph is called $k-path\ transitive$ if whenever there is a xy path of length less than or equal to k and a yz path of length less than or equal to k, then there exists a xz path of length less than or equal to k.

Anil Kumar V. and Mohanan T. generalised the concept of k-transitivity as follows[3]: Let m and n be two positive integers such that $m > n$. A digraph G is $(m,n)-transitive$ if whenever there is a directed path of length m from x to y there is a directed path of length n from x to y.

In this paper we study some graph theortic properties in terms of algebraic properties.

2. Main Results

Let G be a group with identity element 1 and let Ω be a subset of $G\setminus\{1\}$. We define

$$A := \{w^n : w \in \Omega,\ n = 1,2,\dots,t-1\}\setminus\{1\}.$$

A t-Cayley Hypergraph $H = t-Cay[G:\Omega]$ is complete if and only if $G = A$.

Proof. First, assume that H be a complete hypergraph. Then for $x \in G$, 1 and x are adjacent. This implies that $1, x \in e = \{gs^i : 0 \leq i \leq t-1\}$, for some $g \in G$, and some $s \in \Omega$. This implies there exists $p,q \in \{0,1,\dots,t-1\}$ such that $1 = gs^p$ and $x = gs^q$. Observe that

$$x = gs^p.s^{q-p} = s^{q-p} \in A.$$

Since $x \in G$ is arbitrary, $G \subseteq A$. Obviously, $A \subseteq G$. Therefore, $G = A$.

Conversely, assume that $G=A$. We want to show that H is complete. Let $x,y\in G$. Then $y=xz$ for some $z\in G$. Since $G=A$, $z\in A$. Then $z=w^r$, for some $w\in\Omega$ and $r\in\{1,2,\dots,t-1\}$. This implies that $y=xw^r$. This means that x,y belongs to an edge $e=\{xw^i:0\leq i\leq t-1\}$. Therefore x and y are adjacent. This completes the proof of the theorem. □

A hypergraph G is a *hasse − diagram* if G is connected and for any path $x_0,x_1,\dots,x_n,\ n\geq 2$ from x_0 to x_n in G, x_0 and x_n are not adjacent.

H is a hasse-diagram if and only if H is connected and $A\cap A^n=\emptyset$ for $n\geq 2$.

Proof. First, assume that H is a hasse-diagram. Let $x\in A^n,\ n\geq 2$. Then there exists $w_1^{r_1},w_2^{r_2},\dots,w_n^{r_n}\in A$ where $w_1,w_2,\dots,w_n\in\Omega$ and $r_1,r_2,\dots,r_n\in\{1,2,\dots,t-1\}$ such that $x=w_1^{r_1}w_2^{r_2}\dots w_n^{r_n}$. Clearly $w_1^{r_1}w_2^{r_2}\dots w_{i-1}^{r_{i-1}}$, $w_1^{r_1}w_2^{r_2}\dots w_i^{r_i}$, $(i\in\{2,3,\dots n\})$, are adjacent. Then $1,w_1^{r_1},w_1^{r_1}w_2^{r_2}$, $\dots,w_1^{r_1}w_2^{r_2}\dots w_n^{r_n}=x$ is a path of length n from 1 to x. But since H is a hasse-diagram 1 and x are not adjacent. That is, there exist no edge $e=\{gw^i:0\leq t-1\},\ g\in G,\ w\in\Omega$ such that $1,x\in e$. This implies $x\neq 1.w^r$ for any $w\in\Omega,\ r\in\{0,1,\dots,t-1\}$ which gives $x\notin A$. That is $x\in A^n$ implies $x\notin A$ for $n\geq 2$. Therefore $A\cap A^n=\emptyset$ for $n\geq 2$.

Conversely assume that H is connected and $A\cap A^n=\emptyset$ for $n\geq 2$. Let $x,y\in G$. Then there exists a path, say, $x=x_0,x_1,\dots,x_n=y$ from x to y of length $n\geq 2$. This implies that there exists $g_i\in G$ and $w_i\in\Omega$ such that $x_{i-1},x_i\in\{g_iw_i^k:0\leq k\leq t-1\}$, $i=1,2,\dots,n$. Observe that $x_i=x_{i-1}w_i^{r_i}$ for some $r_i\in\{1,2,\dots,t-1\}$. Then

$$y=x_n=xw_1^{r_1}w_2^{r_2}\dots w_n^{r_n}\tag{2.1}$$

If x and y are adjacent, then there exist $g\in G$ and $w\in\Omega$ such that $x,y\in\{gw^k:0\leq k\leq t-1\}$. Then

$$y=xw^{k_0}\tag{2.2}$$

for some $k_0\in\{1,2,\dots,t-1\}$. From (2.1) and (2.2), $w^{k_0}=w_1^{r_1}w_2^{r_2}\dots w_n^{r_n}$, which implies $w^{k_0}\in A^n$. This implies that $w^{k_0}\in A\cap A^n$, (since $w^{k_0}\in A$), which is a contradiction to the assumption that $A\cap A^n=\emptyset$ for $n\geq 2$. Hence x and y are not adjacent. Thus H is a hasse-diagram. This completes the proof. □

The hypergraph H is k-transitive if and only if $A^k\subseteq A$.

Proof. Assume that H is k-transitive. Let $x\in A^k$. Then there exists $w_1^{r_1},w_2^{r_2},\dots,w_k^{r_k}\in A$ where $w_1,w_2,\dots,w_k\in\Omega$ and $r_1,r_2,\dots,r_k\in\{1,2,\dots,t-1\}$ such that $x=w_1^{r_1}w_2^{r_2}\dots w_k^{r_k}$.

Obviously, $1,w_1^{r_1},w_1^{r_1}w_2^{r_2},\dots,w_1^{r_1}w_2^{r_2}\dots w_k^{r_k}=x$ is a path from 1 to x of length k. Since H is k-transitive, 1 and x are adjacent. Then there exist $w\in\Omega$ such that $x=1.w^r$ for some $r\in\{1,2,\dots,t-1\}$ which implies that $x=w^r\in A$. Hence $A^k\subseteq A$.

Conversely assume $A^k\subseteq A$. Let $x,y\in V(H)$ be such that there exists a path of length k from x to y, say, $x=x_0,x_1,\dots,x_k=y$. Then we obtain $y=x.w_1^{r_1}w_2^{r_2}\dots w_k^{r_k}$ for some $w_i^{r_i}\in A$, where $w_i\in\Omega$, $r_i\in\{1,2,\dots,t-1\}$, $i=1,2,\dots,k$. Since $A^k\subseteq A$, $w_1^{r_1}w_2^{r_2}\dots w_k^{r_k}\in A$. Then there exist $w\in\Omega$ and $r\in\{1,2,\dots,t-1\}$ such that $w^r=w_1^{r_1}w_2^{r_2}\dots w_k^{r_k}$. This gives, $y=xw^r$ which clearly implies x and y are adjacent. Hence H is k-transitive. □

H is k-path transitive implies

$$SA\cup SA^2\cup\dots\cup SA^k\subseteq S,$$

where $S=A\cup A^2\cup\dots\cup A^k$.

Proof. Let $x\in SA\cup SA^2\cup\dots\cup SA^k$. Then $x\in SA^i$ for some $i\in\{1,2,\dots,k\}$. Then there exist $w_1^{r_1},w_2^{r_2},\dots,w_i^{r_i}\in A$, r_1,r_2, $\dots,r_i\in\{1,2,\dots,t-1\}$, and $a\in S$ such that $x=aw_1^{r_1}w_2^{r_2}\dots w_i^{r_i}$. Clearly $a,aw_1^{r_1},aw_1^{r_1}w_2^{r_2},\dots,aw_1^{r_1}w_2^{r_2}\dots w_i^{r_i}=x$ is a path from a to x of length $i\leq k$. Also $a\in S$ implies that $a\in A^r$ for some integer r such that $1\leq r\leq k$. Then there exist $a_1^{p_1},a_2^{p_2},\dots,a_r^{p_r}\in A$, $(a_i\in\Omega,\ p_1,p_2,\dots,p_r\in\{1,2,\dots,t-1\})$, such that $a=a_1^{p_1}a_2^{p_2}\dots a_r^{p_r}$. This implies that $1,a_1^{p_1},a_1^{p_1}a_2^{p_2},\dots,a_1^{p_1}a_2^{p_2}\dots a_r^{p_r}=a$ is a path from 1 to a of length $r\leq k$. Since H is k-path transitive there exists a path from 1 to x of length q less than or equal to k. Let this path be $1=x_0,x_1,\dots,x_q=x$. This implies that there exists $w_j\in\Omega$ and $s_j\in\{1,2,\dots,t-1\}$, $1\leq j\leq q$, such that

$$\begin{aligned}x_1&=x_0w_1^{s_1}=w_1^{s_1},\\x_2&=w_1^{s_1}w_2^{s_2},\\&\vdots\\x&=w_1^{s_1}w_2^{s_2}\dots w_q^{s_q}.\end{aligned}$$

This implies that $x\in A^q\subseteq A\cup A^2\cup\dots\cup A^k=S$. Equivalently,

$$SA\cup SA^2\cup\dots\cup SA^k\subseteq S.$$

This completes the proof. □

H is k-path transitive if and only if

$$(A\cup A^2\cup\dots\cup A^k)^2\subseteq A\cup A^2\cup\dots\cup A^k$$

Proof. Assume H is k-path transitive. Let $x\in(A\cup A^2\cup\dots\cup A^k)^2$. Then $x=a_1a_2$, such that $a_1,a_2\in A\cup A^2\cup\dots\cup A^k$, implies $a_1\in A^p$, $a_2\in A^q$ for some $p,q\in\{1,2,\dots,k\}$. Then $a_1=x_1x_2\dots x_p$ and $a_2=y_1y_2\dots y_q$ where $x_1,x_2,\dots,x_p,y_1,y_2,\dots,y_q\in A$. Then $x=a_1y_1y_2\dots y_q=a_1w_1^{r_1}w_2^{r_2}\dots w_q^{r_q}$, $w_i\in\Omega$, $r_i\in\{1,2,\dots,t-1\}$ for all $i\in\{1,2,\dots,q\}$. Clearly there exists a path from a_1 to x of length $q\leq k$. Again $a_1=x_1x_2\dots x_p$ implies that there exist a path from 1 to a_1 of length $p\leq k$. Now since H is k-path transitive there exists a path from 1 to x of length $m\leq k$. Then there exist $s_1,s_2,\dots,s_m\in\Omega$ such that $x=s_1^{p_1}s_2^{p_2}\dots s_m^{p_m}$, $p_1,p_2,\dots,p_m\in\{1,2,\dots,t-1\}$. This implies that $x\in A^m\subseteq A\cup A^2\cup\dots\cup A^k$. Hence $(A\cup A^2\cup\dots\cup A^k)^2\subseteq A\cup A^2\cup\dots\cup A^k$.

Conversely, assume $(A \cup A^2 \cup \ldots \cup A^k)^2 \subseteq A \cup A^2 \cup \ldots \cup A^k$. Let there exist a path from x to y of length $i \leq k$ and a path from y to z of length $j \leq k$. Then there exists $w_1, w_2, \ldots, w_i$, $v_1, v_2, \ldots, v_j \in \Omega$ and $r_1, r_2, \ldots, r_i, p_1, p_2, \ldots, p_j \in \{1, 2, \ldots, t-1\}$ such that $y = xw_1^{r_1} w_2^{r_2} \ldots w_i^{r_i}$ and $z = yv_1^{p_1} v_2^{p_2} \ldots v_j^{p_j}$. Then $z = xw_1^{r_1} w_2^{r_2} \ldots w_i^{r_i} v_1^{p_1} v_2^{p_2} \ldots v_j^{p_j} = xa_1a_2$, where $a_1 \in A^i$ and $a_2 \in A^j$. This implies $z = xa_0$ where $a_0 = a_1a_2 \in (A \cup A^2 \cup \ldots \cup A^k)^2$. Then by assumption $a_0 \in A \cup A^2 \cup \ldots \cup A^k$ and hence $a_0 \in A^p$, for some $p \leq k$. Then $z = xu_1^{q_1} u_2^{q_2} \ldots u_p^{q_p}$, $u_1, u_2, \ldots, u_p \in \Omega, q_1, q_2, \ldots q_p \in \{1, 2, \ldots, t-1\}$. This implies that there exist a path from x to z of length $p \leq k$. Hence H is k-path transitive. □

H is (m,n)-transitive if and only if $A^m \subseteq A^n$.

Proof. Suppose H is (m,n)-transitive. Let $x \in A^m$. Then there exist $w_1, w_2, \ldots, w_m \in \Omega$ and $r_1, r_2, \ldots, r_m \in \{1, 2, \ldots, t-1\}$ such that $x = w_1^{r_1} w_2^{r_2} \ldots w_m^{r_m}$. Then $1, w_1^{r_1}, w_1^{r_1} w_2 r_2, \ldots, w_1^{r_1} w_2^{r_2} \ldots w_m^{r_m}$ is a path from 1 to x of length m. Since H is (m,n)-transitive, there exist a path $1 = x_0, x_1, \ldots, x_n = x$ from 1 to x of length n. Then there exist $g_i \in G$ and $w_i \in \Omega$ for $i \in \{1, 2, \ldots, n\}$ such that $x_{i-1}, x_i \in e_i = \{g_i w_i^k : 0 \leq k \leq t-1\}$. Then $x_i = x_{i-1} w_i^{k_i}$ for some $k_i \in \{1, 2, \ldots, t-1\}$. Then $x = x_n = 1.w_1^{k_1} w_2^{k_2} \ldots w_n^{k_n}$, implies $x \in A^n$. Hence $A^m \subseteq A^n$.

Conversely assume that $A^m \subseteq A^n$. Let $x, y \in G$ such that there exist a path from x to y of length m. Then there exists $w_1, w_2, \ldots, w_m \in \Omega$ and $r_1, r_2, \ldots, r_m \in \{1, 2, \ldots, t-1\}$ such that $y = x.w_1^{r_1} w_2^{r_2} \ldots w_m^{r_m}$. Since $A^m \subseteq A^n$, $w_1^{r_1} w_2^{r_2} \ldots w_m^{r_m} \in A^n$. This implies that there exists $v_1, v_2, \ldots, v_n \in \Omega$ and $k_1, k_2, \ldots, k_n \in \{1, 2, \ldots, t-1\}$ such that $w_1^{r_1} w_2^{r_2} \ldots w_m^{r_m} = v_1^{r_1} v_2^{r_2} \ldots v_n^{r_n}$. Then $y = xv_1^{r_1} v_2^{r_2} \ldots v_n^{r_n}$. Clearly $x, xv_1^{k_1}, xv_1^{k_1} v_2^{k_2}, \ldots, xv_1^{r_1} v_2^{r_2} \ldots v_n^{r_n} = y$ is a path from x to y of length n. Hence H is (m,n)-transitive.

□

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